$au_x + bu_y = c$

$u = g$ ON γ .... SMOOTH CURVE IN $\mathbb{R}^2$

(a, b, c ... FUNCTIONS OF (x, y, u))

$\Big\downarrow$ $(x(\tau), y(\tau))$ ... CURVE IN $\mathbb{R}^2$, $z(\tau) = u(x(\tau), y(\tau))$

$$z'(\tau) = u_x(x(\tau), y(\tau))\,x'(\tau) + u_y(x(\tau), y(\tau))\,y'(\tau)$$
$$z(0) = u(x(0), y(0)) = g(x(0), y(0))$$

$\Big\downarrow$ $x'(\tau) = a(x(\tau), y(\tau), z(\tau))$, $y'(\tau) = b(x(\tau), y(\tau), z(\tau))$

$$x'(\tau) = a(x(\tau), y(\tau), z(\tau))$$
$$y'(\tau) = b(x(\tau), y(\tau), z(\tau))$$
$$z'(\tau) = c(x(\tau), y(\tau), z(\tau))$$
$$z(0) = g(x(0), y(0)), \quad (x(0), y(0)) \in \gamma$$

$\Big\downarrow$ $a, b, c \in C^1(\mathbb{R})$

UNIQUE (LOCAL) SOLUTION

$(x(\tau), y(\tau), z(\tau))$

$\Big\downarrow$

$u(x(\tau), y(\tau)) = z(\tau)$ SOLUTION ALONG $(x(\tau), y(\tau))$

$(x(\tau,\sigma), y(\tau,\sigma))$ ... CURVE $(x(\tau), y(\tau))$ WITH $(x(0), y(0)) = (r_1(\sigma), r_2(\sigma)) \in \gamma$

$z(\tau,\sigma)$ ... $z(\tau)$ WITH $z(0) = u(x(0), y(0))$

$\left|\det\begin{pmatrix} x_\sigma & x_\tau \\ y_\sigma & y_\tau \end{pmatrix}\right| \neq 0 \quad \forall (\tau,\sigma)$

$\Big\downarrow$ INVERSE FCT THM

u ... WELL-DEFINED

$\left|\det\begin{pmatrix} x_\sigma & x_\tau \\ y_\sigma & y_\tau \end{pmatrix}\right| = 0$ FOR SOME (τ,σ)

$\Big\downarrow$

u MIGHT NOT BE WELL-DEFINED

$$u_{tt} - c^2 u_{xx} = 0$$
$$u(0,x) = g(x) \in C^2(\mathbb{R})$$
$$u_t(0,x) = h(x) \in C^1(\mathbb{R})$$

$$\Big\downarrow$$

$$u_{tt} - c^2 u_{xx} = (\partial_t + c\partial_x)\underbrace{(\partial_t - c\partial_x)u}_{w}$$

$$\Big\downarrow \quad w(t,x) = u_t(t,x) - c u_x(t,x)$$

$$w_t(t,x) + c w_x(t,x) = 0$$
$$w(0,x) = h(x) - c g'(x)$$

$$\Big\downarrow$$

$$w(t,x) = h(x-ct) - c g'(x-ct)$$

$$\Big\downarrow \quad u_t(t,x) - c u_x(t,x) = w(t,x)$$

$$u(t,x) = u(0,x+ct) + \int_0^t w(s, x+c(t-s))\,ds$$

$$\Big\downarrow \quad u(0,x) = g(x)$$

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(z)\,dz.$$

D'ALEMBERT'S FORMULA