$$u_{tt} - c^2 u_{xx} = 0$$
$$u(0,x) = g(x) \in C^2(\mathbb{R})$$
$$u_t(0,x) = h(x) \in C^1(\mathbb{R})$$

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(z)\,dz.$$

D'ALEMBERT'S FORMULA

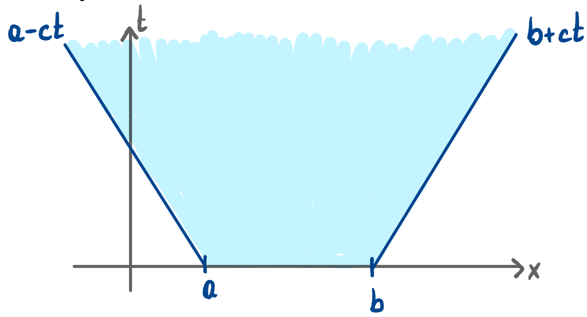
$$u_1(x) = \frac{1}{2}g(x) + \frac{1}{2c}\int_a^x h(y)\,dy$$
$$u_2(x) = \frac{1}{2}g(x) - \frac{1}{2c}\int_a^x h(y)\,dy$$

$$u(t,x) = u_1(x+ct) + u_2(x-ct)$$

TRAVELING TO THE LEFT / RIGHT WITH SPEED c.

$\mathrm{supp}(g), \mathrm{supp}(h) \subseteq [a,b]$

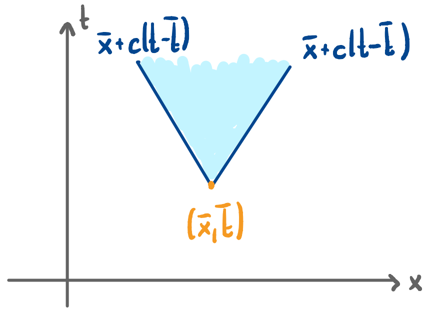
$$\mathrm{supp}(u) = \{(t,x) \in \mathbb{R}^+ \times \mathbb{R}^2 \mid a - ct \leq x \leq b + ct\}$$



$(\bar{t}, \bar{x}) \in \mathbb{R}^+ \times \mathbb{R}$

$$\{(t,x) \in \mathbb{R}^+ \times \mathbb{R} \mid \bar{x} - c(t-\bar{t}) \leq x \leq \bar{x} + c(t-\bar{t})\}$$

RANGE OF INFLUENCE



$$v_{tt} - c^2 v_{xx} = f$$
$$v(0,x) = v_t(0,x) = 0$$
$$(f \in C^1(\mathbb{R}^2))$$

$$\Big\downarrow \; s>0$$

$\eta(t,x;s)$ SOLUTION TO

$$\eta_{tt} - c^2\eta_{xx} = 0 \qquad (t>s)$$
$$\eta(s,x;s) = 0, \; \eta_t(s,x;s) = f(s,x)$$

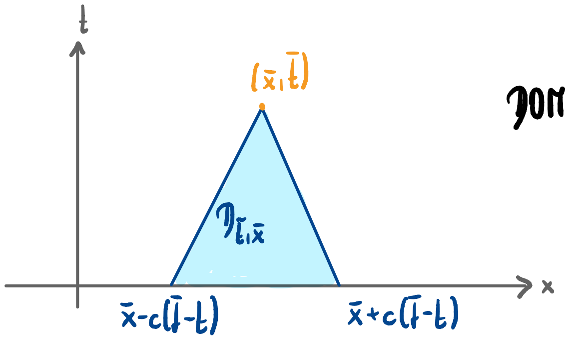
$$\Big\downarrow \; \text{D'ALEMBERT'S FORMULA}$$

$$\eta(t,x;s) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(s,x')\,dx'$$

$$\Big\downarrow$$

$$v(t,x) = \int_0^t \eta(t,x;s)\,ds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(s,x')\,dx'\,ds = \frac{1}{2c}\iint_{D_{t,x}} f(s,x')\,dx'\,ds$$

$D_{t,x} \downarrow$ DOMAIN OF DEPENDENCE



$$\Big\downarrow \; w \text{ SOLUTION TO } \begin{cases} w_{tt} - c^2 w_{xx} = 0 \\ w(0,x) = g(x),\; w_t(0,x) = h(x) \end{cases}$$

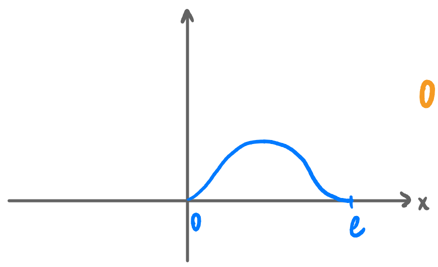
$u = v + w$ SOLUTION TO

$$u_{tt} - c^2 u_{xx} = f$$
$$u(0,x) = g(x),\; u_t(0,x) = h(x)$$

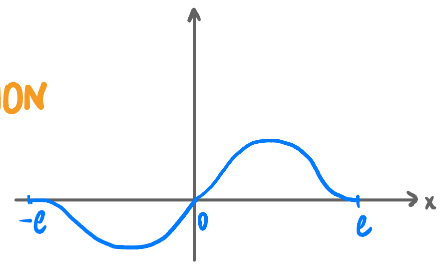
$$u_{tt}-c^2u_{xx}=0 \quad \text{ON } [0,\ell]\times\mathbb{R}^+$$
$$u(t,0)=0=u(t,\ell) \quad \forall t>0$$
$$u(0,x)=g(x),\ u_t(0,x)=h(x) \quad x\in[0,\ell]$$

$$\Big\downarrow \ \text{ASSUME} \quad u(t,x)=\frac{1}{2}\left[g(x+ct)+g(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}h(y)\,dy$$

$$g(x)=-g(-x),\ h(x)=-h(-x) \quad \forall x\in[0,\ell]$$
$$\Downarrow$$
$$u(t,0)=0 \quad \forall t>0$$

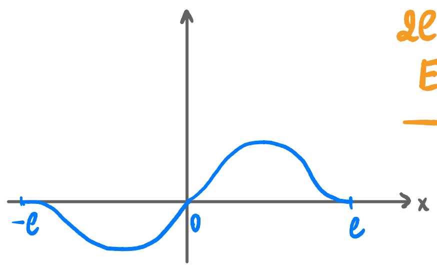


ODD EXTENSION

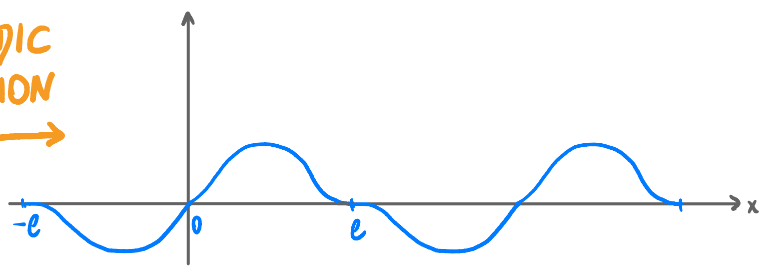


$$\Big\downarrow$$

$$g(\ell-x)=-g(\ell+x),\ h(\ell-x)=-h(\ell+x) \quad \forall x\in[-\ell,\ell]$$
$$\Downarrow$$
$$u(t,\ell)=0 \quad \forall t>0$$



2ℓ PERIODIC EXTENSION



$$\Big\downarrow$$

$$u(t,x)=\frac{1}{2}\left[g(x+ct)+g(x-ct)\right]+\frac{1}{2c}\int_{x-ct}^{x+ct}h(z)\,dz$$

g... ODD, 2ℓ PERIODIC EXTENSION

h... ODD, 2ℓ PERIODIC EXTENSION