

Formula sheet

Radial integral formula: Let $\mathbb{S}^{n-1} = \{\mathbf{x} \in \mathbb{R}^n \mid |\mathbf{x}| = 1\}$ and $w(\mathbf{y})$ a parametrisation of \mathbb{S}^{n-1} , then

$$\int_{B(\mathbf{0};R)} f(\mathbf{x})d^n\mathbf{x} = \int_{\mathbb{S}^{n-1}} \int_0^R f(r\omega(\mathbf{y}))r^{n-1}drdS(\mathbf{y}).$$

Divergence theorem: Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary. For a vector field $F \in C^1(\overline{\Omega}; \mathbb{R}^n)$,

$$\int_{\Omega} \nabla \cdot F d^n\mathbf{x} = \int_{\partial\Omega} F \cdot \nu dS,$$

where ν is the outward unit normal to $\partial\Omega$.

Green's first identity: If $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary, then for $u \in C^2(\overline{\Omega})$ and $v \in C^1(\overline{\Omega})$,

$$\int_{\Omega} (\nabla v \cdot \nabla u + v\Delta u) d^n\mathbf{x} = \int_{\partial\Omega} v \frac{\partial u}{\partial \nu} dS,$$

where ν is the outward unit normal to $\partial\Omega$.

d'Alembert's formula: The unique classical solution to the initial value problem

$$u_{tt} - c^2 u_{xx} = 0, \quad u(0, x) = g(x), \quad u_t(0, x) = h(x),$$

for $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, is given by

$$u(t, x) = \frac{1}{2} (g(x+ct) + g(x-ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} h(z) dz.$$

Inhomogeneous wave equation: For $f \in C^1(\mathbb{R}^2)$, the unique classical solution to the initial value problem

$$u_{tt} - c^2 u_{xx} = f, \quad u(0, x) = 0, \quad u_t(0, x) = 0$$

is given by

$$u(t, x) = \frac{1}{2c} \int_{\mathcal{D}_{t,x}} f(s, z) dz ds,$$

where

$$\mathcal{D}_{t,x} = \{(s, z) \in \mathbb{R}_+ \times \mathbb{R} \mid x - c(t-s) \leq z \leq x + c(t-s)\}.$$

Kirchhoff's integral formula: The unique solution $u \in C^2([0, \infty) \times \mathbb{R}^3)$ to the initial value problem

$$u_{tt} - \Delta u = 0, \quad u|_{t=0} = g \quad u_t|_{t=0} = h$$

in three space dimensions is given by

$$u(t, \mathbf{x}) = \frac{\partial}{\partial t} \tilde{g}(\mathbf{x}; t) + \tilde{h}(\mathbf{x}; t),$$

where

$$\tilde{g}(\mathbf{x}; \rho) = \frac{1}{4\pi\rho} \int_{\partial B(\mathbf{x}; \rho)} g(\mathbf{w}) dS(\mathbf{w}) \quad \text{for } \rho > 0$$

and likewise for $\tilde{h}(\mathbf{x}; \rho)$.

Poisson's integral formula: The unique solution $u \in C^2([0, \infty) \times \mathbb{R}^2)$ to the initial value problem

$$u_{tt} - \Delta u = 0, \quad u|_{t=0} = g \quad u_t|_{t=0} = h$$

in two space dimensions is given by

$$u(t, \mathbf{x}) = \frac{\partial}{\partial t} \left(\frac{t}{2\pi} \int_{\mathbb{D}} \frac{g(\mathbf{x} + t\mathbf{y})}{\sqrt{1 - |\mathbf{y}|^2}} d^2\mathbf{y} \right) + \frac{t}{2\pi} \int_{\mathbb{D}} \frac{h(\mathbf{x} + t\mathbf{y})}{\sqrt{1 - |\mathbf{y}|^2}} d^2\mathbf{y}.$$

Homogeneous heat equation: For a bounded function $g \in C^0(\mathbb{R}^n)$, the heat equation

$$u_t - \Delta u = 0, \quad u|_{t=0} = g$$

admits a classical solution given by

$$u(t, \mathbf{x}) = H_t * g(\mathbf{x}) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} g(\mathbf{y}) d^n\mathbf{y}.$$

Inhomogeneous heat equation: For $f \in C_{\text{cpt}}^2([0, \infty) \times \mathbb{R}^n)$, the initial value problem

$$u_t - \Delta u = f, \quad u|_{t=0} = 0$$

admits a classical solution given by

$$u(t, \mathbf{x}) = \int_0^t \int_{\mathbb{R}^n} H_{t-s}(\mathbf{x} - \mathbf{y}) f(s, \mathbf{y}) d^n\mathbf{y} ds.$$

Volume and surface: If A_n denotes the surface area of \mathbb{S}^{n-1} , the unit sphere in \mathbb{R}^n , then

$$A_n r^{n-1} \quad \text{and} \quad \frac{A_n r^n}{n}$$

denote the surface area and the volume of $B(\mathbf{x}; r)$, respectively.

Fundamental solution for $-\Delta$: On \mathbb{R}^n the operator $-\Delta$ has the fundamental solution

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \ln |\mathbf{x}|, & n = 2, \\ \frac{1}{(n-2)A_n |\mathbf{x}|^{n-2}}, & 3 \leq n, \end{cases}$$

where A_n denotes the surface area of \mathbb{S}^{n-1} , the unit sphere in \mathbb{R}^n .

Green's representation formula: Suppose $\Omega \subset \mathbb{R}^n$ is a bounded domain with piecewise C^1 boundary. For $u \in C^2(\overline{\Omega})$

$$u(\mathbf{y}) = - \int_{\Omega} \Phi_{\mathbf{y}} \Delta u d^n \mathbf{x} + \int_{\partial\Omega} \left[\Phi_{\mathbf{y}} \frac{\partial u}{\partial \nu} - u \frac{\partial \Phi_{\mathbf{y}}}{\partial \nu} \right] dS \quad \text{for all } \mathbf{y} \in \Omega,$$

where $\Phi_{\mathbf{y}}(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{y})$ for $\mathbf{y} \in \mathbb{R}^n$ and Φ denotes the fundamental solution of $-\Delta$ on \mathbb{R}^n .

Dominated convergence theorem: Suppose $\Omega \subset \mathbb{R}^n$ is a domain. Let $\{f_j\}$ be a sequence in $L^1(\Omega)$ such that

- $f_j \rightarrow f$ a.e. and
- there exists a nonnegative $g \in L^1(\Omega)$ such that $|f_j| \leq g$ a.e. for all j .

Then $f \in L^1(\Omega)$ and $\int_{\Omega} f(\mathbf{x}) d^n \mathbf{x} = \lim_{j \rightarrow \infty} \int_{\Omega} f_j(\mathbf{x}) d^n \mathbf{x}$.