M6: Linear mixed models (LMM) 30.10.2017
week 1

Beaches example: random intercept pr beach

Notation :

$Y_{i1}, Y_{i2}, \dots, Y_{in_i}$: response for cluster i (beach), $i = 1, \dots, m$

Y_{ij} for sample j, $j = 1, \dots, n_i$

$x_{i1}, x_{i2}, \dots, x_{in_i}$: covariates ($p \times 1$)

Random intercept model:

$$Y_{ij} = \beta_0 + \beta_1 x_{ij}' + \underbrace{\gamma_{0i}}_{RV} + \underbrace{\varepsilon_{ij}}_{RV}$$

γ_{0i}: $N(0, \tau_0^2)$

ε_{ij}: i.i.d $N(0, \sigma^2)$

$\gamma_{0i}, \varepsilon_{ij}$ independent

Q: unknown parameters:

$\beta_0, \beta_1, \sigma^2, \tau_0^2$

$\hat{\beta}_0 = 6,5819$, $\hat{\beta}_1 = -2.5084$, $\hat{\tau}_0^2 = 8.668$ (Beaches variance), $\hat{\sigma}^2 = 9.362$ (Residual variance)

random effects

Correlations

$$Y_{ij} = \beta_0 + \beta_1 x_{ij} + \underbrace{\gamma_{0i}}_{N(0,\tau_0^2)} + \underbrace{\varepsilon_{ij}}_{N(0,\sigma^2)} \sim N(\beta_0 + \beta_1 x_{ij},\ \sigma^2 + \tau_0^2)$$

independent

$$\mathrm{Var}(Y_{ij}) = \sigma^2 + \tau_0^2$$

$$\mathrm{Cov}(Y_{ij}, Y_{ik}) = E[(Y_{ij} - \underbrace{\mu_{ij}}_{\beta_0+\beta_1 x_{ij}})(Y_{ik} - \underbrace{\mu_{ik}}_{\beta_0+\beta_1 x_{ik}})]$$

beach i
sample j, k

$$= E[(\beta_0 + \underbrace{\beta_1 x_{ij} + \gamma_{0i} + \varepsilon_{ij} - \beta_0}_{\gamma_{0i}+\varepsilon_{ij}} - \beta_1 x_{ij})(\beta_0 + \underbrace{\beta_1 x_{ik} + \gamma_{0i} + \varepsilon_{ik} - \beta_0}_{\gamma_{0i}+\varepsilon_{ik}} - \beta_1 x_{ik})]$$

$$= E[\gamma_{0i}^2 + \gamma_{0i}\cdot\varepsilon_{ik} + \gamma_{0i}\cdot\varepsilon_{ij} + \varepsilon_{ij}\cdot\varepsilon_{ik}]$$

$$= \underbrace{E(\gamma_{0i}^2)}_{} + E(\gamma_{0i})\cdot E(\varepsilon_{ik}) + E(\gamma_{0i})E(\varepsilon_{ij}) + E(\varepsilon_{ij})\cdot E(\varepsilon_{ik})$$

independent

$$= \underset{\mathrm{Var}(\gamma_{0i})}{\underset{\parallel}{\tau_0^2}} + 0 = \tau_0^2$$

Two beaches: Y_{ij} vs Y_{lk}

$$\mathrm{Cov}(Y_{ij}, Y_{lk}) = \dots = 0$$

$$\mathrm{Cov}(Y_i) = \mathrm{Cov}\begin{bmatrix} Y_{i1} \\ Y_{i2} \\ \vdots \\ Y_{in_i} \end{bmatrix} = \begin{bmatrix} \sigma^2+\tau_0^2 & \tau_0^2 & \cdots & \tau_0^2 \\ \tau_0^2 & \sigma^2+\tau_0^2 & & \\ & & \ddots & \\ & & & \sigma^2+\tau_0^2 \end{bmatrix}$$

$$= \sigma^2 \cdot I + \tau_0^2 \, 1 1^T$$

compound symmetry

$$\mathrm{Corr}(Y_{ij}, Y_{ik}) = \frac{\tau_0^2}{\sigma^2+\tau_0^2}$$

Beeches example: $\mathrm{ICC} = \dfrac{\hat{\tau}_0^2}{\hat{\sigma}^2+\hat{\tau}_0^2}$

$\hat{\tau}_0^2 = 8.668 \leftarrow$ between

$\hat{\sigma}^2 = 9.362 \leftarrow$ within

$$\mathrm{ICC} = \frac{8.668}{9.362+8.668} = 0.48$$

important to model

Measurement model $\quad i = 1, \ldots, n$

$$\underset{n_i \times 1}{Y_i} = \underset{n_i \times p}{X_i} \underset{p \times 1}{\beta} + \underset{n_i \times (q+1)}{U_i} \underset{(q+1) \times 1}{\gamma_i} + \underset{n_i \times 1}{\varepsilon_i}$$

Random intercept model
$q = 0$

$$U_i = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \gamma_i = \gamma_{0i} \leftarrow \text{scalar}$$

Distributional assumptions

$$\gamma_i \sim N(0, \underset{(q+1) \times (q+1)}{Q}), \quad \varepsilon_i \sim N(\underset{n_i \times 1}{0}, \sigma^2 \underset{n_i \times n_i}{I})$$

Random intercept model

$$\underset{1 \times 1}{Q} = \tau_0^2$$

Marginal distribution of Y_i

$$Y_i \sim N_{n_i}\left(X_i\beta,\ U_i Q U_i^T + \sigma^2 I = V_i \right)$$

$$Cov(Y_i) = Cov(X_i\beta + U_i\gamma_i + \varepsilon_i)$$

$$= 0 + Cov(U_i\gamma_i) + Cov(\varepsilon_i)$$

$$= 0 + U_i \underbrace{Cov(\gamma_i)}_{Q} U_i^T + \sigma^2 I$$

Parameter estimation for fixed effects β

$$Y \sim N_N(X\beta,\ V)$$

$$N = \sum_{i=1}^{m} n_i \qquad \sigma^2 I + \underbrace{UGU^T}_{\text{randompart}}$$

ML estimation:

$$f_Y(y) = (2\pi)^{-\frac{N}{2}} \left[\det(V)\right]^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}(Y - X\beta)^T V^{-1} (Y - X\beta) \right\}$$

$$\hat{\beta} = (X^T V^{-1} X)^{-1} X V^{-1} Y$$

$$\hat{\beta} = (X^TX)^{-1}X^TY \quad \text{MLR}$$

$$\hat{\beta} \sim N(\beta, (X^TX)^{-1}\sigma^2)$$

Properties of $\hat{\beta}$: $\hat{\beta} \sim N_p\left(\beta,\ (X^T V^{-1} X)^{-1}\right)$

When inserting $\hat{V}$ $\Rightarrow$ still (asymptotically) normal

$$\hat{\beta} \approx N_p\left(\beta, (X^T \hat{V}^{-1} X)^{-1}\right)$$