M7: Generalized linear mixed models GLMM
13.11.2017

Aim: analyse correlated observations with non-normal response.

GLM: (Y_i, x_i) $i = 1, .., n$ independent pairs

1. Random component: exponential family

Beeches: $Y_i \sim \text{Poisson}(\lambda_i)$, $\mu_i = E(Y_i) = \lambda_i$

2. Systematic component: linear predictor

$$\eta_i = x_i^T \beta$$

3. Link/response function

$$\eta_i = g(\mu_i) \quad \text{or} \quad \mu_i = h(\eta_i)$$

Beeches: $\eta_i = \ln(\mu_i) = \ln(\lambda_i)$, $\lambda_i = \exp(\eta_i)$

Beeches

i) fitall: black line $\hat{\mu}_j = \exp(\hat{\eta}_j) = \exp(\hat{\beta}_0 + \hat{\beta}_1 \underbrace{x_j}_{NAP_j})$

ii) fitbeach: many $\hat{\mu}_{ij} = \exp(\hat{\eta}_{ij}) = \exp(\hat{\beta} + \beta_{beach\,i} + \hat{\beta}_1 x_{ij})$

beach obs

GLMM: we add random component to <u>linear predictor</u> AND let $\gamma_i \sim N(0, Q)$ <u>as</u> for LMM

+ distribution of Y_{ij} is conditional on γ_i

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GLMM:

Distributional assumptions

$Y_{ij} \mid \gamma_i, x_{ij}, u_{ij}$ conditionally independent

response cluster i obs j — Y_{ij}; random effect — γ_i; design vector — x_{ij}, u_{ij}

$i = 1,.., m$

$j = 1,.., n_i$

$\sum n_i = N$

$f(y_{ij} \mid \gamma_i, x_{ij}, u_{ij})$

exponential family

Structural assumptions

$\mu_{ij} = E(Y_{ij} \mid \gamma_i, x_i, u_i)$ conditional mean

η_{ij} = linear predictor $= x_{ij}^T \beta + u_{ij}^T \gamma_i$

$\eta_{ij} = g(\mu_{ij})$ link function

$\mu_{ij} = h(\eta_{ij})$ response function

Distributional assumption for random effects

γ_i i.i.d $\qquad i = 1, .., m$

$$\gamma_i \sim N(0, Q)$$

$(q+1) \times (q+1)$ positive definite

How to simulate data: [not in lecture]

- decide on β and elements in Q
- draw $\gamma_i \sim N(0, Q)$
- $x_{ij}, u_{ij} \leftarrow$ draw or fixed
- $\eta_{ij} = x_{ij}^T \beta + u_{ij}^T \gamma_i$ and $\mu_{ij} = h(\eta_{ij})$
- draw $Y_{ij} \sim f(\mu_{ij})$

Beeches: $\text{Poisson}(\exp(x_{ij}^T \beta + u_{ij}^T \gamma_i))$

GLMM with random intercept

$$\eta_{ij} = x_{ij}^T \beta + \gamma_{0i}$$

$N(0, \tau_0^2)$

Beeches:

- ML & Laplace
- $\hat{\mu}_{ij} = \exp(\hat{\beta}_0 + \hat{\beta}_1 x_{ij} + \overset{\text{MAP}}{\gamma_{0i}})$

$$= \exp(\hat{\beta}_0 + \hat{\beta}_1 x_{ij}) \cdot \underbrace{\exp(\gamma_{0i})}$$

Not just a shift in intercept on exp scale!
but a mult. effect!

remember LMM: $\mu_{ij} = \hat{\beta}_0 + \hat{\beta}_1 x_{ij} + \gamma_{0i}$

Conditional model:

$$f(y_{ij} \mid \gamma_i) \quad \text{and} \quad f(\gamma_i) \in \text{normal}$$

Poisson
binomial
gamma
normal

Joint model: $f(y_{ij}, \gamma_i) = f(y_{ij} \mid \gamma_i) \cdot f(\gamma_i)$

Marginal model:

$$f(y_{ij}) = \int_{\gamma_i} f(y_{ij}|\gamma_i)\, f(\gamma_i)\, d\gamma_i$$

N

? Poisson N N N ← closed form expression only in very special cases!

Parameter estimation:

↑

ML, but performed numerically

parameter estim becomes challenging

⇓

Compulsory 3e