Basic5

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(1) THE MODEL (x_i, Y_i) independent pairs

LM: $Y_i = x_i^T\beta + \varepsilon_i$ (1×1)

$\varepsilon_i \sim N(0,\sigma^2)$

that is $Y_i \sim N(\mu_i = x_i^T\beta, \sigma_i^2 = \sigma^2)$

GLM: 1) $Y_i \sim f(x_i)$ univ. exp. fam (1×1)

$E(Y_i)=\mu_i$, $Var(Y_i)=\sigma_i^2$

2) $\eta_i = x_i^T\beta$

3) $\eta_i = g(\mu_i)$

mvGLM

1) $Y_i \sim f(x_i)$ mv. exp. fam ($C\times1$)

(we only) multinomial $\mu_i = \pi_i$

2) $\eta_i = \begin{bmatrix}\eta_{i1}\\ \eta_{i2}\\ \vdots \\ \eta_{ic}\end{bmatrix} = \begin{bmatrix}x_i^T\beta_1\\ \vdots \\ x_i^T\beta_c\end{bmatrix}$

3) $g(\mu_i)=\eta_i$ ($C\times1$)

LMM: $Y_i = X_i\beta + Z_i\gamma_i + \varepsilon_i$ ($n_i\times1$)

$\varepsilon_i \sim N(0,\sigma^2 I)$

$\gamma_i \sim N(0,Q)$

GLMM:

1) $Y_{ij}|\gamma_i \sim f(y_{ij}|\gamma_i)$ univ. exp. fam (1×1, $(q+1)\times1$)

$\mu_{ij} = E(Y_{ij}|\gamma_i)$

2) $\eta_{ij} = x_{ij}^T\beta + u_{ij}^T\gamma_i$ linked to μ_{ij} ($1\times p$, $p\times1$, $1\times(q+1)$, $(q+1)\times1$)

$\eta_{ij} = g(\mu_{ij})$

3) γ_i i.i.d $N(0,Q)$ ($(q+1)\times1$, $(q+1)\times(q+1)$)

(4) Inference

(R-print-out)

(5) Model assessment and model choice

- deviance test
- residual plot (LM & LMM)
- AIC

(3) Asymptotic distribution of $\hat\beta$

$\hat\beta \approx N(\beta, F^{-1}(\hat\beta))$

$LRT = -2[\ln L(\hat\beta_B) - \ln L(\hat\beta_A)] \approx \chi^2_{\#A-\#B}$

B nested within A

(0)

Underlying: univariate exponential family

$$f(y_i) = \exp\left(\frac{y_i\theta_i - b(\theta_i)}{\phi}w_i + c(y_i,w_i,\phi)\right)$$

$E(Y_i)=b'(\theta_i)$, $Var(Y_i) = b''(\theta_i)\cdot\frac{\phi}{w_i}$

Canonical link for GLM

$\eta_i = \theta_i$

- concave log-likelihood and
- $H(\beta) = F(\beta)$

Find $\hat\beta$ maximizing $L(\beta)$

(2) Likelihood GLM

$L(\beta) = \prod_{i=1}^n L_i(\beta)$

log

a) $l(\beta) = \sum_{i=1}^n \ln L_i(\beta) = \sum_{i=1}^n l_i(\beta) = \sum_{i=1}^n \frac{1}{\phi}(y_i\theta_i - b(\theta_i))w_i + \sum_{i=1}^n c(y_i,\phi,w_i)$

$\sum_{i=1}^n y_i\theta_i = \sum_{j=1}^p \beta_j \sum_{i=1}^n y_i x_{ij}$ concave

b) $s(\beta) = \frac{\partial l}{\partial\beta} = \sum_{i=1}^n \frac{(y_i-\mu_i)x_i h'(\eta_i)}{Var(Y_i)} = X^T D\Sigma^{-1}(y-\mu) \Rightarrow$ Solve $s(\hat\beta)=0$ but not closed form solution

$\sum_{i=1}^n \frac{(y_i-\mu_i)x_i w_i}{\phi}$ because $\frac{\partial\mu_i}{\partial\eta_i} = b''(\theta_i)$ ($h(\eta_i)$)

$D = diag(h'(\eta_i))$

$\Sigma = diag(Var(Y_i))$

c) $F(\beta) = Cov(s(\beta)) = X^T W X$

$W = diag\left(\frac{h'(\eta_i)^2}{Var(Y_i)}\right)$

also $F(\beta) = E(H(\beta)) = E\left(\frac{\partial^2 l}{\partial\beta\partial\beta^T}\right)$

no RV's so $H(\beta)=F(\beta)$

d) Fischer scoring & IRWLS

$$\beta^{(t+1)} = \beta^{(t)} + F(\beta^{(t)})^{-1}s(\beta^{(t)}) = (X^T W(\beta^{(t)})X)^{-1}X^T W(\beta^{(t)})\tilde{y}^{(t)}$$

$$\tilde{y}_i^{(t)} = x_i^T\beta^{(t)} + \frac{y_i - h(x_i^T\beta^{(t)})}{h'(x_i^T\beta^{(t)})}$$

Newton-Raphson: replace F with H.