# The exponential family & properties

$$f(y) = \exp\left( (y\theta - b(\theta)) \cdot \frac{w}{\phi} + c(y, \phi, w) \right)$$

$\theta$ : (natural) canonical parameter (of interest)

$\phi$ : dispersion (nuisance) parameter

$b(\theta)$ : $b'(\theta)$ and $b''(\theta)$ must exist

$w$ : weight ($w = 1$ for "our" distributions)

$c(y, \phi, w)$ : not a function of $\theta$

We have seen:

| | $\theta$ | $b(\theta)$ | $\phi$ | $w$ |
|---|---|---|---|---|
| normal | $\mu$ | $\frac{1}{2}\theta^2$ | $\sigma^2$ | 1 |
| binomial | $\ln\left(\frac{p}{1-p}\right)$ | $n\ln(1+e^{\theta})$ | 1 | 1 |
| Poisson | $\ln \mu$ | $\exp(\theta)$ | 1 | 1 |
| gamma | $-\frac{1}{\mu}$ | $-\ln(-\theta)$ | $\frac{1}{\upsilon}$ | 1 |

## Two properties:

$$E(Y) = b'(\theta) \qquad \left[ b'(\theta) = \frac{db}{d\theta} \right]$$

$$Var(Y) = b''(\theta) \cdot \frac{\phi}{w} \qquad \left[ b''(\theta) = \frac{d^2 b}{d\theta^2} \right]$$

## Proof of E(Y) and Var(Y):

$$f(y) = \exp\left( (y\theta - b(\theta)) \cdot \frac{w}{\phi} + c(y, \phi, w) \right)$$

(1) $$\frac{df}{d\theta} = f(y) \cdot \left( (y - b'(\theta)) \cdot \frac{w}{\phi} \right) = (u) \cdot (v) \qquad (1)$$

(2) $$\frac{d^2 f}{d\theta^2} = \underbrace{f(y) \cdot (y - b'(\theta)) \frac{w}{\phi}}_{u'} \cdot \underbrace{(y - b'(\theta)) \frac{w}{\phi}}_{v}$$

$$+ \underbrace{f(y)}_{u} \cdot \underbrace{(0 - b''(\theta)) \frac{w}{\phi}}_{v'}$$

$$= f(y) \left[ \left(\frac{w}{\phi}\right)^2 (y - b'(\theta))^2 - \frac{w}{\phi} b''(\theta) \right] \qquad (2)$$

We will use these in combination with:

$$\int_y f(y)\, dy = 1$$

$$\frac{d}{d\theta} \int_y f(y)\, dy = \int_y \frac{d}{d\theta} f(y)\, dy = \frac{d}{d\theta} 1 = 0$$

so: $$\underbrace{\int_y \frac{d}{d\theta} f(y)\, dy = 0}_{(*)}$$ if we may change $\frac{d}{d\theta}$ and $\int$

and $$\frac{d^2}{d\theta^2} \int_y f(y)\, dy = \int_y \frac{d^2}{d\theta^2} f(y)\, dy = \frac{d^2}{d\theta^2} 1 = 0$$

$$\underbrace{\int_y \frac{d^2}{d\theta^2} f(y)\, dy = 0}_{(**)}$$

(1) and (*) to get $E(Y) = b'(\theta)$

$$\int_y \frac{df}{d\theta}\, dy = 0 \quad \text{and} \quad \frac{df}{d\theta} = f(y)\left(y - b'(\theta)\right)\frac{w}{\phi}$$

(1) (*)

$$\int_y \frac{df}{d\theta}\, dy = \int_y f(y)\,(y - b'(\theta))\,\frac{w}{\phi}\, dy$$

$$= \frac{w}{\phi} \underbrace{\int_y y\, f(y)\, dy}_{E(Y)} - \frac{w}{\phi}\, b'(\theta) \underbrace{\int_y f(y)\, dy}_{1}$$

$$= \cancel{\frac{w}{\phi}}\, E(Y) - \cancel{\frac{w}{\phi}}\, b'(\theta) = 0$$

and $\int_y \frac{df}{d\theta}\, dy = 0$

$$\underline{\underline{E(Y) = b'(\theta)}}$$

(2) and (**) to get Var(Y)

$$\int_y \frac{d^2 f}{d\theta^2}\, dy = 0 \quad \text{and}$$

(**)

$$\frac{d^2 f}{d\theta^2} = f(y)\left[\left(\frac{w}{\phi}\right)^2 \left(y - b'(\theta)\right)^2 - \frac{w}{\phi} b''(\theta)\right]$$

(2)

$$\int_y \frac{d^2 f}{d\theta^2}\, dy = \int_y f(y) \frac{w}{\phi}\left[\frac{w}{\phi}\left(y - \underbrace{b'(\theta)}_{E(Y)}\right)^2 - b''(\theta)\right] dy$$

remark: $\int_y f(y)\left(y - E(Y)\right)^2 dy = Var(Y)$

$$\int_y \frac{d^2 f}{d\theta^2}\, dy = \underline{\left(\frac{w}{\phi}\right)^2 Var(Y)} - \frac{w}{\phi} b''(\theta) \underbrace{\int_y f(y)\, dy}_{1}$$

$$= \frac{w}{\phi}\left[\frac{w}{\phi} Var(Y) - b''(\theta)\right] = 0$$

and combine with (**)

$$\frac{w}{\phi} Var(Y) - b''(\theta) = 0$$

$$Var(Y) = b''(\theta) \cdot \frac{\phi}{w}$$