When ϕ needs to be estimated → how about $Cov(\hat{\beta})$ then?

For some GLMs (normal, gamma) our parameter vector is partitioned into $(\beta, \phi) = \beta^*$

We should then solve $\frac{\partial l}{\partial \beta^*} = 0$ to find $(\hat{\beta}, \hat{\phi})$

and then $$\begin{bmatrix} \hat{\beta} \\ \hat{\phi} \end{bmatrix} \approx N\left(\begin{bmatrix} \beta \\ \phi \end{bmatrix}, F^{-1}(\hat{\beta}^*)\right)$$

where here $$F(\beta^*) = \begin{bmatrix} F_{\beta\beta} & F_{\beta\phi} \\ F_{\phi\beta} & F_{\phi\phi} \end{bmatrix}$$

$F_{\beta\beta}$ has elements from $E\left(-\frac{\partial l}{\partial \beta_n} \frac{\partial l}{\partial \beta_k}\right)$ ← we have so far only looked at this

$F_{\beta\phi}$ $E\left(-\frac{\partial l}{\partial \beta_n} \cdot \frac{\partial l}{\partial \phi}\right)$

$F_{\phi\beta}$ $E\left(-\frac{\partial l}{\partial \phi} \frac{\partial l}{\partial \beta_n}\right)$

$F_{\phi\phi}$ $E\left(-\frac{\partial l}{\partial \phi} \frac{\partial l}{\partial \phi}\right)$

What is $F_{\beta\phi}$ $\left(F_{\beta\phi} = F_{\phi\beta}^T\right)$

$$\frac{\partial l}{\partial \beta_h} = \sum_{i=1}^{n} \frac{(y_i - \mu_i)\, x_{ih}}{\underbrace{\mathrm{Var}(Y_i)}_{b''(\theta_i)\cdot \frac{\phi}{w_i}}} \frac{\partial \mu_i}{\partial \eta_i} = \sum_{i=1}^{n} \frac{(y_i-\mu_i)\, x_{ih}}{b''(\theta_i)\cdot \frac{\phi}{w_i}} \left(\frac{\partial \mu_i}{\partial \eta_i}\right)$$

$[F_{\beta\phi}]_h$

$$\frac{\partial^2 l}{\partial \phi\, \partial \beta_h} = \sum_{i=1}^{n} w_i \frac{(y_i-\mu_i)\, x_{ih}}{b''(\theta_i)} \left(\frac{\partial \mu_i}{\partial \eta_i}\right) \cdot \underbrace{\frac{\partial}{\partial \phi}\left(\phi^{-1}\right)}_{-\frac{1}{\phi^2}}$$

$$E(\;\cdot\;) = \sum_{i=1}^{n} w_i \frac{x_{ih}}{b''(\theta_i)} \frac{\partial \mu_i}{\partial \eta_i} \left(-\frac{1}{\phi^2}\right) \cdot \underbrace{E(Y_i - \mu_i)}_{0} = 0$$

Therefore

$$F = \begin{bmatrix} F_{\beta\beta} & 0 \\ 0 & F_{\phi\phi} \end{bmatrix} \text{ and}$$

$$F^{-1} = \begin{bmatrix} F_{\beta\beta}^{-1} & 0 \\ 0 & F_{\phi\phi}^{-1} \end{bmatrix}$$

and

$$\begin{bmatrix} \hat\beta \\ \hat\phi \end{bmatrix} \approx N\left[\begin{pmatrix} \beta \\ \phi \end{pmatrix}, \begin{bmatrix} \overset{F(\beta)^{-1}}{\overset{\shortparallel}{F_{\beta\beta}^{-1}}} & 0 \\ 0 & F_{\phi\phi}^{-1} \end{bmatrix}\right]$$

so $\hat\beta \approx N(\beta, F(\beta)^{-1})$ is the same whether ϕ is fixed or estimated

We say that the parameters β and ϕ are orthogonal and for our exp. fam GLM this is always the case.

Remark : observed fisher info not so?