M7W1 : Linear mixed effects models (LMM) 01.11.2018

Beaches: random intercept pr beach

Notation:

$$Y_i = \begin{bmatrix} Y_{i1} \\ Y_{i2} \\ \vdots \\ Y_{in_i} \end{bmatrix} \quad (n_i \times 1)$$

Y_{ij} response for cluster i (beach) for sample j $\quad i = 1, \dots, m$

$j = 1, \dots, n_i$

with covariates $x_{i1}, \dots, x_{in_i}$ ($p \times 1$; x_{ij} element)

$$X_i = \begin{bmatrix} x_{i1}^T \\ \vdots \\ x_{in_i}^T \end{bmatrix} \quad (n_i \times p)$$

Beaches: $m = 9$
$n_i = 5$
$p = 2$

Random intercept model

$$Y_{ij} = \underbrace{\beta_0 + \beta_1}_{\text{population fixed effects}} \overset{p=2 \text{, one covariate}}{x_{ij}} + \underbrace{\gamma_{0i}}_{N(0,\tau_0^2)} + \underbrace{\varepsilon_{ij}}_{\text{i.i.d } N(0,\sigma^2)}$$

γ_{0i}: RV, deviation from the pop. intercept

ε_{ij}: RV

Variance components

Q: What are the unknown parameters?

A: $\beta_0, \beta_1, \tau_0^2, \sigma^2$ $\quad$ Print-out: $\hat{\beta}_0 = 6.58$

$\hat{\beta}_1 = -2.57$

$\hat{\tau}_0^2 = 8.67$

$\hat{\sigma}^2 = 9.36$

Correlation

WANT: $\boxed{\underset{n_i \times n_i}{\mathrm{Cov}(Y_i)}}$

$$Y_{ij} = \beta_0 + \beta_1 x_{ij} + \overset{N(0,\tau_0^2)}{\gamma_{0i}} + \overset{N(0,\sigma^2)}{\varepsilon_{ij}} \sim N_1(\beta_0+\beta_1 x_{ij}, \tau_0^2+\sigma^2)$$

independent

$$\mathrm{Var}(Y_{ij}) = \tau_0^2 + \sigma^2$$

$$\mathrm{Cov}(Y_{ij}, Y_{ik}) = E\left[(Y_{ij} - \underset{\beta_0+\beta_1 x_{ij}}{\mu_{ij}})(Y_{ik} - \underset{\beta_0+\beta_1 x_{ik}}{\mu_{ik}})\right]$$

$$= E\left[\left(\cancel{\beta_0+\beta_1 x_{ij}} + \gamma_{0i} + \varepsilon_{ij} - \cancel{(\beta_0+\beta_1 x_{ij})}\right)\left(\cancel{\beta_0+\beta_1 x_{ik}} + \gamma_{0i} + \varepsilon_{ik} - \cancel{(\beta_0+\beta_1 x_{ik})}\right)\right]$$

$$= E\left((\gamma_{0i}+\varepsilon_{ij})(\gamma_{0i}+\varepsilon_{ik})\right) = E\left[\gamma_{0i}^2 + \varepsilon_{ij}\cdot\gamma_{0i} + \gamma_{0i}\cdot\varepsilon_{ik} + \varepsilon_{ij}\cdot\varepsilon_{ik}\right]$$

$$= E(\gamma_{0i}^2) + \underbrace{E(\varepsilon_{ij})}_{0}\cdot\underbrace{E(\gamma_{0i})}_{0} + \underbrace{E(\gamma_{0i})}_{0}\cdot\underbrace{E(\varepsilon_{ik})}_{0} + \underbrace{E(\varepsilon_{ij})}_{0}\cdot\underbrace{E(\varepsilon_{ik})}_{0} = \tau_0^2$$

$$\mathrm{Var}(\gamma_{0i}) = E(\gamma_{0i}^2) - \underbrace{E(\gamma_{0i})}_{0}{}^2 = \tau_0^2$$

$$\mathrm{Cov}(Y_i) = \mathrm{Cov}\begin{bmatrix} Y_{i1} \\ Y_{i2} \\ \vdots \\ Y_{in_i} \end{bmatrix} = \begin{bmatrix} \sigma^2+\tau_0^2 & \tau_0^2 & \cdots & \tau_0^2 \\ \tau_0^2 & \sigma^2+\tau_0^2 & & \\ \vdots & & \ddots & \\ \tau_0^2 & & & \sigma^2+\tau_0^2 \end{bmatrix}$$

$$= \sigma^2 \cdot \underset{n_i\times n_i}{I} + \tau_0^2 \cdot \underbrace{11^T}_{n_i \times n_i \text{ with 1s everywhere}}$$

compound symmetry

$$\mathrm{Corr}(Y_{ij}, Y_{ik}) = \frac{\tau_0^2}{\underbrace{\tau_0^2+\sigma^2}_{\sqrt{\sigma^2+\tau_0^2}\cdot\sqrt{\sigma^2+\tau_0^2}}}$$ ← intraclass correlation ICC

Beaches:

$\hat{\tau}_0^2 = 8.67$ ← between

$\hat{\sigma}^2 = 9.36$ ← within

$$ICC = \frac{\hat{\tau}_0^2}{\hat{\tau}_0^2 + \hat{\sigma}^2} = \frac{8.67}{8.67 + 9.36} = \underline{\underline{0.48}}$$

$\mathrm{Corr}(Y_{ij}, Y_{ik})$ two obs from the same beach

Linear mixed effects model

Measurement model

$$\underset{n_i \times 1}{Y_i} = \underset{\substack{n_i \times p \\ \text{design matrix}}}{X_i} \underset{\substack{p \times 1 \\ \text{fixed effects}}}{\beta} + \underset{\substack{(n_i \times (q+1)) \\ \text{design matrix random effects}}}{U_i} \cdot \underset{\substack{(q+1) \times 1 \\ \text{vector of random effects}}}{\gamma_i} + \underset{\substack{n_i \times 1 \\ \text{random error}}}{\varepsilon_i}$$

$$U_i = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \text{ if random intercept model } (q=0)$$

$$U_i = \begin{bmatrix} 1 & u_{i1} \\ 1 & \vdots \\ \vdots & \\ 1 & u_{in_i} \end{bmatrix}$$

NAP for beach i, $u_{ij} = x_{ij}$

Distributional assumptions

For cluster i $\quad \gamma_i \sim N_{(q+1)}(0, \underset{(q+1)\times(q+1)}{Q})$ and $\varepsilon_i \sim N(0, \sigma^2 I)$

random intercept $\quad \underset{1\times 1}{Q} = \tau_0^2$

random intercept and slope $\quad \underset{2\times 2}{Q} = \begin{bmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{bmatrix}$

Two views: conditional and marginal

$$\boxed{Y_i = X_i\beta + U_i\gamma_i + \varepsilon_i \quad \gamma_i \sim N(0, Q),\ \varepsilon_i \sim N(0, \sigma^2 I)}$$

The conditional distr. of Y_i

$$Y_i \mid \gamma_i \sim N(X_i\beta + U_i\gamma_i, \sigma^2 I)$$

The marginal distr. of Y_i:

$$Y_i = X_i\beta + \varepsilon_i^* \quad \text{where} \quad \varepsilon_i^* = U_i\gamma_i + \varepsilon_i$$

ε_i^* is mvN with $E(\varepsilon_i^*) = U_i \cdot E(\gamma_i) + E(\varepsilon_i) = 0$

$$V_i = Cov(\varepsilon_i^*) = Cov(U_i\gamma_i) + Cov(\varepsilon_i)$$

$$= U_i \underbrace{Cov(\gamma_i)}_{Q} U_i^T + \sigma^2 I$$

$$= U_i Q U_i^T + \sigma^2 I$$

Y_i has $E(Y_i) = X_i\beta$, $Cov(Y_i) = V_i$

$$Y_i \sim N_{n_i}(X_i\beta, V_i)$$

Parameter estimation $\rightarrow$ β now

Global model $\qquad Y = X\beta + \varepsilon^*$ $\qquad \varepsilon^* \sim N(0, V)$

$$\varepsilon^* \leftarrow U\gamma + \varepsilon$$

See module page

Home work: Find $E(\hat\beta)$, $Cov(\hat\beta)$

$$\hat\beta = \overbrace{(X^T V^{-1} X)^{-1} X^T V^{-1}}^{A} Y$$

$$= \left(\sum_{i=1}^{m} X_i^T V_i^{-1} X_i\right)^{-1} \sum_{i=1}^{m} X_i V_i^{-1} Y_i$$

$$\hat\beta \sim N\left(\beta, (X^T V^{-1} X)^{-1}\right)$$