M7. W2 : More advanced issues for LMM

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linear mixed effects models

Notation

$$n_i \rightarrow \underset{n_i \times 1}{Y_i} = X_i \beta + U_i \gamma_i + \varepsilon_i \qquad \text{cluster specific}$$

$$N = \sum_{i=1}^{m} n_i \rightarrow \underset{N \times 1}{Y} = X \beta + U \gamma + \varepsilon \qquad \text{global model}$$

measurement model

$$\gamma_i \sim N(0, Q) \qquad \varepsilon_i \sim N(0, \sigma^2 I)$$

$$\gamma \sim N(0, G) \qquad \varepsilon \sim N(0, \sigma^2 I)$$

distributional assumption

Marginal model:

$$U_i = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \quad \gamma_i = \gamma_{i0}$$

random intercept model

$$Q = \tau_0^2$$

$$\underbrace{Y_i}_{n_i \times 1} \sim N(\underbrace{X_i}_{n_i \times p} \underbrace{\beta}_{p \times 1}, \ V_i = \sigma^2 I + U_i Q U_i^T)$$

$$Y \sim N(X\beta, \ V = \sigma^2 I + U G U^T)$$

PARAMETER ESTIMATION : $\beta, \sigma^2, G \leftarrow Q \rightarrow V \leftarrow \vartheta$ = parameter in V

script theta

random intercept (σ^2, τ_0^2)

1) $\hat{\beta} = (X^T V^{-1} X)^{-1} X^T V^{-1} Y$

$\sim N(\beta, (X^T V^{-1} X)^{-1})$

2) Finding $V(\vartheta)$ with REML

not on closed form

restricted / residual

→ transformation

→ integration

Predicting γ_i and ε_i

$$\beta, \underbrace{(\sigma^2, Q)}_{\theta}$$

Random effects γ_i

$$\begin{bmatrix} Y_i \\ \gamma_i \end{bmatrix} \sim N\left(\begin{bmatrix} X_i\beta \\ 0 \end{bmatrix}, \begin{bmatrix} V_i & U_i Q \\ Q U_i^T & Q \end{bmatrix} \right)$$

Conditional mean:

$$E(\gamma_i | Y_i) = 0 + Q U_i^T V_i^{-1} (Y_i - X_i\beta)$$

$$\hat{\gamma}_i = 0 + \hat{Q} U_i^T \hat{V}_i^{-1} (Y_i - X_i\hat{\beta})$$

Residuals

$$Y_i = X_i\beta + U_i\gamma_i + \varepsilon_i$$

Marginal (level 0): $\hat{\mu}_i = X_i\hat{\beta} \rightarrow e_i = Y_i - X_i\hat{\beta}$

Conditional (level 1): $\hat{\mu}_i = X_i\hat{\beta} + U_i\hat{\gamma}_i \rightarrow e_i = Y_i - X_i\hat{\beta} - U_i\hat{\gamma}_i$

Random intercept and slope model

$$Y_{ij} = \underbrace{\beta_0 + \beta_1 x_{ij}}_{\text{population}} + \underbrace{\gamma_{0i} + \gamma_{1i}}_{\text{individual effects}} x_{ij} + \varepsilon_{ij}$$

$$Q = \begin{bmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{bmatrix}$$

Hypothesis tests

Fixed effects:

$$\hat{\beta} \approx N\left(\beta, \left(\sum (X_i^T V_i^{-1} X_i)\right)^{-1}\right)$$

→ χ^2 for $C\beta = d$ with Wald

or LRT (NB use ML not REML)

Random effects: LRT with REML and χ^2-mixture.

Model selection → via AIC!

↖ ML fixed effects
REML random effects