

M8: Generalized linear mixed models GLMM

TMA1315
(5.11.2018)

Aim: analyse correlated observations with non-normal response

GLM: (Y_i, x_i) $i=1, \dots, n$ independent pairs

1. Random component: Y_i from exponential family

Beeches: $Y_i \sim \text{Poisson}(\mu_i)$, $\mu_i = E(Y_i)$

2. Systematic component: linear predictor $\eta_i = x_i^T \beta$

3. Link/response function

$$\eta_i = g(\mu_i) \text{ or } \mu_i = h(\eta_i)$$

Beeches: $\eta_i = \ln(\mu_i)$, $\mu_i = \exp(\eta_i)$

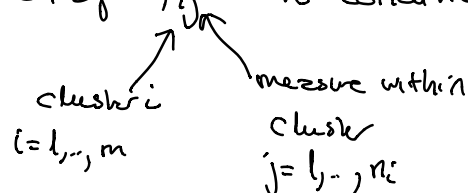
a) Population model: beach curve: $\hat{\mu} = \exp(\hat{\eta}) = \exp(\hat{\beta}_0 + \hat{\beta}_1 x)$

b) Each beach fixed intercept: $\hat{\mu}_{ij} = \exp(\hat{\eta}_{ij}) = \exp(\hat{\beta}_0 + \hat{\beta}_{\text{beach}, i} + \hat{\beta}_1 x_{ij})$

GLMM: we add a random component to the linear

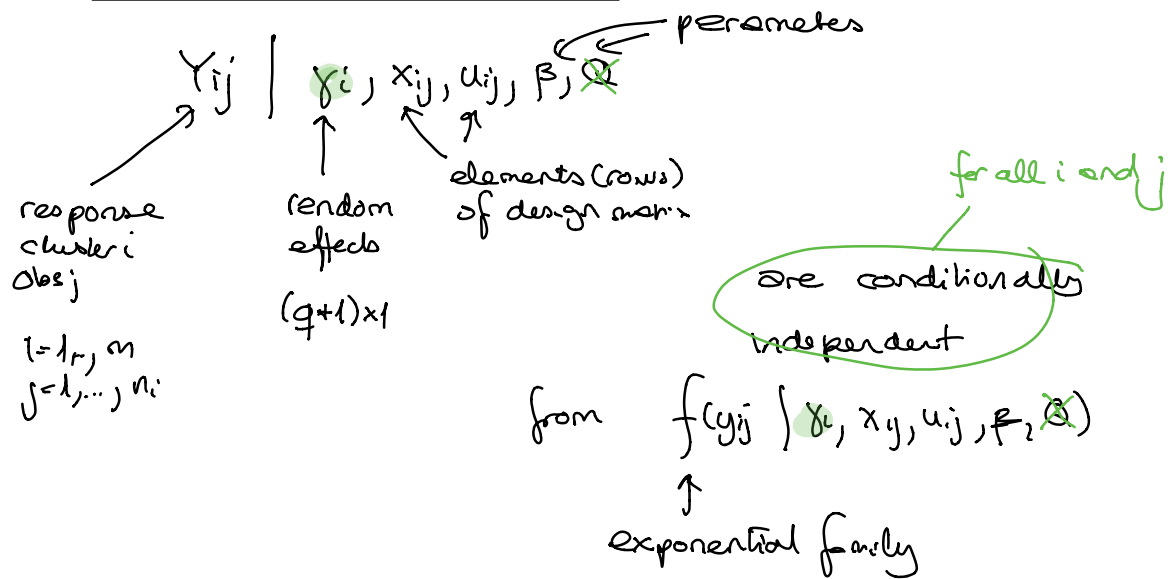
predictor and let $\gamma_i \sim N(0, Q)$ as for LMM.

+ distribution of Y_{ij} is conditional on γ_i .



GLMM

Distributional assumptions



Structural assumptions

$$\mu_{ij} = E(Y_{ij} | \gamma_i, x_{ij}, u_{ij}, \beta) \quad \text{conditional mean}$$

$$\eta_{ij} = x_{ij}^T \beta + u_{ij}^T \gamma_i$$

$$\eta_{ij} = g(\mu_{ij}) \quad \text{and} \quad \mu_{ij} = h(\eta_{ij})$$

NB: now conditional mean is linked to linear predictor

Distributional assumptions on random effects

$$\gamma_i \text{ i.i.d} \quad \gamma_i \sim N(0, \Sigma)$$

$i = 1, \dots, m$ $\uparrow$
 $(q+1) \times (q+1)$

How to simulate data (to be analyzed with GLMM)?

- 1) Decide on β and elements of Q
 β_0, β_1 random intercept τ_0^2
- 2) Draw $\gamma_i \sim N(0, Q)$
- 3) $x_{ij}, u_{ij} \leftarrow$ fixed or generate $\leftarrow$ choose some values
- 4) $\eta_{ij} = x_{ij}^T \beta + u_{ij}^T \gamma_i$ and $\mu_{ij} = h(\eta_{ij})$
- 5) Draw $Y_{ij} \sim f(\mu_{ij})$
Beaches: $\text{Poisson}(\exp(x_{ij}^T \beta + u_{ij}^T \gamma_i))$

GLMM with random intercept

Distributional assumptions: $Y_{ij} \mid \gamma_i, x_{ij}, u_{ij}$

[Poisson for Beaches] $\mu_{ij} = E(Y_{ij} \mid \gamma_i, x_{ij}, u_{ij})$

Structural assumptions:

$$\eta_{ij} = \underbrace{x_{ij}^T \beta}_{\text{NAP}} + \gamma_{0i} \quad \text{and} \quad \eta_{ij} = \log(\mu_{ij})$$

[Beaches: $\beta_0 + \beta_1 \text{NAP}_{ij} + \gamma_{0i}$]

Distribution of $\gamma_{0i} \sim N(0, \tau_0^2)$

Q: Beaches with NAP: # parameters to estimate: $\beta_0, \beta_1, \tau_0^2$ 3

Beeches (print-out): ML (Wepiece approx)

$$\hat{\mu}_{ij} = \exp(\hat{\beta}_0 + \hat{\beta}_1 x_{ij} + \hat{\gamma}_{i0}) = \exp(\hat{\beta}_0 + \hat{\beta}_1 x_{ij}) \cdot \exp(\hat{\gamma}_{i0})$$

Not just a shift in the intercept on exp. scale

→ but a multiplicative effect

(for GLM: $\hat{\mu}_{ij} = \hat{\beta}_0 + \hat{\beta}_1 x_{ij} + \hat{\gamma}_{i0}$)

Parameter estimation

Conditional model: $f(y_{ij} | x_i)$ and $f(x_i)$

$\uparrow$
 Normal, binomial
 Poisson
 Gamma

$\uparrow$
 Normal

Joint model: $f(y_{ij}, x_i) = f(y_{ij} | x_i) \cdot f(x_i)$

Marginal model: $f(y_{ij}) = \int f(y_{ij} | x_i) f(x_i) dx_i$

$\uparrow$
 N
 ?
 ?

$\uparrow$
 N
 Poisson
 binomial

$\uparrow$
 N

for only special cases this is in closed form

Likelihood: $L(\beta, Q) = \prod_{i=1}^m f(y_i | \beta, Q)$

$$= \prod_{i=1}^m \int \prod_{j=1}^{n_i} f(y_{ij} | x_i, \beta) \cdot f(x_i | Q) dx_i$$

Inference: as for LRM (and GLM)

$$\hat{\beta} \approx N(\beta, F(\beta)^{-1})$$

and deviance for model fit and AIC for model selection.

S