Fourier series

Complex inner products and orthogonal systems

Definition (Complex inner product)
Let V be a complex vector space. Then a mapping $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{C}$ is called an inner product if the following assumptions are satisfied:

1) linearity in the first argument:

$$\langle \lambda f + \mu g, h \rangle = \lambda \langle f, h \rangle + \mu \langle g, h \rangle$$

for all $\lambda, \mu \in \mathbb{C}$ and $f, g, h \in V$.

2) Conjugate symmetry

$$\langle f, g \rangle = \overline{\langle g, f \rangle} \quad \forall f, g \in V$$

3) Positive definite

$$\langle f, f \rangle \geq 0 \text{ and } \langle f, f \rangle = 0 \Leftrightarrow f = 0.$$

Example: Let V be the space of complex, continuous 2π periodic functions. Then

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f \bar{g} \, dx$$

is a complex inner product on V.

Observation: As for a real vector space, a complex inner product induces a norm on V via

$$\|f\| := \sqrt{\langle f, f \rangle}.$$

- We have $|\langle f, g \rangle| \leq \|f\| \|g\|$, Cauchy Schwarz inequality.

Definition: Let V be a complex vector space with an inner product.
A sequence / family $\{\phi_n\}_n$ is called

orthogonal if $\langle \phi_n, \phi_m \rangle = \begin{cases} c_n \neq 0 & \text{if } n = m \\ 0 & \text{if } n \neq m, \end{cases}$

orthonormal if in addition $c_n = \langle \phi_n, \phi_n \rangle = \|\phi_n\|^2 = 1$. We then write $\langle \phi_n, \phi_m \rangle = \underbrace{\delta_{nm}}_{\text{Kronecker delta.}} = \begin{cases} 1 & n = m \\ 0 & n \neq m. \end{cases}$

- **Example**: $\{e^{inx}\}_{n \in \mathbb{Z}}$ is an orthogonal system since

$$\langle e^{inx}, e^{imx} \rangle = \int_{-\pi}^{\pi} e^{inx} \overline{e^{imx}} \, dx = \int_{-\pi}^{\pi} e^{inx} e^{-imx} \, dx$$

$$= \int_{-\pi}^{\pi} e^{i(n-m)x} \, dx = \begin{cases} 2\pi & n = m \\ \frac{1}{i(n-m)} e^{i(n-m)x} \Big|_{x=-\pi}^{x=\pi} \end{cases}$$

$$= \frac{2 \cdot 1}{2 \cdot i(n-m)} \left(e^{i(n-m)\pi} - e^{i(n-m)\pi} \right) = \frac{2}{(n-m)} \sin (n-m)\pi = 0.$$

- By the previous calculation we see that

$$\left\{ \frac{e^{inx}}{\sqrt{2\pi}} \right\}_{n \in \mathbb{Z}} \text{ is a orthonormal system.}$$

Example: $\{\sin nx\}_{n=1}^{\infty} \cup \{\cos nx\}_{n=0}^{\infty}$ is an orthogonal system. Then

$$\langle \sin nx, \sin mx \rangle = \langle \cos nx, \cos mx \rangle = \begin{cases} \pi & \text{if } n = m \\ 0 & \text{else,} \end{cases}$$

and $\langle \cos nx, \sin mx \rangle = 0 \quad \forall n, m \in \mathbb{N}$.

Proof: See exercise set 1, problem 3. ■

- Last time we considered the vector spaces

$$V_N = \{ f(x) = a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx), \; a_n, b_n \in \mathbb{C} \}$$

and

$$\tilde{V}_N = \{ f(x) = \sum_{n=-N}^{N} c_n e^{inx} \quad c_n \in \mathbb{C} \}.$$

Note that $\tilde{V}_N = V_N$ thanks to Euler's formula.

Orthogonal projection (again)

Questions: For a general $f \notin V_N$, can we find an $f_N \in V_N$ which is "close" to f? Or said differently: Which $f_N \in V_N$ is the closest one to f, regarding all choices in V_N we have? How to compute them e.g. c_n or a_n, b_n?

Let's start with the simple example, assuming that f is already in V!

Assumptions for remaining lecture: V complex vector space with inner product $\langle \cdot, \cdot \rangle$, $\{\phi_n\} \subseteq V$ orthogonal system,

$V_N := \{ g_N = \sum_{n=1}^{N} d_n \phi_n \}$ N-dimensional subspace spanned by the first N members of $\{\phi_n\}$.

Proposition: Let V be complex vector space with an inner product $\langle \cdot, \cdot \rangle$. Let $\{\phi_n\}$ be an orthogonal family. Assume that f is given by

$$f(x) = \sum_{n=1}^{N} c_n \phi_n. \quad \text{Then } \boxed{c_n = \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2}}.$$

In particular if $\{\phi_n\}$ is an orthonormal system then $c_n = \langle f, \phi_n \rangle$.

Proof: Take inner product between f and ϕ_n:

$$\langle f, \phi_n \rangle = \Big\langle \sum_{m=1}^{N} c_m \phi_m , \phi_n \Big\rangle$$

$$= \sum_{m=1}^{N} c_m \langle \phi_m, \phi_n \rangle$$

$$= c_n \underbrace{\langle \phi_n, \phi_n \rangle}_{= \|\phi_n\|^2}$$

thanks to the orthogonality property ■

Definition: For an orthogonal system $\{\phi_n\}$ and given vector/function $f \in V$, the coefficients

$$c_n := \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2}$$

are called the (general) Fourier coefficients. Sometimes we write $\hat{f}(n) := c_n$.

Definition (orthogonal projection)

For $f \in V$ we define the projection $\Pi_N f \in V_N$ by

$$\Pi_N f := \sum_{n=1}^{N} c_n \phi_n \quad \text{with } c_n := \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2}.$$

Observations:

- Π_N is a linear mapping $V \ni f \longmapsto \Pi_N f \in V_N$.
- Π_N is a indeed a projection, that is $\Pi_N f = f$ whenever f is already in V_N, thanks to the previous proposition.
- $\Pi_N f$ is orthogonal is the sense that $\langle f - \Pi_N f, g \rangle \ \forall g \in V_N$, that is the error $f - \Pi_N f$ is orthogonal to V_N.

Proof: Since $\{\phi_n\}_{n=1}^{N}$ is a basis for V_N it is enough to show that $\langle f - \Pi_N f, \phi_m \rangle = 0 \quad m = 1, .., N$.

$$\langle f - \Pi_N f, \phi_m \rangle = \Big\langle f - \sum_{n=1}^{N} \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} \phi_n , \phi_m \Big\rangle$$

$$= \langle f, \phi_m \rangle - \sum_{n=1}^{N} \frac{\langle f, \phi_n \rangle}{\|\phi_n\|^2} \langle \phi_n, \phi_m \rangle$$

$\{\phi_n\}$ is orthogonal.

$$= \langle f, \phi_m \rangle - \frac{\langle f, \phi_m \rangle}{\cancel{\langle \phi_m, \phi_m \rangle}} \cdot \cancel{\langle \phi_m, \phi_m \rangle} = 0.$$

■

Theorem: (Best approximation property)

Let $f \in V$. Then $\pi_W f$ satisfies

$$\boxed{\|f - \pi_W f\| = \min_{g \in V_W} \|f - g\|.}$$

Proof: Exactly the same as in lecture 1, but we repeat it here to extract an important corollary.
Let $g \in V_W$. Then

$$\|f - \pi_W f\|^2 = \langle f - \pi_W f, f \overset{-g+g}{-} \pi_W f \rangle$$

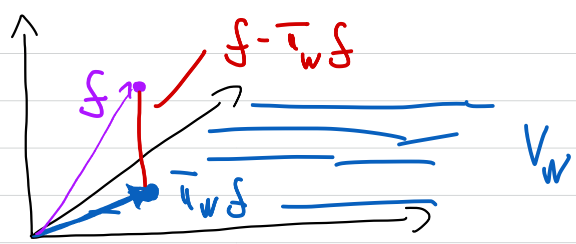
$$= \langle f - \pi_W f, f - g \rangle + \underbrace{\langle f - \pi_W f, \underbrace{g - \pi_W f}_{\in V_W} \rangle}_{= 0 \quad \pi_W f \text{ is orthogonal projection}}$$

$$= \langle f - \pi_W f, f - g \rangle$$

$$\underset{\text{Cauchy Schwarz}}{\le} \boxed{\|f - \pi_W f\|} \, \|f - g\|.$$ Divide by $\|f - \pi_W f\|$ (if not 0, otherwise trivial since $\|f - \pi_W f\| = 0$ is clearly the minimimum.) and get

$$\|f - \pi_W f\| \le \|f - g\| \text{ no matter what } g \in V_W$$

we close. ■



Proposition: $\|\pi_W f\| \le \|f\|$.

· Intuitively $f = \pi_W f + f - \pi_W f$ and $f - \pi_W f \perp \pi_W f$ so Pythagorean theorem $a^2 + b^2 = c^2$ gives

$$\|f\|^2 = \|\pi_W f\|^2 + \|f - \pi_W f\|^2 \ge \|\pi_W f\|^2.$$

A proof in the spirit of the previous proof is as follows:

$$\cdot \ \overbrace{\langle f, f \rangle}^{c^2} = \langle \pi_W f + (f - \pi_W f), \pi_W f + (f - \pi_W f) \rangle$$

$$= \underbrace{\langle \pi_W f, \pi_W f \rangle}_{= \|\pi_W f\|^2 = a^2} + \underbrace{\langle f - \pi_W f, f - \pi_W f \rangle}_{\|f - \pi_W f\|^2 = b^2}$$

$$+ \underbrace{\langle \pi_W f, f - \pi_W f \rangle}_{= 0} + \underbrace{\langle f - \pi_W f, \pi_W f \rangle}_{= 0}$$

since $\pi_W f$ is orthogonal.

Corollary (Bessel's inequality)

Let $\{\phi_n\}$ be an orthonormal system, then for any N we have

$$\sum_{n=1}^{N} |\hat{f}(n)|^2 \leq \|f\|^2.$$

Proof: Thanks to the last proposition we have

$$\|f\|^2 \geq \|\Pi_N f\|^2$$

$$= \left\langle \sum_{n=1}^{N} \hat{f}(n)\,\phi_n \,, \sum_{m=1}^{N} \hat{f}(m)\,\phi_m \right\rangle$$

$$= \sum_{n=1}^{N} \sum_{m=1} \hat{f}(n)\hat{f}(m) \underbrace{\langle \phi_n, \phi_m \rangle}_{=\delta_{nm}}$$

$$= \sum_{n=1}^{N} |\hat{f}(n)|^2 \qquad \blacksquare$$

Corollary (Riemann-Lebesgue)

If $\|f\| < \infty$ and $\{\phi_n\}$ be an orthonormal family, then

$$\lim_{n \to \infty} \hat{f}(n) = 0.$$

Proof: Since $\|f\| < \infty$ we have

$$\sum_{n=1}^{N} |\hat{f}(n)|^2 \leq \|f\| < \infty \quad \text{for any } N > 0,$$

thus $\sum_{n=1}^{N} |\hat{f}(n)|^2$ converges to some finite number which means that $|\hat{f}(n)|^2 \to 0$ $n \to \infty$, and thus $|\hat{f}(n)| \to 0$ $n \to \infty$.

Real and complex Fourier series

Definition (General Fourier series for an orthog. system)

Let $\{\phi_n\}_{n=1}^{\infty}$ be a orthogonal system. Then the formal expression

$$\sum_{n=1}^{\infty} \hat{f}(n)\,\phi_n$$ is called a *general Fourier series*.

Remark: Since $\Pi_N f = \sum_{n=1}^{N} \hat{f}(n)\,\phi_n$ is made up from partial sums of the general Fourier series, we often write

$$\boxed{S_N f := \Pi_N f}$$

After the general considerations, we return to our original questions.

- We define now $\boxed{V := \{ f : [-\pi,\pi] \to \mathbb{C} \mid \int_{-\pi}^{\pi} |f|^2 < \infty \}}$

where we assumed that the integral $\int_{-\pi}^{\pi} |f|^2$ exists.

Such functions are said to be "*square-integrable*".

The set V is thus the set of all square-integrable functions, which is in fact a vector space (Exercise: Prove this). This vector space is also denoted by $L^2(-\pi,\pi)$, read "L two".

Definition (complex trigonometric series / Fourier series)

Let f be a complex, 2π periodic functions and consider the orthogonal system

$$\{e^{inx}\}_{n\in\mathbb{Z}}.$$

The the formal series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx}$$

is called the (complex) Fourier series associated with f, we write:

$$\boxed{f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx}}$$

Often, we denote $\hat{f}(n)$ for the complex Fourier series as c_n; that is,

$$c_n := \hat{f}(n) = \frac{\langle f, e^{inx}\rangle}{\|e^{inx}\|^2} = \frac{\int_{-\pi}^{\pi} f\, \overline{e^{inx}}\, dx}{\int_{-\pi}^{\pi} |e^{inx}|^2\, dx} = \boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx =: c_n}$$

Definition (real Fourier series)

Now we consider the orthogonal system on $(-\pi,\pi)$

$$\{1\} \cup \{\cos nx\}_{n=1}^{\infty} \cup \{\sin nx\}_{n=1}^{\infty} \quad \text{instead,}$$

and compute the corresponding (general) Fourier coefficients:

$$\boxed{a_0 :=} \frac{\langle f,1\rangle}{\langle 1,1\rangle} = \boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx}$$

$$\boxed{a_n :=} \frac{\langle f,\cos nx\rangle}{\langle \cos nx,\cos nx\rangle} = \frac{\int_{-\pi}^{\pi} f(x)\cos nx\,dx}{\int_{-\pi}^{\pi}(\cos nx)^2\,dx}$$

$$\boxed{= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx}$$

and similarly,

$$\boxed{b_n :=} \frac{\langle f,\sin nx\rangle}{\langle \sin nx,\sin nx\rangle} \boxed{= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx}.$$

Note that if f is real then $a_n, b \in \mathbb{R}$.

The expression

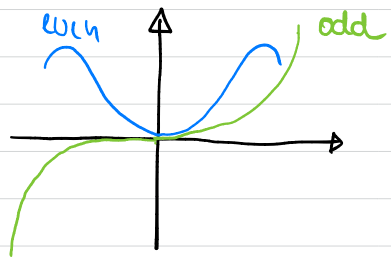
$$\boxed{f \sim a_0 + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)}$$

is called the (real) Fourier series of f.

Before we start to loo. at a number of examples we introduce the concept of even and odd functions

Definition: (odd and even functions)

A function f on a symmetric interval $J = [-d,d]$ is called even if $f(x) = f(-x)$ $\forall x \in J$, and is called odd if $f(-x) = -f(x)$ $\forall x \in J$.



Observation: For the product $f\cdot g$ of two functions, we have

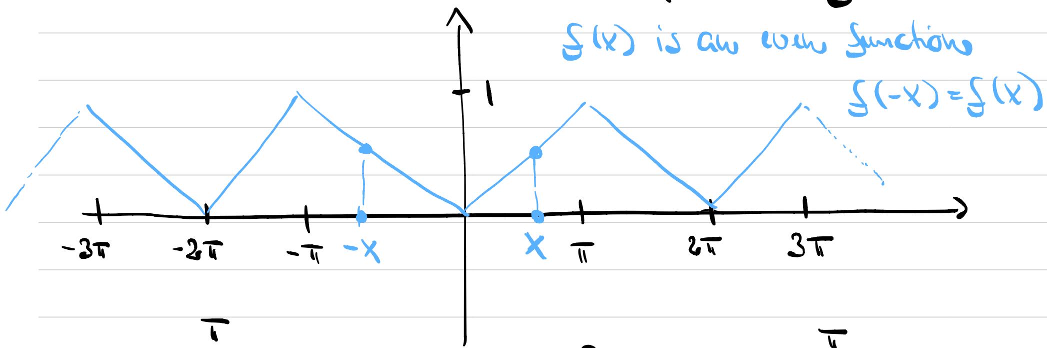
f	g	$f\cdot g$
even	even	even
odd	even	odd
even	odd	odd
odd	odd	even

Exercise. Show that

$$f \text{ even} \Rightarrow \int_{-d}^{d} f(x)\,dx = 2\int_{0}^{d} f(x)\,dx, \quad f \text{ odd} \Rightarrow \int_{-d}^{d} f(x)\,dx = 0.$$

Example 1.

$f(x) = |x|$ on $[-\pi, \pi]$ and periodically extended



$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |x| \cdot 1 = \frac{1}{2\pi}\cdot\left(\int_{-\pi}^{0}(-x)\,dx + \int_{0}^{\pi} x\,dx\right)$$

$$= \frac{1}{2\pi}\cdot\left(-\frac{x^2}{2}\Big|_{-\pi}^{0} + \frac{x^2}{2}\Big|_{0}^{\pi}\right) = \frac{1}{2\pi}\cdot\frac{2\pi^2}{2} = \frac{\pi}{2}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\,dx$$ f and $\cos nx$ are even functions

$$= \frac{1}{\pi}\int_{-\pi}^{0} -x\cos x\, nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx = \frac{2}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

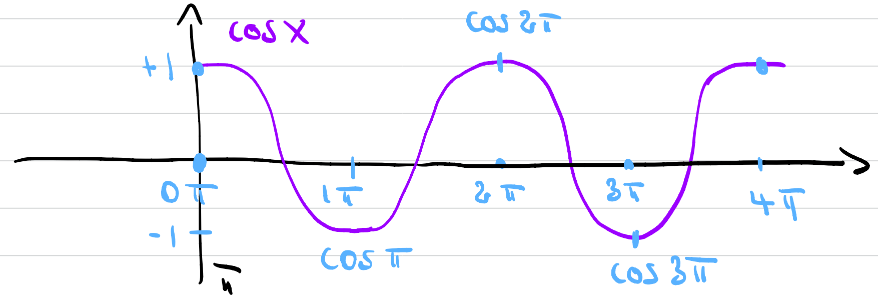
How to compute $\int_a^b x\cos nx\,dx$? Integration by parts!

$$\int_a^b \underbrace{x}_{f}\underbrace{\cos nx}_{g'} = \underbrace{x}_{f}\,\underbrace{\frac{\sin nx}{n}}_{g}\Big|_a^b - \int_a^b \underbrace{\frac{\sin nx}{n}}_{f'g}$$

$$= \frac{x}{n}\sin nx\Big|_a^b + \frac{\cos nx}{n^2}\Big|_a^b$$

Thus

$$a_n = \frac{2}{\pi}\left(\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right) = \frac{2}{\pi n^2}\left((-1)^n - 1\right)$$



$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cdot \sin nx$$

Note that $\sin x$ is an "odd" function, i.e

$\sin(-x) = -\sin(x)$, and so is $\sin(nx)$ and thus $|x|\sin(nx)$. Thus

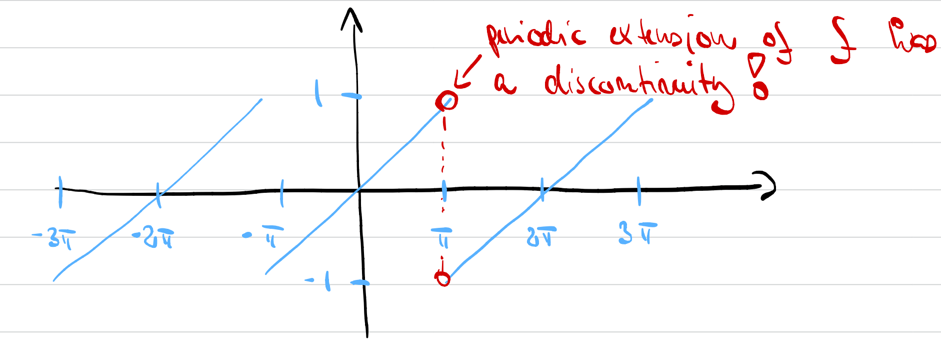
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx = 0.$$

Fourier series for $f(x) = |x|$:

$$f \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n}\left((-1)^n - 1\right).$$

- For a plot of the partial sums $S_N f$ we refer to the supplemental jupyter notebook.

Example 2: $f(x) = x$ on $[-\pi, \pi]$, extended periodically to entire real line.



a) Complex Fourier series

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \cdot 1 \, dx = 0$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, e^{-inx}$$

integration by parts

$$= \frac{1}{2\pi} \left(\frac{x\, e^{-inx}}{-in} \Bigg|_{-\pi}^{+\pi} - \underbrace{\int_{-\pi}^{\pi} \frac{e^{inx}}{-in} dx}_{=0} \right)$$

$$= \frac{e^{in\pi} + e^{-in\pi}}{-2in} = -\frac{\cos n\pi}{in} = \frac{(-1)^{n+1}}{in}.$$

$$\Rightarrow f \sim \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n+1}}{in} e^{inx}$$

- We can write this Fourier series as follows

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n+1}}{in} e^{inx} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{in} \left(e^{inx} - e^{inx} \right)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot 2 \sin nx. \quad (*)$$

b) Exercise. Now compute the real Fourier series

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x = \dots$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos nx = \dots$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx = \frac{2}{\pi} \int_{0}^{\pi} x \sin nx \, dx$$

$$= \dots$$

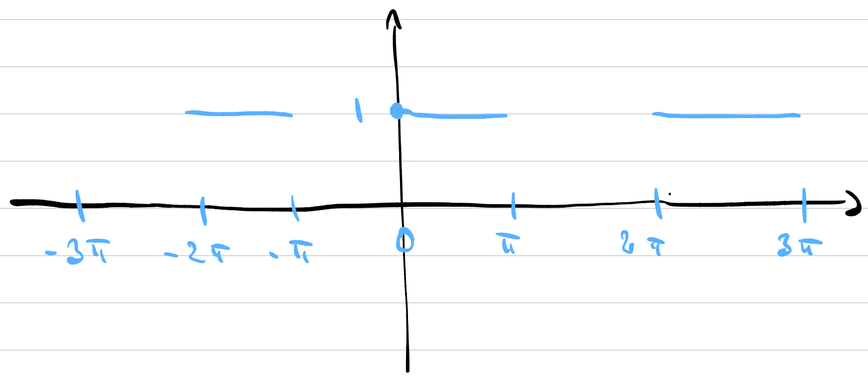
and verify that you end up with $(*)$.

- When you plot the partial sums $S_N f$, you see that approximate $f(x)$ quite well for increasing N except for close to the interval endpoints, where $S_N f$ overshoots / show large oscillations. This phenomena is called Gibbs phenomena and caused by the discontinuity of the function.

Example 3 Consider the Heaviside function

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 1 \end{cases}$$

Consider this as a function on $[-\pi, \pi]$ and extend it to $\mathbb{R}$ periodically.



Compute the real Fourier series

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(x)\,dx = \frac{1}{2\pi} \int_0^{\pi} 1\,dx = \frac{1}{2}.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \cos nx\,dx = \frac{1}{\pi} \int_0^{\pi} \cos nx\,dx$$

$$= 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} u(x) \sin nx\,dx = \int_0^{\pi} \sin nx\,dx$$

$$= -\frac{1}{\pi} \left(\frac{\cos nx}{n} \Big|_0^{\pi} \right)$$

$$= -\frac{1}{n\pi} \left(\cos n\pi - 1 \right)$$

$$= -\frac{1}{n\pi} \left((-1)^n - 1 \right)$$

$$= \frac{1}{n\pi} \left(1 + (-1)^{n+1} \right) = \begin{cases} 0 & n \text{ even} \\ \frac{2}{n\pi} & n \text{ odd.} \end{cases}$$

$$\Rightarrow u(x) \sim \sum_{n=1}^{\infty} \frac{1}{n\pi} \left(1 + (-1)^{n+1} \right) \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{1}{(2n-1)\pi} \sin((2n-1)x).$$

Complex vs. real Fourier series

If f is a real function, then

- $$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, \overline{e^{inx}}\, dx$$

$$= \overline{\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{inx}\, dx}$$

$$= \overline{c_{-n}}$$

. Thus all sums

$$c_n e^{inx} + c_{-n} e^{-inx}$$

$$= c_n e^{inx} + \overline{c_n e^{inx}}$$

$$= z + \bar{z} = \operatorname{Re}(z)$$

are real, so entire "complex" F.S. is a real function

- Transitions between real and complex form

$$c_n + c_{-n} = \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} f(x) \left(e^{-inx} + e^{inx} \right) dx \right)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = a_n$$

$$c_n - c_{-n} = -i \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = -i\, b_n$$

$$c_n + c_{-n} = a_n$$

$$c_n - c_{-n} = -i\, b_n$$

$$c_n = \frac{a_n - i\, b_n}{2}$$

$$c_{-n} = \frac{a_n + i\, b_n}{2}$$

Convergence of Fourier series

Question: We introduced formal Fourier series

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx}$$

with $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$

When / for which x does

$$f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx} = \lim_{N\to\infty} \underbrace{\sum_{n=-N}^{N} \hat{f}(n)\, e^{inx}}_{=:\, S_N f(x)}$$

hold?

- We start by rewriting $S_N f$. To do so we define the so-called N-th Dirichlet-kernel

$$D_N(x) := \sum_{n=-N}^{N} e^{inx} = 1 + \sum_{n=1}^{N}\left(e^{inx} + e^{-inx}\right)$$

$$= 1 + 2\sum_{n=1}^{N} \cos nx$$

Note that $\int_{-\pi}^{\pi} D_N(x)\, dx = 2\pi \quad \forall N$.

- Because of this, one sometimes normalizes the Dirichlet kernel:

$$\tilde{D}_N(x) = \frac{1}{2\pi} D_N(x), \quad \int_{-\pi}^{\pi} \tilde{D}_N(x)\, dx = 1.$$

- We can rewrite D_N further

$$\sum_{n=-N}^{N} e^{inx} = \sum_{n=0}^{2N} e^{inx}\, e^{-iNx}$$

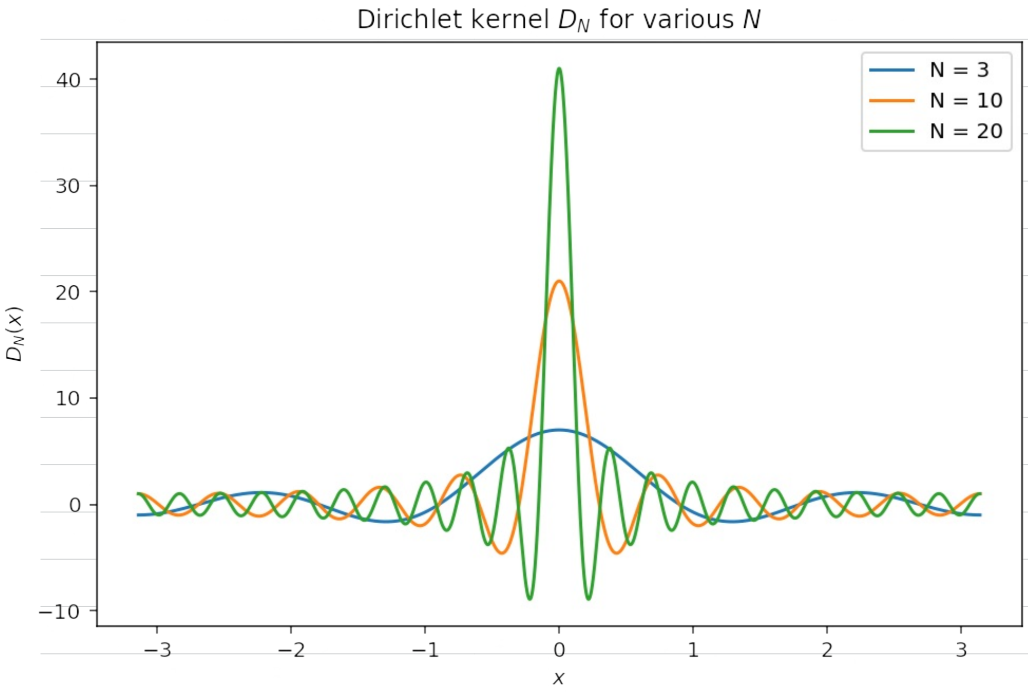
$$= e^{-iNx} \sum_{n=0}^{2N} q^n \quad \text{geometric sum with } q = e^{ix}$$

$$= e^{-iNx} \cdot \frac{1 - q^{2N+1}}{1-q} = e^{-iNx}\, \frac{\left(1 - e^{i(2N+1)x}\right)}{1 - e^{ix}}$$

$$= \frac{e^{-iNx} - e^{iNx}\, e^{ix}}{1 - e^{ix}} \cdot \frac{e^{-i\frac{x}{2}}}{e^{-i\frac{x}{2}}}$$

$$= \frac{e^{-i(N+\frac{1}{2})x} - e^{i(N+\frac{1}{2})x}}{e^{-i\frac{x}{2}} - e^{i\frac{x}{2}}} =: \boxed{\frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)} = D_N(x)}$$

• look at plots for $D_N(x)$



• Now we rewrite the partial sum $S_N f(x)$ as follows

$$S_N f(x) = \sum_{-N}^{N} \hat{f}(n)\, e^{inx}$$

$$= \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y)\, e^{-iny}\, dy\; e^{inx}$$

$$= \int_{-\pi}^{\pi} f(y) \cdot \underbrace{\frac{1}{2\pi} \sum_{n=-N}^{N} e^{in(x-y)}}_{= \tilde{D}_N(x-y)}\, dy$$

$$= \int_{-\pi}^{\pi} f(y)\, \tilde{D}_N(x-y)\, dy .$$

This kind of representation of some function via such integral is so common in mathematics that it deserves its own name: "Convolution".

Definition: For two 2π-periodic functions f, g the convolution $f * g$ is defined by

$$(f * g)(x) := \int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy .$$

Lemma (Commutativity of convolution)

For two 2π periodic functions it holds that

$$(f * g)(x) := \int_{-\pi}^{\pi} f(y)\, g(x-y)$$

$$= \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy =: (g * f)(x).$$

Proof:

$$(f * g)(x) = \int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy$$

(Substitution rule $t = x - y$, that is $t(y) = x - y$ and $y = y(t) = x - t$

$$= \int_{t(-\pi)}^{t(\pi)} f(x-t)\, g(t)\, \overbrace{\frac{dy}{dt}}^{=-1}\, dt$$

$$= - \int_{x+\pi}^{x-\pi} f(x-t)\, g(t)\, dt$$

$$= - \int_{\pi}^{-\pi} f(x-t)\, g(t)\, dt$$

(since integrand is 2π periodic, so it doesn't matter on which interval of length 2π we integrate)

$$= \int_{-\pi}^{\pi} f(x-t)\, g(t)\, dt$$

$$= (g * f)(x). \quad \blacksquare$$

Interlude: Understanding Convolutions

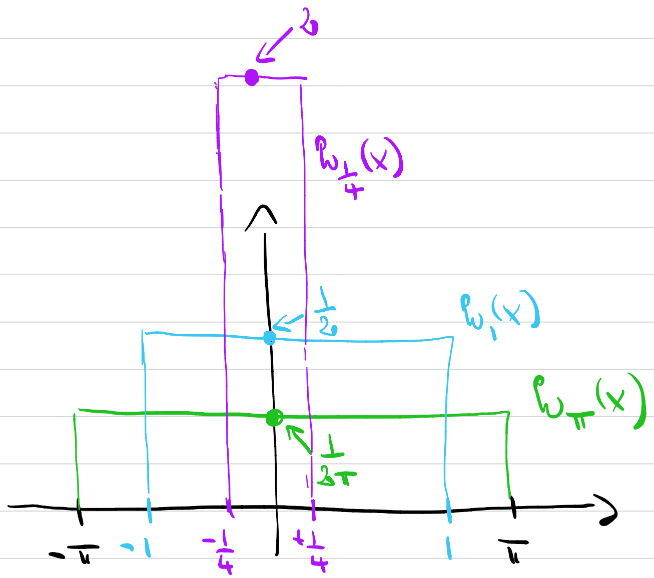
- At first, convolution might a difficult operation to grasp. We thus pause here to investigate the meaning of taking convolutions further.

- Let's take a look at the meaning of

$$g * w_\varepsilon$$

where for $0 < \varepsilon \le \pi$ we define

$$w_\varepsilon(x) = \frac{1}{2\varepsilon} \cdot \chi_{[-\varepsilon,\varepsilon]} = \begin{cases} \frac{1}{2\varepsilon}, & |x| \le \varepsilon \\ 0 & \text{else} \end{cases}$$



- Note that we always have that

$$\int_{-\pi}^{\pi} w_\varepsilon(x)\,dx = 1 \qquad \forall \varepsilon \in (0,\pi]$$

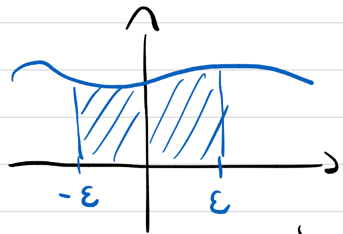
so the "total mass" under the integral is always 1.

Now we have a look at $g * w_\varepsilon$; specifically we look at

$$(g * w_\varepsilon)(0) = \frac{1}{2\varepsilon}\int_{-\pi}^{\pi} g(y)\, w_\varepsilon(0-y)\,dy$$

Since w_ε is even $= \frac{1}{2\varepsilon}\int_{-\pi}^{\pi} g(y)\,\chi_{[-\varepsilon,\varepsilon]}(y)\,dy$

Definition of $\chi_{[\varepsilon,\varepsilon]}$ $= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} g(y)\,dy$



So we integrate $g(y)$ over an interval of length 2ε centered at 0 and then we divide by the length of the interval.

> So $g * w_\varepsilon(0)$ represents a sort of an average value of g taken over an interval centered at 0.

· For a 'nice' function g, we actually expect then that $(g * w_\varepsilon)(0) \longrightarrow g(0)$, $\varepsilon \to 0$, that is average values over smaller and smaller intervals centred at 0 should converge to $g(0)$, if e.g. g is continuous at 0. This is just the classical fundamental theorem of calculus: If g is continuous, then

$$\int_a^b g(y)\,dy = G(a) - G(b) \quad \text{where } G'(y) = g(y).$$

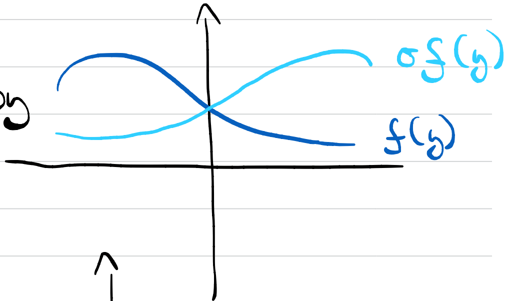
Now
$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} g(y)\,dy = \frac{1}{2\varepsilon}\big(G(\varepsilon) - G(-\varepsilon)\big)$$
$$= \frac{1}{2}\left(\frac{G(\varepsilon)-G(0)}{\varepsilon} + \frac{G(-\varepsilon)-G(0)}{-\varepsilon}\right) \xrightarrow{\varepsilon \to 0} \frac{1}{2}\big(G'(0) + G'(0)\big)$$
$$= g(0).$$

· Now look at

$$(g * w_\varepsilon)(x) = \frac{1}{2\varepsilon}\int_{-\pi}^{\pi} g(y)\, w_\varepsilon(x-y)\,dy$$

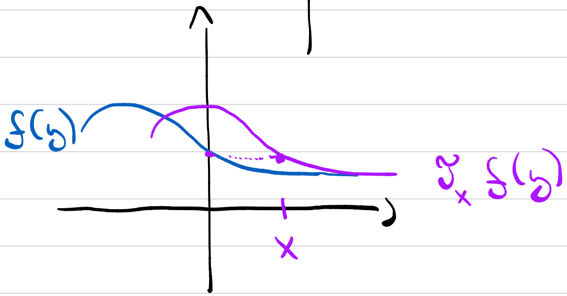
· Introduce mirror operator
$\sigma : f \longmapsto \sigma f$ defined by

$$\sigma f(y) = f(-y)$$



· Translation operator
$f \longmapsto \tau_x f$ given by

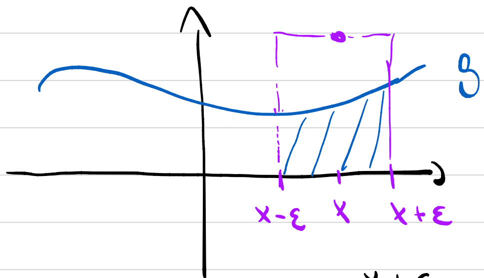
$$\tau_x f(y) := f(y-x)$$



Now we realize that $w_\varepsilon(x-y)$ as a function of y is just

$$w_\varepsilon(x-y) = w_\varepsilon(-(y-x))$$
$$= \sigma w_\varepsilon(y-x)$$
$$= \tau_x \sigma w_\varepsilon(y)$$

So $w_\varepsilon(x-y)$ is just w_ε mirrored first and then translated to x. In pictures



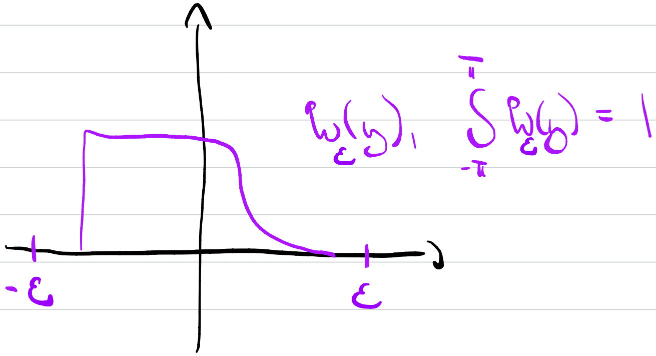
Thus
$$g * w_\varepsilon(x) = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{x+\varepsilon} w(y)\,dy$$

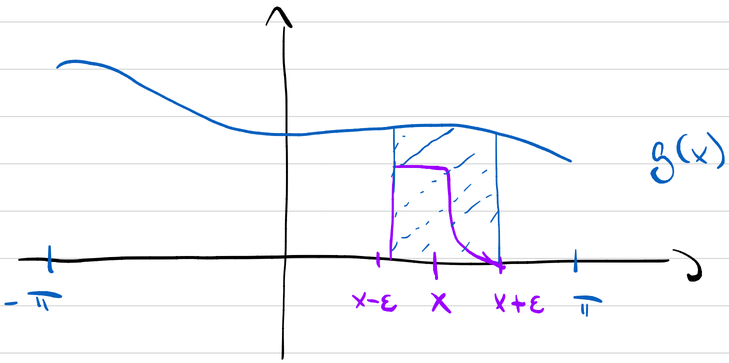
So $g * w_\varepsilon(x)$ represents a sort of an average value of g taken over an interval centred at x.

• As in the case for $x = 0$, we can show that

$$g * \varphi_\varepsilon(x) \longrightarrow g(x) \quad \text{if } g \text{ is continuous at } x.$$

• Let's consider a more general case





Here the functions $\varphi_\varepsilon(x)$ does not weight all contributions to f close to x in the same way, here we put more weight towards the left contribution. And not all values of $g(y)$ are weighted the same way.

After this interlude we reiterate:

We observed that the partial sums

$$S_N f(x) = T_N f(x) := \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

are nothing else but a convolution of f with the (normalized) Dirichlet kernel

$$\tilde{D}_N(y) := \frac{1}{2\pi} \sum_{n=-N}^{N} e^{iny}.$$

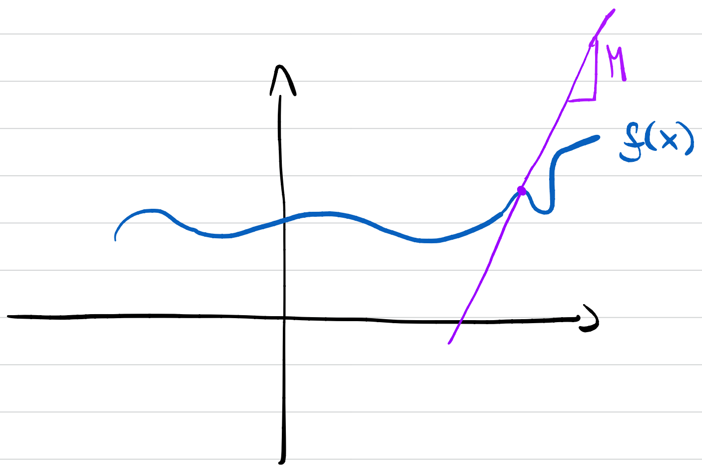
For any N: $\int_{-\pi}^{\pi} \tilde{D}_N(y)\, dy = 1$

and for $N \to \infty$, the 'mass' concentrates more and more around 0. Thus it might come as a complete surprise, that for "nice" functions f we have that $\lim_{N \to \infty} S_N f(x) = f(x)$.

Warning: The Dirichlet kernel is quite nasty, in particular it is very oscillatory for large N. In particular, there exists example of functions f where f is continuous at x, but $S_N f$ does not converge to f at x.

But if we assume a slightly stronger condition one can prove convergence: We say that f is Lipschitz continuous in x if

$$|f(x-y) - f(x)| \leq M|y| \quad \forall y \in (-2\pi, 2\pi)$$



Remark: The condition above implies that f is continuous at x, since $f(x-y) \to f(x)$ if $y \to 0$.

Theorem (Pointwise convergence of Fourier series for Lipschitz continuous functions)

Let f be a 2π periodic function which is square-integrable and let $x \in (-\pi, \pi)$ be a point such that

$$|f(x-y) - f(x)| \leq M|y| \quad \forall y \in (-2\pi, 2\pi)$$

for some constant M. Then

$$S_N f(x) := \sum_{n=-N}^{N} c_n e^{inx} \longrightarrow f(x) \text{ for } N \to \infty.$$

Proof: A detailed proof for the mathematically interested can be found on the next two pages.

Note: Proof is not relevant for exam!

Proof: We want to show that $f(x) - S_N f(x) \to 0$, $N \to \infty$.

Let's start from there:

$$f(x) - S_N f(x) = f(x) - f * \tilde{D}_N(x)$$

$$= f(x) - \int_{-\pi}^{\pi} f(y)\tilde{D}_N(x-y)\,dy$$

(using Lemma 1)

$$= f(x) - \int_{-\pi}^{\pi} f(x-y)\tilde{D}_N(y)\,dy$$

(using $1 = \int_{-\pi}^{\pi} \tilde{D}_N(y)dy$ for all N)

$$= f(x)\cdot\underbrace{\int_{-\pi}^{\pi} \tilde{D}_N(y)\,dy}_{=1} - \int_{-\pi}^{\pi} f(x-y)\tilde{D}_N(y)\,dy$$

$$= \int_{-\pi}^{\pi} \big(f(x) - f(x-y)\big)\,\tilde{D}_N(y)\,dy$$

(Representation of normalized Dirichlet kernel, see page 1)

$$= \int_{-\pi}^{\pi} \boxed{\frac{(f(x)-f(x-y))}{\pi\cdot\sin\frac{y}{2}}}\,\sin((N+\tfrac{1}{2})y)\,dy \quad =: g(y)$$

Let's pause for a moment and have a closer look at

$$g(y) := \frac{f(x)-f(x-y)}{\pi\,\sin\frac{y}{2}}.$$

Away from the origin, this function is "nice" since $\sin\frac{y}{2}$ is not zero on $(-\pi,\pi)\setminus\{0\}$. The question is whether $g(y)$ blows up for $y \to 0$ since $\frac{1}{\sin\frac{y}{2}} \to \infty$ for $y \to 0$.

But thanks to our condition, we have that

$$\pi\cdot|g(y)| = \frac{|f(x)-f(x-y)|}{|\sin\frac{y}{2}|} \le \frac{M|y|}{|\sin\frac{y}{2}|} \le C$$

for some constant, since we know (using de l'Hospital's rule) that

$$\lim_{y\to 0} \frac{y}{\sin\frac{y}{2}} = \lim_{y\to 0} \frac{1}{\frac{1}{2}\cos\frac{y}{2}} = 2.$$

So we proceed with

$$f(x) - S_N f(x) = \int_{-\pi}^{\pi} g(y)\,\sin((N+\tfrac{1}{2})y)\,dy$$

(using $\sin(a+b) = \sin a\cos b + \cos a \sin b$)

$$= \int_{-\pi}^{\pi} \underbrace{\boxed{g(y)\cos\tfrac{y}{2}}}_{=:\,g_1(y)}\,\sin Ny\,dy \qquad \mathrm{I}$$

$$+ \int_{-\pi}^{\pi} \underbrace{\boxed{g(y)\sin\tfrac{y}{2}}}_{=:\,g_2(y)}\,\cos Ny\,dy \qquad \mathrm{II}$$

We want to show that both I and II $\to 0$ for $N \to \infty$.

Looking closer at I, we realize that it is (modulo a factor 2π) the real part of the Fourier coefficient c_n associated with g_1:

$2\pi\cdot\mathrm{I} = \mathrm{Re}(c_n(g_1))$. Similarly

$2\pi\cdot\mathrm{II} = \mathrm{Im}(c_n(g_2))$. Because of the nice behavior of g, both g_1 and g_2 are square-integrable ($\int_{-\pi}^{\pi} |g_1|^2 dx < \infty$, $\int_{-\pi}^{\pi} |g_2|^2 dx < \infty$) so by Bessel's inequality and our observations on p.3, $c_n(g_1) \to 0$, $c_n(g_2) \to 0$ for $n \to \infty$ and so do I and II. $\blacksquare$

Theorem (Pointwise convergence of Fourier series for functions with jumps).

Let f be piecewise continuously differentiable function on $[-\pi, \pi]$, where both the left and right derivative exist at any point of discontinuity.

If f is continuous at a point x, then

$$\lim_{N \to \infty} S_N f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} = f(x).$$

If f is not continuous at x, the Fourier series

converges to mean value of the left-hand-side and right-hand-side limit of f at x, i.e.,

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx} = \frac{f_+(x) + f_+(x)}{2} \quad \text{where}$$

$$f_+(x) = \lim_{h \to 0^+} f(x+h), \quad f_-(x) = \lim_{h \to 0^+} f(x-h).$$

If f is C^1 (= continuously differentiable) on $(-\pi, \pi)$ (excluding endpoints) then $f(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$ $\forall x \in (-\pi, \pi)$. If f is C^1 on the entire real-line and 2π periodic, then the Fourier series converges to f also at $-\pi$ and π.

Proof: Omitted.

- Note that the last 2 theorems were about pointwise convergence. But if we can also talk about convergence in the $\|\cdot\| = \left(\int_{-\pi}^{\pi} |\cdot|^2 \, dx\right)^{\frac{1}{2}}$

This is an integral norm and doesn't look at pointwise convergence.

$$V = \left\{ f : [-\pi,\pi] \to \mathbb{C} \,,\ \|f\| = \left(\int_{-\pi}^{\pi} |f|^2\right) < \infty \right\}$$

$= L^2((-\pi,\pi)) =$ "space of all square-integrable functions", $\|f\|$ are called L^2-norm

Theorem 3

Let $f \in L^2((-\pi,\pi))$ then $S_N f = T_N f \to f$ in the L^2-norm, that is

$$\lim_{N\to\infty} \|f - S_N f\| = 0.$$

Proof. Omitted ∎

Todo: Add outline for the mathematical interested, needs approximation result by Weierstraß.

Fourier series on $[-L, L]$

We now consider $J = [-L, L]$ instead of $[-\pi, \pi]$.

Then $[-\pi,\pi]$, $\sin x$ $\quad$ $[-L,L]$ $\sin \frac{\pi}{L} x$

$$\left\{ e^{inx\cdot\frac{\pi}{L}} \right\}_{n\in\mathbb{Z}} \quad \text{and} \quad \left\{ \cos \frac{n\pi x}{L} \right\}_{n=0}^{\infty} \cup \left\{ \sin \frac{n\pi x}{L} \right\}_{n=1}^{\infty}$$

are orthogonal systems with respect to the scalar product

$$\langle f, g \rangle = \int_{-L}^{L} f(x)\overline{g(x)}\, dx.$$

Exercise: Verify this.

The corresponding complex Fourier series is defined by

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}} \quad \text{with} \quad c_n = \frac{1}{2L}\int_{-L}^{L} f(y) e^{\frac{-in\pi y}{L}}\, dy,$$

Similarly we have

$$f \sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \quad \text{with}$$

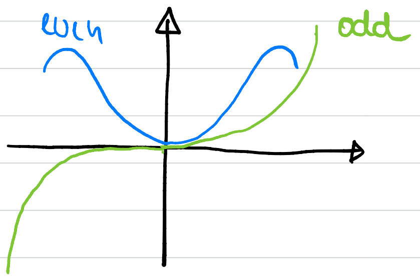
$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(y)\, dy, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(y)\cos\frac{n\pi y}{L}\, dy$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(y)\sin\frac{n\pi y}{L}\, dy.$$

3. Even and odd extension

Definition:

A function f on a symmetric interval $I = [-\ell, \ell]$ is called even if $f(x) = f(-x)$ $\forall x \in I$, and is called odd if $f(-x) = -f(x)$ $\forall x \in I$.



Observation: For the product $f \cdot g$ of two functions, we have

f	g	$f \cdot g$
even	even	even
odd	even	odd
even	odd	odd
odd	odd	even

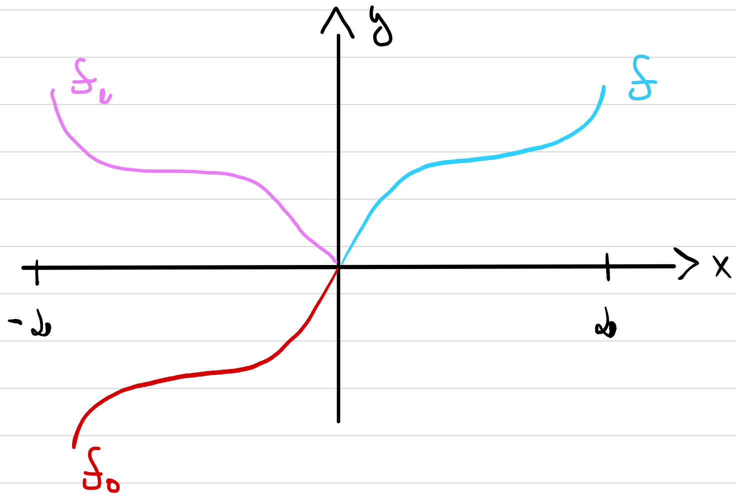
Definition:

"odd utvidelse"

If $f : [0, \ell) \to \mathbb{R}$, we define its odd extension f_o and its even extension f_e ("jevn utvidelse") by

$$f_o(x) = \begin{cases} f(x) & x \in [0, \ell) \\ -f(-x) & x \in (-\ell, 0), \end{cases}$$

$$f_e(x) = \begin{cases} f(x) & x \in [0, \ell) \\ f(-x) & x \in (-\ell, 0). \end{cases}$$



Note: Both f_e and f_o extend f from the original domain of definition $[0, \ell)$ to $(-\ell, \ell)$.

Observations: We can now compute the Fourier series for f_o and f_e. Since $f_o = f_e = f$ on $(0,L)$ they Fourier series must converge to f.

For f_o we have that

$$a_n = \frac{1}{L}\int_{-L}^{L} f_o(y)\cos\frac{n\pi y}{L}\,dy = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_o(y)\sin\frac{n\pi y}{L}\,dy = \frac{2}{L}\int_0^L f(y)\sin\frac{n\pi y}{L}\,dy$$

so we obtain a sinus series for f.

For f_e we get

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f_e(y)\,dy = \frac{1}{L}\int_0^L f(y)\,dy$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f_e(y)\cos\frac{n\pi y}{L}\,dy = \frac{2}{L}\int_0^L f(y)\cos\frac{n\pi y}{L}\,dy,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f_e(y)\sin\frac{n\pi y}{L}\,dy = 0$$

Thus we obtain a cosinus series for f.

Exercise: Compute the cosinus series to $f(x) = x$ on $(0,\pi)$.

Hint: This should be same as the Fourier series for $f(x) = |x|$.

See also Example 8.23 in Morten's lecture notes.

Parseval' identity

Theorem

Assume that $\int_{-L}^{L} |f(x)|^2\,dx$ exists and is finite, i.e. $f \in L^2((-L,L))$. Let

$$f \sim \sum_{n=-\infty}^{\infty} c_n e^{inx\frac{\pi}{L}}$$

$$\sim a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\frac{\pi}{L}x + b_n \sin n\frac{\pi}{L}x\right).$$ Then

$$\boxed{\frac{1}{2L}\int_{-L}^{L} |f(x)|^2\,dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}$$

Proof: We only prove the first identity, the second can be derived in the same way or via the relation between a_n, b_n and c_n and c_{-n}.

- Since $f \in L^2((-\pi,\pi))$, we know that

$\lim_{N\to\infty} \|f - S_N f\| = 0$ and therefore $\lim_{N\to\infty} \|S_N f\| = \|f\|$, or equivalently, $\lim_{N\to\infty} \|S_N f\|^2 = \|f\|^2 = \int_{-L}^{L} |f|^2\,dx$.

Now,

$$\|S_N f\|^2 = \int_{-L}^{L} \Big(\overbrace{\sum_{n=-N}^{N} c_n e^{inx\frac{\pi}{L}}}^{S_N f}\Big)\overline{\Big(\sum_{m=-N}^{N} c_m e^{imx\frac{\pi}{L}}\Big)}\,dx$$

$$= \sum_{n,m=-N}^{N} c_n \bar{c}_m \underbrace{\int_{-L}^{L} e^{inx\frac{\pi}{L}} e^{-imx\frac{\pi}{L}}}_{2\cdot L\cdot \delta_{nm}}$$

$$= \sum_{n=-N}^{N} |c_n|^2 \cdot 2 \cdot L$$

$N \to \infty$

$$\longrightarrow \quad \|f\|^2 = \int_{-L}^{L} |f|^2\,dx. \quad \blacksquare$$

Remark

Parseval's identity is just an infinite dimensional version of what we already have seen for $\mathbb{R}^n$.

Here we write $\vec{x} = (x_1, \dots, x_n)$ which is just the coordinate representation of

$$\vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i$$

, $\vec{e}_i$ are our standard unit basis vectors.

Now $\|\vec{x}\|^2 = \langle \vec{x}, \vec{x} \rangle = \sum_{i=1}^{n} x_i^2$.

For $L^2((-L,L))$ (an infinite dimensional space), the orthonormal system $\left\{\frac{e^{in\frac{\pi}{L}x}}{\sqrt{2L}}\right\}_{n\in\mathbb{Z}}$ plays a similar role as $\{\vec{e}_i\}_{i=1}^{n}$ in $\mathbb{R}^n$.

Applications: Computation of infinite series

Take

$$w(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Heaviside function.

$$w(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x$$

$w(x)$ is smooth in $x = \frac{\pi}{2}$ satisfying assumptions of Thm.

Thus

$$1 = w\left(\frac{\pi}{2}\right) = \frac{1}{2\pi} + \frac{2}{\pi} \cdot \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin\left((2n-1)\frac{\pi}{2}\right)$$

$$\overbrace{\sin\left(\frac{\pi}{2}\right)}^{1},\ \overbrace{\sin\left(\frac{3}{2}\pi\right)}^{-1},\ \overbrace{\sin\left(\frac{5}{2}\pi\right)}^{1}$$

$$= \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \cdot (-1)^{n+1}$$

Thus

$$\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right) = \frac{\pi}{2}\left(1 - \frac{1}{2}\right) = \frac{\pi}{4}.$$

Spectrum of periodic functions

• We started with a $2L$ periodic function $f:(-L,L)\to\mathbb{C}$ and defined its corresponding Fourier series

$$f \sim \sum_{n=-\infty}^{\infty} \hat{f}(n)\, e^{inx\frac{\pi}{L}}$$

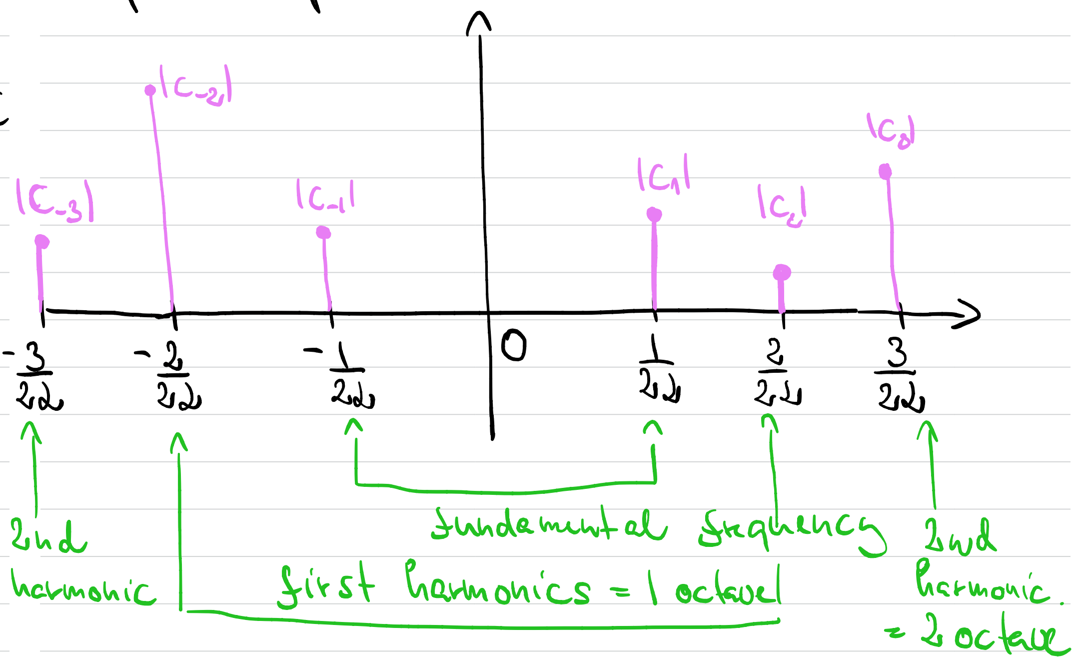
where $\hat{f}(n) = \frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-inx\frac{\pi}{L}}\, dx \in \mathbb{C}$.

So we can now associate with f a sequence of pairs $\left(c_n, \frac{n}{2L}\right)_{n\in\mathbb{Z}}$ which is called the spectrum of f.

• Note that $\hat{f}(n) = c_n \in \mathbb{C}$ can also be written as $c_n = |c_n|\, e^{i\theta_n}$ where $|c_n| \in \mathbb{R}$ is the amplitude and $\theta_n \in [-\pi,\pi)$ is the phase.

• $\frac{1}{2L} = \frac{1}{\text{fundamental period}}$ is called the fundamental frequency.

Amplitude spectrum



Phase spectrum