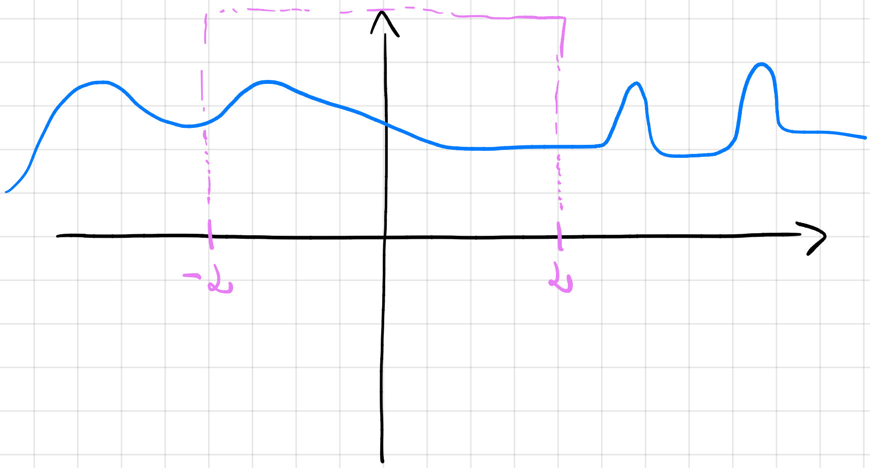
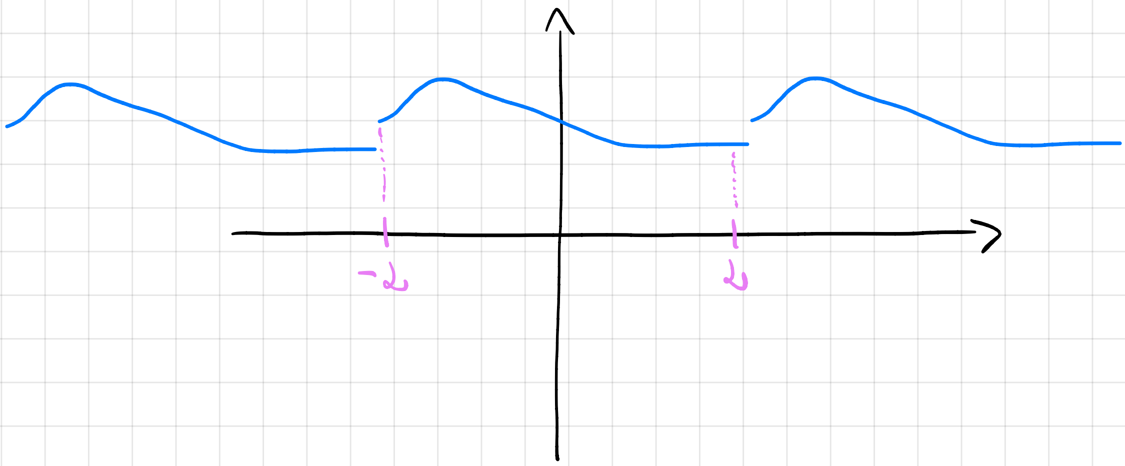
Fourier transform

Questions: Fourier series allows us to handle $f: \mathbb{R} \to \mathbb{C}$ which are $2L$ periodic. What about non-periodic functions?

Idea: Restrict f to interval $(-L, L)$, treat it as "$2L$" periodic function using Fourier series and try to understand what happens if $L \to \infty$.





Let's have a look at the resulting Fourier series:

$$f|_{(-L,L)} \sim \sum_{n=-\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} f(y)\, e^{-iy\boxed{n\frac{\pi}{L}}}\, dy\; e^{ix\boxed{n\frac{\pi}{L}}}$$

$\Delta\omega := \frac{\pi}{L}$

$\omega_n := n \cdot \Delta\omega$

$$= \sum_{n=-\infty}^{\infty} \Delta\omega \frac{1}{2\pi} \int_{-L}^{L} f(y)\, e^{-iy\boxed{\omega_n}}\, dy\; e^{ix\boxed{\omega_n}}$$

This looks like a Riemann sum for the infinite integral of $\hat{f}_L(\omega) := \frac{1}{2\pi} \int_{-L}^{L} f(y)\, e^{-iy\omega}\, dy$

Function in ω!

So with a lot of handwaving, we expect that for $L \to \infty$ we obtain

$$(*) \quad f(x) \sim \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(y) e^{-i\omega y} \, dy \, e^{i\omega x} \, d\omega .$$

→ we then say f is absolutely integrable or $f \in \mathcal{L}^1(\mathbb{R}) = \{ f : \int_{-\infty}^{\infty} |f| < \infty \}$

Definition (Fourier transform)

Assume that $\int_{-\infty}^{\infty} |f(x)| \, dx$ exists and $< \infty$. Then the Fourier transform $\mathcal{F}(f)$ or $\hat{f}$ is defined by

$$\boxed{\hat{f}(\omega) := \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx}$$

If $g(\omega)$ is absolutely integrable, then the inverse Fourier transform is defined by

$$\boxed{\check{g}(x) = \mathcal{F}^{-1}(g) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\omega) e^{i\omega x} \, d\omega .}$$

Now $(*)$ translates to

Theorem

Let f and $\hat{f}$ be absolutely integrable, then

$$f(x) = \mathcal{F}^{-1}(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega) \, e^{i\omega x} \, d\omega .$$

Proof. Omitted.

Warning (Different versions of $\mathcal{F}, \mathcal{F}^{-1}$ ...)

There exists a lot of different definitions for $\mathcal{F}$, as there is no agreement on whether / how to absorb the $\frac{1}{2\pi}$ in the definition.

Common other choices:

$\mathcal{F}$: $\quad \int_{-\infty}^{\infty} f(x) e^{-i\omega x} \, dx \qquad \mathcal{F}^{-1}$: $\quad \boxed{\frac{1}{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{i\omega x} \, d\omega$

Either in front of $\mathcal{F}$ or $\mathcal{F}^{-1}$.

$$\int_{-\infty}^{\infty} f(x) e^{-i2\pi\omega x} \, dx \qquad \int_{-\infty}^{\infty} f(\omega) e^{i2\pi\omega x} \, d\omega$$

So if you see a Fourier transformed function, check which def. was used!

Examples of Fourier transforms

Example 1: $f(x) = \chi_{[a,b]} = \begin{cases} 1 & \text{if } x \in [a,b] \\ 0 & \text{else.} \end{cases}$

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{[a,b]} \cdot 1 \, dx = \frac{1}{\sqrt{2\pi}} (b-a).$$

If $\omega \neq 0$ we compute that

$$\hat{f}(\omega) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \chi_{[a,b]}(x) \, e^{-i\omega x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_a^b e^{-i\omega x} \, dx = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{-i\omega} \, e^{-i\omega x} \Big|_{x=a}^{b}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{-i\omega} \cdot \left(e^{-i\omega b} - e^{-i\omega a} \right) \underbrace{e^{i\omega\frac{(a+b)}{2}} \cdot e^{-i\omega\frac{(a+b)}{2}}}_{=1}$$

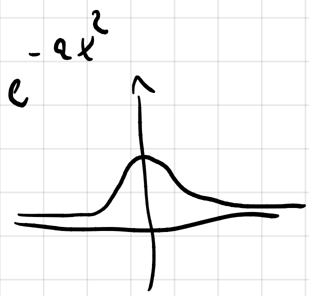
$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1 \cdot 2}{\omega} \left(\frac{e^{-i\omega\frac{(b-a)}{2}} - e^{i\omega\frac{(b-a)}{2}}}{-i \cdot 2} \right) \cdot e^{-i\omega\frac{(a+b)}{2}}$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sin\left(\omega \cdot \frac{(b-a)}{2}\right)}{\omega} \cdot e^{-i\omega\frac{(a+b)}{2}}.$$

Example 2: Let $a > 0$, and set $f(x) = e^{-a|x|}$. Then

$$\mathcal{F}(e^{-a|x|}) = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + \omega^2}$$

See plenary exercise for complete calculation



Example 3: Let $a > 0$, then

$$\mathcal{F}(e^{-ax^2}) = \frac{1}{\sqrt{2a}} \, e^{-\omega^2/2a}.$$

See plenary exercise

"The Fourier transform of a Gaussian is again a Gaussian"

Important rules for computing the Fourier transform

Theorem (Linearity of $\mathcal{F}$)

Let $a, b \in \mathbb{C}$ and $f, g \in \mathcal{L}^1(\mathbb{R})$. Then

$$\boxed{\mathcal{F}(af + bg) = a\,\mathcal{F}(f) + b\,\mathcal{F}(g)}$$

Proof. Exercise, follows directly from linearity of the integral.

Theorem

Assume that both f and $f' \in \mathcal{L}^1$ and that $f(x) \to 0$ $x \to \pm\infty$.

Then it holds that

$$\boxed{\mathcal{F}(f') = i\omega\,\mathcal{F}(f).}$$

Proof:

$$\sqrt{2\pi}\;\mathcal{F}(f') = \lim_{L\to\infty} \int_{-L}^{L} f'(x)\,e^{-i\omega x}\,dx$$

integration by part

$$= \lim_{L\to\infty}\left(f(x)\,e^{-i\omega x}\Big|_{-L}^{L} - \int_{-L}^{L} f(x)\,(-i\omega)\,e^{-i\omega x}\,dx\right)$$

$$= i\omega \int_{-\infty}^{\infty} f(x)\,e^{-i\omega x}\,dx = \sqrt{2\pi}\; i\omega\,\mathcal{F}(f).$$

Definition (convolution)

Let $f, g \in \mathcal{L}^1(\mathbb{R})$. Then the convolution $f * g$

$$(f*g)(x) := \int_{-\infty}^{\infty} f(y)\,g(x-y)\,dx$$

is well-defined and $f*g \in \mathcal{L}^1(\mathbb{R})$.

Remark: That $f*g$ is well-defined needs actually a little proof which we skip here.

Theorem (Commutativity of the convolution).

$$\boxed{(f * g) = (g * f).}$$

Proof: Similar as in the 2π periodic case.

Theorem

Let $f, g \in \mathcal{L}^1(\mathbb{R})$. Then

$$\boxed{\mathcal{F}(f * g) = \sqrt{2\pi} \cdot \mathcal{F}(f) \cdot \mathcal{F}(g)}$$

We can also write this as

$$\boxed{(f * g)^\wedge(\omega) = \sqrt{2\pi}\ \hat{f}(\omega) \cdot \hat{g}(\omega).}$$

Remark: Last theorem tells us that on the ω-side / in the frequency domain, convolution translates into multiplication.