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SIMULATION OF PHASE SEPARATION IN BINARY MIXTURES

Or: How to solve the Cahn-Hilliard equation with 40 lines of code

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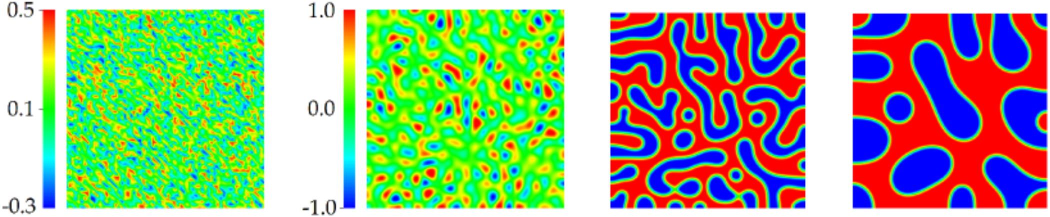
March 21, 2025

The Cahn-Hilliard equation is ubiquitous in science and engineering



Phase separation in metall alloys during rapid quenching Cahn and Hilliard [1958] (Example taken from Bosch [2016])

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(a) Destroyed image.

(b) Reconstructed image.

Image impainting Bertozzi et al. [2007] (Example taken from Bosch [2016])

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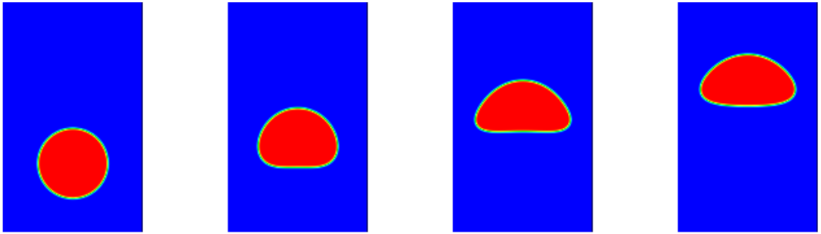

Phase-field/diffuse interface methods for moving interface Du and Feng [2020], e.g. for two-phase flows Yue et al. [2004] (Example taken from Bosch [2016])

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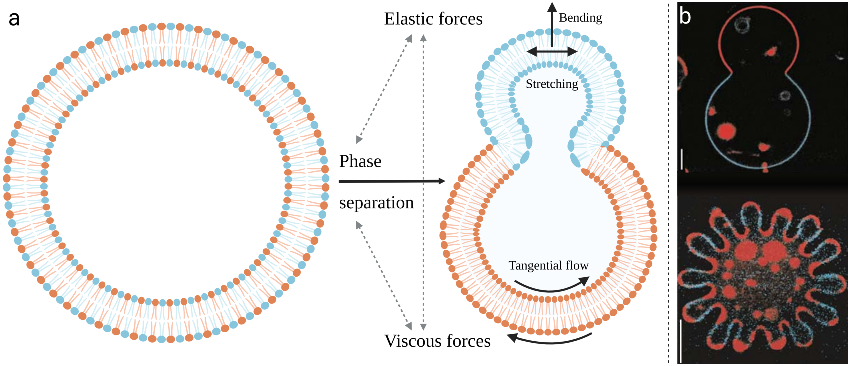
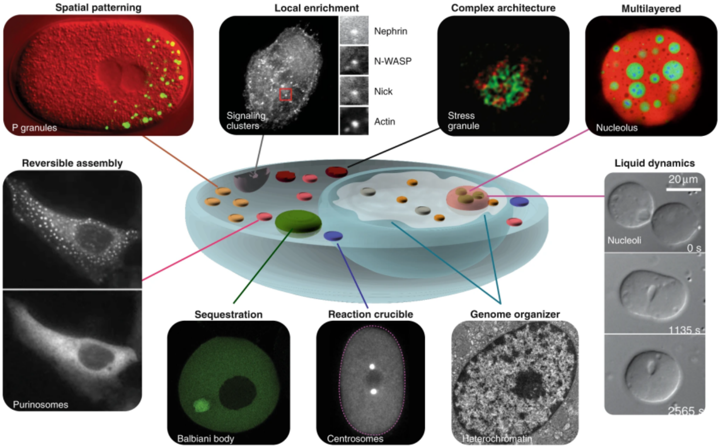


Phase separation of lipids mixture in cell membranes, Baumgart et al. [2003], Wang and Du [2008], Du et al. [2011], Elliott and Ranner [2015], Yushutin et al. [2019]

Liquid-liquid phase separation plays also a crucial role in the intracellular organization and dynamics



Bracha et al. [2019]

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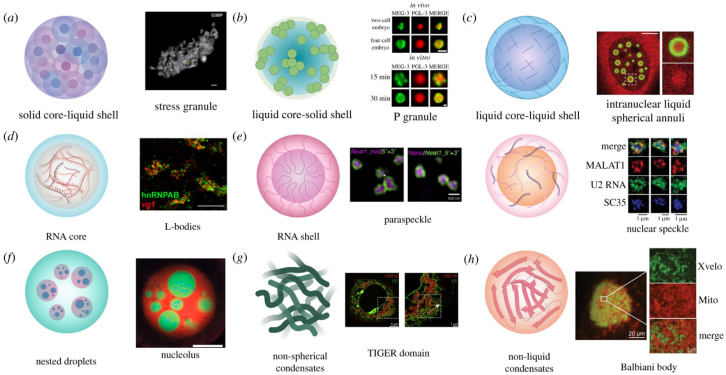

Fare et al. [2021]

Shin and Brangwynne [2017], Hyman et al. [2014], Weber et al. [2019]

The Cahn-Hilliard equation is a 4th order nonlinear parabolic problem

Mass-conservative two-component system described by (rescaled) concentration u of component 1 and $-u$ of component 2:

$$\partial_t u = \nabla \cdot (M \nabla \mu), \quad \mu = \frac{\delta \Psi_\Omega}{\delta u} = -\kappa \Delta u + f(u) \quad \text{on } \Omega \times (0,T)$$

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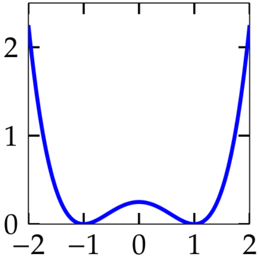
Ginzburg-Landau free energy

$$\Psi_\Omega(u, \nabla u) = \int_\Omega \left(F(u) + \frac{\kappa}{2} |\nabla u|^2 \right) \mathrm{d}x$$

Mobility M (constant and $= 1$)

Gradient energy coefficient $\kappa = \epsilon^2$



$$F(u) = \frac{1}{4}(u^2 - 1)^2$$

References I

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