Oppgave 6 *Du trenger ikke å grunngi svarene dine i denne oppgaven*

f) La $\mathbf{u}$ og $\mathbf{v}$ være vektorer i $\mathbb{R}^n$, begge ulik null, og la r være en skalar. For hver av de følgende fire påstandene, avgjør om den er sann eller ikke.

1. $\|r\mathbf{v}\| = r\|\mathbf{v}\|$, bortsett fra hvis $r = 0$.
2. Hvis $\mathbf{u}$ and $\mathbf{v}$ er ortogonale, så er $\{\mathbf{u}, \mathbf{v}\}$ lineært uavhengige.
3. Hvis $\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, så er $\mathbf{u}$ og $\mathbf{v}$ ortogonale.
4. Hvis $\|\mathbf{u}-\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, så er $\mathbf{u}$ og $\mathbf{v}$ ortogonale.

1) $\|r\vec{v}\| = \sqrt{\langle r\vec{v}, r\vec{v}\rangle} = \sqrt{r^2\langle \vec{v}, \vec{v}\rangle} = \sqrt{r^2}\sqrt{\langle \vec{v}, \vec{v}\rangle}$

$= |r| \cdot \|\vec{v}\|$ (alt.: $\|r\vec{v}\| = \sqrt{(rv_1)^2 + \dots + (rv_n)^2}$)

<u>usant</u> (for $r < 0$)

2) $a\vec{u} + b\vec{v} = \vec{0}$

$0 = \langle \vec{u}, a\vec{u} + b\vec{v}\rangle = a\langle \vec{u}, \vec{u}\rangle + b\cancel{\langle \vec{u}, \vec{v}\rangle}^{0}$

$= a \underbrace{\|\vec{u}\|^2}_{\neq 0 \text{ siden } \vec{u} \neq \vec{0}}$

$\Rightarrow a = 0$

Tilsvarende: $0 = \langle \vec{v}, a\vec{u} + b\vec{v}\rangle = b\underbrace{\|\vec{v}\|^2}_{\neq 0}$

$\Rightarrow b = 0$

<u>sant</u>.

3) $$\|\vec{u}+\vec{v}\|^2 = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2}$$

$$\|\vec{u}+\vec{v}\|^2 = \langle \bar{u}+\vec{v}, \vec{u}+\vec{v} \rangle = \langle \bar{u},\vec{u}\rangle + \langle \vec{v},\vec{u}\rangle + \langle \bar{u},\vec{v}\rangle + \langle \vec{v},\vec{v}\rangle = \cancel{\|\vec{u}\|^2} + 2\langle \bar{u},\vec{v}\rangle + \cancel{\|\vec{v}\|^2}$$

$$\Rightarrow 2\langle \bar{u},\vec{v}\rangle = 0$$

<u>sant</u>

4) $$\|\bar{u}-\vec{v}\|^2 = \cancel{\|\vec{u}\|^2} + \cancel{\|\vec{v}\|^2}$$

$$\|\bar{u}-\vec{v}\|^2 = \cdots = \cancel{\|\vec{u}\|^2} - 2\langle \bar{u},\vec{v}\rangle + \cancel{\|\vec{v}\|^2}$$

$$\Rightarrow -2\langle \bar{u},\vec{v}\rangle = 0$$

<u>sant</u>

(<u>alt.</u>: fra (3) $\|\bar{u}+(-\bar{v})\|^2 = \|\bar{u}\|^2 + \|-\bar{v}\|^2$
$\Rightarrow \{\bar{u},-\bar{v}\}$ er ortogonale)

Eksamen TMA4115 kont '24

Oppgave 10 La $A = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix}$. Vis direkte ved bruk av definisjoner at funksjonen

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \boldsymbol{u}^T A \boldsymbol{v}, \qquad \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^2$$

er et indreprodukt i $\mathbb{R}^2$.

Må vise: for alle $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^2$ og $a, b \in \mathbb{R}$

1) $\langle a\vec{u} + b\vec{v}, \vec{w} \rangle = a\langle \vec{u}, \vec{w} \rangle + b\langle \vec{v}, \vec{w} \rangle$

2) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

3) $\langle \vec{u}, \vec{u} \rangle \geq 0$ og $\langle \vec{u}, \vec{u} \rangle = 0 \Rightarrow \vec{u} = \vec{0}$

1) $\langle a\vec{u} + b\vec{v}, \vec{w} \rangle = \underbrace{(a\vec{u} + b\vec{v})^T}_{= a\vec{u}^T + b\vec{v}^T} A \vec{w}$

$= a\vec{u}^T A \vec{w} + b\vec{v}^T A \vec{w}$

(matrisemult. er lineært)

$= a\langle \vec{u}, \vec{w} \rangle + b\langle \vec{v}, \vec{w} \rangle$

2) $\langle \vec{u}, \vec{v} \rangle = [u_1 \; u_2] \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} 2v_1 - v_2 \\ -v_1 + 4v_2 \end{bmatrix}$$

$$= u_1(2v_1 - v_2) + u_2(-v_1 + 4v_2)$$

$$\langle \vec{v}, \vec{u} \rangle = v_1(2\underline{u_1} - \underline{u_2}) + v_2(-\underline{u_1} + 4\underline{u_2})$$

$$= \underline{u_1}(2v_1 - v_2) + \underline{u_2}(-v_1 + 4v_2)$$

$$= \langle \vec{u}, \vec{v} \rangle$$

3) $\langle \vec{u}, \vec{u} \rangle = u_1(2u_1 - u_2) + u_2(-u_1 + 4u_2)$

$$= 2u_1^2 - 2u_1u_2 + 4u_2^2$$

$$= (u_1 - u_2)^2 + u_1^2 + 3u_2^2 \geq 0$$

og $= 0 \iff \begin{cases} (u_1 - u_2)^2 = 0 \\ u_1^2 = 0 \\ 3u_2^2 = 0 \end{cases}$

$\Rightarrow u_1 = 0$ og $u_2 = 0$, altså $\vec{u} = \vec{0}$

Eksamen TMA4115 vår '22

Oppgave 4 Bestem om funksjonene

$$f_1(x) = 1, \quad f_2(x) = x, \quad f_3(x) = \sin(x),$$

er ortogonale i indreproduktsrommet $\mathcal{C}([-\pi, \pi])$, når indreproduktet er definert ved

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx.$$

Hvis ikke, finn en ortogonal basis for det lineære spennet $\text{span}(f_1, f_2, f_3)$ ved å bruke Gram-Schmidts ortogonaliseringsmetode.

Må regne ut $\langle f_1, f_2 \rangle$, $\langle f_1, f_3 \rangle$ og $\langle f_2, f_3 \rangle$

odde funksjon på symmetrisk intervall

$$\langle f_1, f_2 \rangle = \int_{-\pi}^{\pi} 1 \cdot x \, dx = \left[\tfrac{1}{2} x^2\right]_{-\pi}^{\pi} = 0$$

$$\langle f_1, f_3 \rangle = \int_{-\pi}^{\pi} 1 \cdot \sin x \, dx = 0$$

$$\langle f_2, f_3 \rangle = \int_{-\pi}^{\pi} \underbrace{x \sin(x)}_{f \cdot g'} dx$$

$f(x) = x$
$g'(x) = \sin x$
$f'(x) = 1$
$g(x) = -\cos x$

$$= \underbrace{\left[-x \cdot \cos(x)\right]_{-\pi}^{\pi}}_{2\pi} - \underbrace{\int_{-\pi}^{\pi} (-\cos x) \, dx}_{=0}$$

$= 2\pi$

$\rightarrow$ f_2 og f_3 står ortogonalt på f_1, men ikke på hverandre.

Gram-Schmidt

$g_1 = f_1$, $\quad \tilde{g}_1 = \frac{g_1}{\|g_1\|} = \frac{f_1}{\|f_1\|} \Rightarrow \boxed{\tilde{g}_1(x) = \frac{1}{\sqrt{2\pi}}}$

$$\|f_1\|^2 = \langle f_1, f_1 \rangle = \int_{-\pi}^{\pi} 1 \cdot 1 \, dx = 2\pi$$

$$g_2 = f_2 - \cancelto{0}{\langle \tilde{g}_1, f_2 \rangle} \tilde{g}_1 = f_2$$

$$\langle \tilde{g}_1, f_2 \rangle = \left\langle \frac{f_1}{\|f_1\|}, f_2 \right\rangle = \frac{1}{\|f_1\|} \langle f_1, f_2 \rangle = 0$$

$\tilde{g}_2 = \frac{g_2}{\|g_2\|} = \frac{f_2}{\|f_2\|} \Rightarrow \boxed{\tilde{g}_2(x) = \frac{x}{\sqrt{\frac{2}{3}\pi^3}}}$

$$\|f_2\|^2 = \int_{-\pi}^{\pi} x \cdot x \, dx = \frac{2}{3}\pi^3$$

$$g_3 = f_3 - \cancel{\langle \tilde{g}_1, f_3 \rangle}^{0} \tilde{g}_1 - \langle \tilde{g}_2, f_3 \rangle \tilde{g}_2$$

$$\langle \tilde{g}_1, f_3 \rangle = \left\langle \frac{f_1}{\|f_1\|}, f_3 \right\rangle = \frac{1}{\|f_1\|} \langle f_1, f_3 \rangle = 0$$

$$\langle \tilde{g}_2, f_3 \rangle = \frac{1}{\|f_2\|} \langle f_2, f_3 \rangle = \frac{1}{\sqrt{\frac{2}{3}\pi^3}} 2\pi$$

$$\Rightarrow g_3(x) = \sin(x) - \frac{2\pi}{\sqrt{\frac{2}{3}\pi^3}} \frac{x}{\sqrt{\frac{2}{3}\pi^3}}$$

$$= \sin(x) - \frac{3}{\pi^2} x$$

$\{\tilde{g}_1, \tilde{g}_2, g_3\}$ er en <u>ortogonal</u> basis

← ortonormal ville krevd $\tilde{g}_3$

(Det er også $\{g_1, g_2, g_3\}$)