Oppgave 8 Finn taylorrekken til

$$g(x) = \int_0^{x^2} f(t)\,dt$$

om $x = 0$, der

$$f(x) = 4x + 8x^3 + 12x^5 + \cdots = \sum_{n=0}^{\infty} 4(n+1)x^{2n+1}, \qquad -1 < x < 1.$$

$$g(x) = \int_0^{x^2} f(t)\,dt = \int_0^{x^2} \sum_{n=0}^{\infty} 4(n+1)t^{2n+1}\,dt$$

$$= \sum_{n=0}^{\infty} \int_0^{x^2} 4(n+1)t^{2n+1}\,dt$$

leddvis integrasjon (gyldig for $-1<x<1$, for da er $x^2<1$)

$$= \sum_{n=0}^{\infty} 4(n+1)\left[\frac{1}{2n+2}t^{2n+2}\right]_0^{x^2}$$

$$= \sum_{n=0}^{\infty} 2x^{4n+4} = 2(x^4 + x^8 + x^{12} + \dots)$$

Dette er taylorrekka til g i punktet 0 siden taylorrekker er unike.

(se Teorem 1 Kap. 10.5)

Oppgave 5

a) Avgjør om rekkene konvergerer betinget, konvergerer absolutt eller divergerer

$$(i) \quad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n+\sqrt{n}} \qquad\qquad (ii) \quad \sum_{n=1}^{\infty}\frac{(2n)!}{n!\,(2n)^n}$$

b) Bestem konvergensintervallet for rekken

$$\sum_{n=0}^{\infty}\left(-\frac{1}{4}\right)^n\frac{(x+4)^n}{2n+1}$$

(a) (i) er alternerende, $a_n = (-1)^{n+1}\frac{1}{n+\sqrt{n}}$

$$|a_{n+1}| = \frac{1}{n+1+\sqrt{n+1}} \leq \frac{1}{n+\sqrt{n}} = |a_n|$$

$n+1 \geq n$ & $\sqrt{n+1} \geq n$

og $|a_n| \longrightarrow 0$ når $n \to \infty$, så (i) konvergerer (ved alternerende rekke test).

• absolutt konvergens?

vet: $\sum_{n=1}^{\infty} \frac{1}{n}$ divergerer

og $\frac{|a_n|}{\left(\frac{1}{n}\right)} = \frac{n}{n+\sqrt{n}} = \frac{1}{1+\frac{1}{\sqrt{n}}} \longrightarrow 1 > 0$

så $\sum_{n=1}^{\infty} |a_n|$ divergerer ved grense-sammenligningstesten

Dermed: (i) konvergerer betinget.

(ii) $a_n = \dfrac{(2n)!}{n!\,(2n)^n}$

Prøver forholdstesten:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+2)!}{(n+1)!\,(2n+2)^{n+1}} \cdot \frac{n!\,(2n)^n}{(2n)!}$$

$$= \cancel{(2n+2)}(2n+1)\,\frac{1}{n+1} \cdot \frac{1}{\cancel{(2n+2)}} \cdot \left(\frac{2n}{2n+2}\right)^n$$

$$= \frac{2n+1}{n+1}\left(\frac{1}{1+\frac{1}{n}}\right)^n$$

$$= \frac{2+\frac{1}{n}}{1+\frac{1}{n}} \cdot \frac{1}{\left(1+\frac{1}{n}\right)^n} \xrightarrow{n\to\infty} \frac{2}{e} < 1$$

$e > 2$

så (ii) konvergerer absolutt.

(b) $\sum_{n=0}^{\infty} \left(-\frac{1}{4}\right)^n \frac{(x+4)^n}{2n+1}$

Forholdstest:

$$\left|\frac{a_{n+1}(x)}{a_n(x)}\right| = \frac{1}{4}|x+4| \frac{2n+1}{2n+3} \longrightarrow \frac{1}{4}|x+4|$$

- konvergerer absolutt hvis

$$\frac{1}{4}|x+4| < 1 \iff |x+4| < 4$$

$$\iff \begin{cases} x + \cancel{4} < \cancel{4} \\ \text{og } -x-4 < 4 \end{cases}$$

$$\iff -8 < x < 0$$

- divergerer hvis

$$\frac{1}{4}|x+4| > 1 \iff \dots \iff \begin{cases} x < -8 \\ \text{eller } x > 0 \end{cases}$$

Endepunkter?

- $\underline{x = -8}$: $a_n = \frac{1}{2n+1}$

Grensesammenligningstesten:

$$\frac{a_n}{\left(\frac{1}{n}\right)} = \frac{n}{2n+1} \longrightarrow \frac{1}{2} > 0$$

$\Rightarrow \sum \frac{1}{2n+1}$ div. siden $\sum \frac{1}{n}$ div.

- $\underline{x = 0:}$ $a_n = (-1)^n \frac{1}{2n+1}$

$\rightarrow$ konvergerer (betinget) ved alternerende rekke test

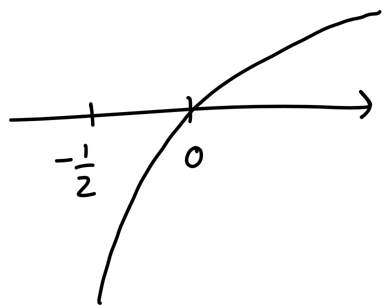
Konvergensintervallet er $(-8, 0]$

Oppgave 6 Hva er konvergensradien til Taylor-rekken til $f(x) = \ln(1+2x)$ rundt 0? Svaret skal begrunnes.

LF bruker at $f(x) = \int_0^x \frac{2}{1+2t}\,dt$

$f(x) = \ln(1+2x)$ (definert for $x > -\frac{1}{2}$)

$f'(x) = \frac{1}{1+2x} \cdot 2$



$f''(x) = (-1)\frac{1}{(1+2x)^2} 2^2$

$f'''(x) = (-1)(-2)\frac{1}{(1+2x)^3} 2^3$

gjetter: $f^{(n)}(x) = (-1)^{n-1}(n-1)!\,\frac{2^n}{(1+2x)^n}$

- stemmer for $n = 1, 2, 3$
- får $f^{(n+1)}(x) = (-1)^n\, n!\, \frac{2^{n+1}}{(1+2x)^{n+1}}$ ved å derivere gjetningen
- stemmer for alle $n \geq 1$ ved induksjon

$\Longrightarrow$ Taylor:

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0)\, x^n = \cancel{f(0)}^{\,0} + \sum_{n=1}^{\infty} \underbrace{(-1)^{n+1} \frac{2^n}{n}}_{a_n(x)} x^n$$

Forholdstesten:

$$\left| \frac{a_{n+1}(x)}{a_n(x)} \right| = \left| 2 \frac{n}{n+1} x \right| \longrightarrow |2x|$$

konvergens: $\quad |2x| < 1 \iff |x| < \frac{1}{2}$

divergens: $\quad |2x| > 1 \iff |x| > \frac{1}{2}$

$\Longrightarrow$ Konvergensradien er $\frac{1}{2}$.