SUMMER SCHOOL EXERCISES

(1) Let $\mathcal{A}$ be an abelian category and $0 \to A \to B \to C \to 0$ a short exact sequence in $\mathcal{A}$. Show that $A \to B \to C \xrightarrow{(1)} A$ is a distinguished triangle in $D(\mathcal{A})$.
(2) Describe the derived category of a field.
(3) Understanding the octahedral axiom. Let $\mathcal{T}$ be a triangulated category. Convince yourself that if we assume TR1-TR3 then TR4 is equivalent to the following: given morphisms

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

and triangles

$$X \xrightarrow{f} Y \to Z' \xrightarrow{(1)} X$$
$$X \xrightarrow{gf} Z \to Y' \xrightarrow{(1)} X$$
$$Y \xrightarrow{g} Z \to X' \xrightarrow{(1)} Y$$

we can complete this to a commutative diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \xrightarrow{(1)} & X \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle g} & & \downarrow & & \downarrow \\
X & \xrightarrow{gf} & Z & \longrightarrow & Y' & \xrightarrow{(1)} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & X' & \xrightarrow{\mathrm{id}} & X' & \xrightarrow{(1)} & 0 \\
\downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (1)} & & \downarrow{\scriptstyle (1)} \\
X & \xrightarrow{f} & Y & \longrightarrow & Z' & \xrightarrow{(1)} & X
\end{array}$$

(4) Let $\mathcal{T}$ be a triangulated category and consider a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \to hU$ with $gf = 0$, $hg = 0$. Show that the set $\langle h, g, f \rangle$ is a coset of $h \operatorname{Hom}(X, Z)_{-1} + \operatorname{Hom}(Y, U)_{-1} f$ inside $\operatorname{Hom}(X, U)_{-1}$.
(5) Let k be a field. Compute $HH^n(k[x_1, \dots, x_n])$ and $HH^n(k(x_1, \dots, x_n))$.
(6) Let A be a k-algebra and $\eta : A \otimes A \to M$ a bilinear function. Then η defines a multiplication on $E = A \oplus M$ via

$$(a_1, m_1) \cdot (a_2, m_2) = (a_1 a_2, a_1 m_2 + m_1 a_2 + \eta(a_1, a_2)).$$

Show that this multiplication is associative if and only if η is a Hochschild cocycle i.e. $d_{\mathrm{Hoch}}(\eta) = 0$.
(7) Now let A be a k-algebra and $\eta : A \otimes \ldots \otimes A \to M$ a n-multilinear function. Show that the A_η we defined in class is an A_∞ algebra if and only if $d_{\mathrm{Hoch}}(\eta) = 0$.

(8) Keeping in mind that an A_∞-category is just an A_∞-algebra with several objects, write down the definition of an A_∞-module over an A_∞-category.

(9) Let $\mathcal{A}$ be a DG category, and let f be a closed morphism in $\mathrm{Tw}(\mathcal{A})$, $f \in \mathrm{Tw}(\mathcal{A})((M, \delta_M), (N, \delta_N))$. Then show that $\mathrm{Cone}(f) \in \mathrm{Tw}(\mathcal{A})$.

(10) Show that if $\mathcal{A}$ is an A_n-category and $m \leq n$ then $\mathrm{Tw}_{\leq m}(\mathcal{A})$ is an $A_{\lfloor \frac{n-m}{m+1} \rfloor}$-category.