The Ornstein-Uhlenbeck process

Parametrization

The Ornstein-Uhlenbeck process is defined with (mean zero), as the SDE

\[ dx_t = -\phi x_t + \sigma dW_t \]

where \( \phi > 0 \) and \( W_t \) is the Wiener process. This is the continuous time analogue to the discrete time AR(1) model.

The process has a Markov property. Let \( x = (x_1, x_2, \ldots, x_n) \) be value of the process at increasing time-points \( t = (t_1, t_2, \ldots, t_n) \), then the conditional distribution

\[ x_i \mid x_1, \ldots, x_{i-1}, \quad i = 2, \ldots, n, \]

is Gaussian with mean

\[ x_{i-1} \exp(-\phi \delta_i) \]

and precision

\[ \tau (1 - \exp(-2\phi \delta_i))^{-1} \]

where

\[ \delta_i = t_i - t_{i-1}, \quad i = 2, \ldots, n \]

and

\[ \tau = 2\phi/\sigma^2. \]

The marginal distribution for \( x_1 \) is taken to be the stationary distribution, which is a zero mean Gaussian with precision \( \tau \).

Hyperparameters

The precision parameter \( \tau \) is represented as

\[ \theta_1 = \log(\tau) \]

where \( \tau \) is the *marginal* precision for the Ornstein-Uhlenbeck process given above.

The parameter \( \phi \) is represented as

\[ \theta_2 = \log(\phi) \]

and the prior is defined on \( \theta = (\theta_1, \theta_2) \).

Specification

The Ornstein-Uhlenbeck model is specified inside the \( f() \) function as

\[ f(<\text{whatever}>, \text{model}="ou", \text{values}=<\text{values}>, \text{hyper} = <\text{hyper}>) \]

The optional argument \( \text{values} \) gives the time-points where the process is defined/observed on (default is \( \text{unique(sort(<\text{whatever}>))} \)).
Hyperparameter specification and default values

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hyper

theta1
  hyperid 16001
  name  log precision
  short.name prec
  prior  loggamma
  param 1 5e-05
  initial 4
  fixed FALSE
  to.theta function(x) log(x)
  from.theta function(x) exp(x)

theta2
  hyperid 16002
  name  log phi
  short.name phi
  prior  normal
  param 0 0.2
  initial -1
  fixed FALSE
  to.theta function(x) log(x)
  from.theta function(x) exp(x)

constr FALSE

nrow.ncol FALSE

augmented FALSE

aug.factor 1

aug.constr

n.div.by

n.required FALSE

set.default.values FALSE

pdf ou

Example

## simulate an OU-process and estimate its parameters back.
phi = -log(0.95)
sigma = 1
marg.prec = 2*phi/sigma^2
n = 1000
locations = cumsum(sample(c(1, 2, 5, 20), n, replace=TRUE))
```r
## do it sequentially and slow (for clarity)
x = numeric(n)
x[1] = rnorm(1, mean=0, sd = sqrt(1/marg.prec))
for(i in 2:n) {
    delta = locations[i] - locations[i-1]
    x[i] = x[i-1] * exp(-phi * delta) +
         rnorm(1, mean=0, sd = sqrt(1/marg.prec * (1-exp(-2*phi*delta)))))
}

## observe it with a little noise
y = 1 + x + rnorm(n, sd= 0.01)
plot(locations, x, type="l")

formula = y ~ 1 + f(locations, model="ou", values=locations)
r = inla(formula, data = data.frame(y, locations))
summary(r)

Notes
The Ornstein-Uhlenbeck process is the continuous-time analogue to the discrete AR(1) model (for positive lag-one correlation only), but they are parameterised slightly different.
```