Quasi-generalised KPZ equation

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10th May 2019, Trondheim

The equation

It is given by

$$\partial_t u - a(u)\partial_x^2 u = f(u)\left(\partial_x u\right)^2 + g(u)\xi$$

where ξ is a space-time white noise and a smooth and non degenerate.

- The case $a = 1$ is well-understood.
- The main difficulty is the treatment of $a(u)\partial_x^2 u$ and the need of some renormalisation.

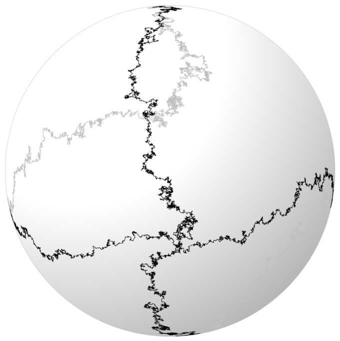
Writing the equation in the divergence form we get:

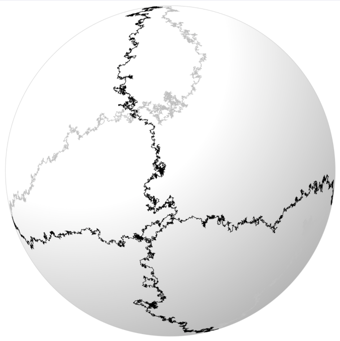
$$\partial_t u - \partial_x(a(u)\partial_x u) = \left(f(u) - a'(u)\right)\left(\partial_x u\right)^2 + g(u)\xi.$$

Generalised KPZ equation

Geometric stochastic heat equations

$$\partial_t u^\alpha = \partial_x^2 u^\alpha + \Gamma^\alpha_{\beta\gamma}(u)\,\partial_x u^\beta \partial_x u^\gamma + \sigma_i^\alpha(u)\xi_i,$$





A local perturbative expansion

$$\partial_t u = \partial_x^2 u + f(u)\,(\partial_x u)^2 + g(u)\xi, \quad \partial_t v = \partial_x^2 v + \xi.$$

The solution u is described by $u = v + w$ but through a Taylor type expansion:

$$u = \sum_{\tau \in \mathcal{T}} c_{\tau,x} u_{\tau,x} + R_{\mathcal{T},x}$$

where

- $\mathcal{T}$ is a finite set of decorated trees
- $u_{\tau,x}$ are recentered (Gaussian) stochastic processes
- $c_{\tau,x}$ are coefficients of the Taylor expansion
- $R_{\mathcal{T},x}$ is a remainder nicer than the $u_{\tau,x}$.

A local perturbative expansion

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The … a Taylor type

exp…

Recentered stochastic processes

$$u_{\text{(tree)},x} = G * \xi - (G * \xi)(x), \qquad u_{X^k,x} = (\cdot - x)^k$$

$$u_{\text{(tree)},x} = G * (\partial_x G * \xi)^2 - (G * (\partial_x G * \xi)^2)(x)$$

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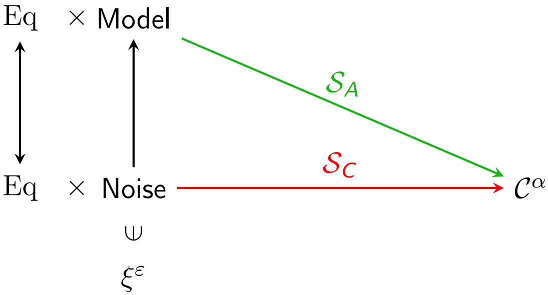
Regularity Structures



$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + (\partial_x u_\varepsilon)^2 + \xi^\varepsilon$$

Regularity Structures

$\mathrm{Eq} \times \mathsf{Model}$

$\mathcal{S}_A$

$\mathcal{S}_C$

$\mathcal{C}^\alpha$

$\mathrm{Eq} \times \mathsf{Noise}$

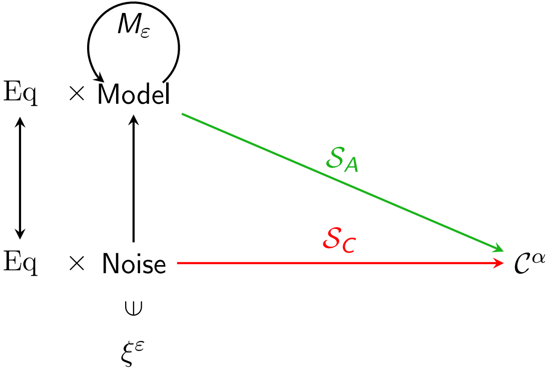
$\Cup$

ξ^ε

$$\textbf{Model} = \left(u_{\circ,x},\, u_{\vee,x},\, \ldots\right)$$

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + (\partial_x u_\varepsilon)^2 + \xi^\varepsilon$$

Regularity Structures



$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + (\partial_x u_\varepsilon)^2 + \xi^\varepsilon$$

Regularity Structures

M_ε

$\mathrm{Eq} \quad \times \text{ Model}$

Renormalised Model $= (u_{\circ,x}, u_{\vee,x} - C_\varepsilon(\vee), \dots)$

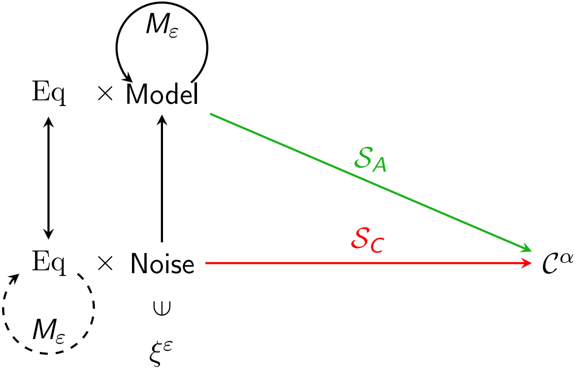
$\mathrm{Eq} \quad \times \text{ Noise}$

$\uplus$

ξ^ε

$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + (\partial_x u_\varepsilon)^2 + \xi^\varepsilon$$

Regularity Structures



$$\partial_t u_\varepsilon = \partial_x^2 u_\varepsilon + (\partial_x u_\varepsilon)^2 + \xi^\varepsilon + C_\varepsilon(\vee)$$

Main results

A series of papers defines a large class of equations which can be solved by regularity structures:

- Martin Hairer (2014), "A theory of regularity structures", Invent. Math.
- **Yvain Bruned**, Martin Hairer, Lorenzo Zambotti (2019), "Algebraic renormalisation of regularity structures", Invent. Math.
- Ajay Chandra, Martin Hairer (2016), "An analytic BPHZ theorem for regularity structures", arxiv.
- **Yvain Bruned**, Ajay Chandra, Ilya Chevyrev, Martin Hairer (2017), "Renormalising SPDEs in regularity structures", to appear in JEMS.

The geometric KPZ is treated in

- **Yvain Bruned**, Franck Gabriel, Martin Hairer, Lorenzo Zambotti (2019), "Geometric stochastic heat equations", arxiv.

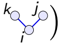
Renormalised equation

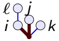
Theorem (B., Chandra, Chevyrev, Hairer 2017)

There exist some constants $(c^{\tau}_{\varrho,\varepsilon})_{\tau\in\mathcal{T}}$ such that the renormalised equation for u_ε is given by

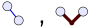
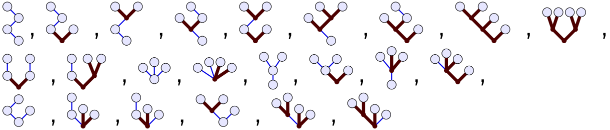
$$\partial_t u^\alpha_\varepsilon = \partial_x^2 u^\alpha_\varepsilon + \Gamma^\alpha_{\beta\gamma}(u_\varepsilon)\,\partial_x u^\beta_\varepsilon \partial_x u^\gamma_\varepsilon + \sigma^\alpha_i(u_\varepsilon)\xi^\varepsilon_i + \sum_{\tau\in\mathcal{T}} c^\tau_{\varrho,\varepsilon}\left(\Upsilon^\alpha_{\Gamma,\sigma}\tau\right)(u_\varepsilon,\partial_x u_\varepsilon)\,.$$

Some examples of coefficients:

$\Upsilon^\alpha_{\Gamma,\sigma}\Big($  $\Big) = \sigma^\beta_j\partial_\beta\sigma^\alpha_i,$ $\qquad$ $\Upsilon^\alpha_{\Gamma,\sigma}\Big($  $\Big) = \sigma^\gamma_k\sigma^\beta_j\partial_\beta\partial_\gamma\sigma^\alpha_i,$

$\Upsilon^\alpha_{\Gamma,\sigma}\Big($  $\Big) = 2\sigma^\eta_k\partial_\eta\Gamma^\alpha_{\beta\gamma}\sigma^\beta_j\sigma^\mu_\ell\partial_\mu\sigma^\gamma_i.$

Decorated trees $\mathcal{T}$

	$\mathcal{T}$
Two Noises	
Four Noises	

Implicit formulation

Write K^c the kernel given by

$$(\partial_t - c\partial_x^2)K^c = \delta_0$$

then one sets

$$I_\alpha(b,f)(z) = (\partial_\alpha K^c * f)(z)|_{c=b(z)} = \int \partial_\alpha K^{b(z)}(z-\bar{z})f(\bar{z})d\bar{z}.$$

The main idea is to enforce a mild formulation and try to find a $\hat{F}$ such that

$$u = I(a(u),\hat{F})$$

where u is the solution of the quasi-generalised KPZ.

Implicit Equation

This equation

$$\partial_t u - a(u)\partial_x^2 u = f(u)\,(\partial_x u)^2 + g(u)\xi$$

is equivalent to

$$u = I(a(u), \hat{F})$$
$$\hat{F} = \hat{f}(u)(\partial_x u)^2 + \hat{g}(u)\xi + 2(aa')(u)(\partial_x u)v_{cx} + a'(u)(\partial_x u)v_x,$$

where $v_c = I_c(a(u), \hat{F})$ and $\hat{f}, \hat{g}$ are explicit coefficients depending on v_c and v_{cc}.

Renormalised equation

Theorem (Gerencser, Hairer 2019)

There exist some constants $(c^{\tau}_{\varrho,\varepsilon})_{\tau\in\mathcal{T}}$ *such that the renormalised equation for* u_ε *is given by*

$$\partial_t u_\varepsilon - a(u_\varepsilon)\partial_x^2 u_\varepsilon = f(u_\varepsilon)\,(\partial_x u_\varepsilon)^2 + g(u_\varepsilon)\xi + \sum_{\tau\in\mathcal{T}_c} c^{\tau}_{\varrho,\varepsilon}\frac{\Upsilon_{\hat{F}}(\tau)}{q}$$

where $q = (1 - a'(u_\varepsilon)v^\varepsilon_c)$, $\mathcal{T}_c$ *is a set of planar trees and* $\Upsilon_{\hat{F}}$ *depends on* $v^\varepsilon_c, v^\varepsilon_{cc}, \ldots$

Conjecture/Main result

Theorem

There exist some constants $(c^{\tau}_{\varrho,\varepsilon})_{\tau\in\mathcal{T}}$ *such that the renormalised equation for* u_ε *is given by*

$$\partial_t u_\varepsilon - a(u_\varepsilon)\partial_x^2 u_\varepsilon = f(u_\varepsilon)\left(\partial_x u_\varepsilon\right)^2 + g(u_\varepsilon)\xi + \sum_{\tau\in\mathcal{T}} c^{\tau}_{\varrho,\varepsilon}\Upsilon_F(\tau)$$

where $F = (f(u) - a'(u))\left(\partial_x u\right)^2 + g(u)\xi.$

Space $\mathcal{T}_{\mathrm{geo}}$

- Space $\mathcal{T}_{\mathrm{geo}}$: $\varphi \cdot (\Upsilon_{\Gamma,\sigma}\tau) = \Upsilon_{\varphi\cdot\Gamma,\varphi\cdot\sigma}\tau$.

Let consider

$$\mathcal{F}_\nabla = \Big\{ \nabla_{\bullet\circ},\ \nabla_\bullet\nabla_\circ\nabla_{\bullet\circ},\ \nabla_\circ\nabla_\bullet\nabla_{\bullet\circ},\ \nabla_\bullet\nabla_\bullet\nabla_{\circ\circ},\ \nabla_\bullet\nabla_{\nabla_{\bullet\circ}\circ},\ \nabla_{\nabla_{\bullet\circ}}\nabla_{\bullet\circ},$$
$$\nabla_{\nabla_{\circ\bullet}}\nabla_{\bullet\circ},\ \nabla_{\nabla_{\bullet\bullet}}\nabla_{\circ\circ},\ \nabla_{\nabla_\bullet\nabla_{\bullet\circ}\circ},\ \nabla_{\nabla_\bullet\nabla_{\circ\circ}\bullet},\ \nabla_{\nabla_{\nabla_{\bullet\circ}\circ}\bullet},$$
$$\nabla_{\nabla_{\nabla_{\bullet\circ}\bullet}\circ},\ \nabla_\circ\nabla_{\nabla_{\bullet\circ}\bullet},\ \nabla_{\nabla_\circ\nabla_{\bullet\circ}\bullet},\ \nabla_{\nabla_{\nabla_{\circ\circ}\bullet}\bullet},\ \nabla_\bullet\nabla_{\nabla_{\circ\circ}\bullet} \Big\}.$$

For example, $\nabla_{\bullet\circ} =$  $+ \frac{1}{2}$  and $\Upsilon_{\Gamma,\sigma}(\nabla_{\bullet\circ}) = \nabla_{\sigma_i\sigma_j}$

Proposition

One has $\mathcal{T}_{\mathrm{geo}} = \langle \mathcal{F}_\nabla \rangle$. In particular, the space of geometric elements has dimension 14.

Geometric KPZ

Under some geometric conditions, we get

Theorem (B., Gabriel, Hairer, Zambotti 2019)

Let ϱ a mollifier, we set $\xi_i^\varepsilon = \varrho_\varepsilon * \xi_i$. Then there exist a unique $\tau \in \mathcal{T}_{\mathrm{geo}}$ such that the solution u_ε of

$$\partial_t u_\varepsilon^\alpha = \partial_x^2 u_\varepsilon^\alpha + \Gamma_{\beta\gamma}^\alpha(u_\varepsilon)\,\partial_x u_\varepsilon^\beta \partial_x u_\varepsilon^\gamma + \sigma_i^\alpha(u_\varepsilon)\xi_i^\varepsilon + (\Upsilon_{\Gamma,\sigma}^\alpha \tau)(u_\varepsilon)\,.$$

converges in probability to a Markov process u when ε goes to zero. The limit does not depend on the mollifier ϱ. Its law satisfies Itô's isometry and behaves well under the change of coordinates.