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Sub-Riemannian geometry and numerics for SDEs

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SDE numerics

The CMT (Cruzeiro-Malliavin-Thalmeier) scheme is a numerical method for simulation of diffusion paths with strong order of convergence 1, that avoids simulation of Lévy area. It is based on the Milstein scheme for SDEs with driving vector fields $A_0, A_1, \ldots, A_n$

$$X_t = X_0 + A_i(X_0)\Delta B_t^i + A_0(X_0)\Delta t + A_i \circ A_j(X_0) \int_0^t B_s^i dB_s^j.$$

Central is the observation that we can avoid simulating the awkward integrals when the vector fields A_i commute as then

$$A_i \circ A_j(X_0) \int_0^t B_s^i dB_s^j + A_j \circ A_i(X_0) \int_0^t B_s^j dB_s^i = A_i \circ A_j(X_0) \left(\Delta B_t^i \Delta B_t^j - h\delta_j^i\right)$$

Diffusions and SDEs

For any Brownian motion B_t, the solution $Y(s,x,t)$ of

$$Y_t = \int_s^t \sigma(t, Y_t)dB_t + \int_s^t b(Y_t)dt, \quad Y_s = x$$

has a density $P(s,x,t)$ that solves the Kolmogorov equation

$$\frac{\partial}{\partial t}P(s,x,t) = A_t^T P(s,x,t), \quad \lim_{t \downarrow s} P(s,x,t) = \delta_x$$

where A^T is the adjoint of the operator

$$A_t = \frac{1}{2}(\sigma\sigma^*)^{ij}(t,x)\frac{\partial^2}{\partial x_i \partial x_j} + b^i(t,x)\frac{\partial}{\partial x_i}$$

Such a process Y is called an A-diffusion.

Diffusions on Riemannian manifolds

If $A_0, A_1, \ldots, A_d$ are vector fields on a manifold M, then the solution of

$$dX_t^i = A_i(X_t) \circ dB_t^i + A_0(X_t)dt$$

is an A-diffusion, where the $\circ$ indicates a Stratonovich differential, and A is the sum of squares operator $Af = \frac{1}{2}A_i(A_i)f + A_0 f$.

The restriction to sum of squares operators is unfortunate as it does not include for instance Riemannian Laplacians:

$$\Delta_M f = g^{ij} \frac{\partial^2}{\partial x^i \partial x^j} - g^{ij} \Gamma_{ij}^k \frac{\partial f}{\partial x^k},$$

where g^{ij} is the cometric and Γ_{ij}^k are the components of the Levi-Civita connection.

Riemannian Brownian motion (with drift)

Recall that for any $\xi \in \mathbb{R}^n$, the basic horizontal field $B(\xi)$ on a principal bundle with respect to a connection with components Γ^q_{kl} in a local coordinate system (x^i, X^i_j) is given by

$$B(\xi) = X^i_j \xi^j \frac{\partial}{\partial x^i} - \Gamma^q_{kl} X^l_p X^k_j \xi^j \frac{\partial}{\partial X^q_p},$$

Let $A = \frac{1}{2}\Delta_M + b$ be the sum of a Riemannian Laplacian and a first order term b. Let $\tilde{L}_i = B(e_i)$ be basic horizontal fields on the orthonormal frame bundle $O(M)$, and $\tilde{L}_0$ the horizontal lift of the vector field b. Suppose r_t solves the following SDE on $O(M)$:

$$dr_t = \tilde{L}_0(r_t)dt + \tilde{L}_i(r_t)dB^i_t,$$

Then the projection $x_t = \pi(r_t)$ is a Markov process, an A-diffusion, and its law depends only on the initial value of x_0 (and not r_0)

A property of horizontal vector fields

Let $L(M)$ be a frame bundle, and θ be the canonical $\mathbb{R}^n$-valued 1-form that reads a tangent vector in the base direction in the given frame. Recall that the torsion form Θ of a connection ω is the exterior covariant derivative of θ. Recall that these are related by the structure equation

$$d\theta = -\omega \wedge \theta + \Theta,$$

an important consequence of which is the following: suppose X, Y are horizontal vector fields with respect to ω on $L(M)$. Then

$$\theta\big([X, Y]\big) = -2\Theta(X, Y).$$

As a consequence, if X, Y are horizontal vectors fields on $O(M)$ with respect to the Levi-Civita connection, then $[X, Y]$ is vertical.

Geometrizing an SDE

Essentially, if we employ the Milstein scheme on the associated equation on the frame bundle, the weak commutativity discussed earlier is sufficient. To geometrize the equation, we let $\sigma_{ij} = A^i_j$, and define a Riemannian metric through the cometric $g^{ij} = \sigma^T \sigma$ (we assume ellipticity so this can be inverted to give a metric). Then the solution of the SDE is an A-diffusion where

$$A = \frac{1}{2}\Delta_M + \left(A_0 + g^{ij}\Gamma^k_{ij}\frac{\partial}{\partial x^k}\right)$$

Let $K^i_{jk}(x)$ be the structure constants of the Lie algebra generated by the $A_1, \ldots, A_n$. Then the components of the Levi-Civita connection for the metric g are given by

$$\Gamma^l_{pq} = \frac{1}{2}(K^l_{pq} + K^p_{lq} + K^q_{lp}).$$

Cruzeiro-Malliavin-Thalmeier scheme

Let $r_t = (x_t, e_t)$ be the solution of the SDE on the orthogonal frame bundle

$$dr_t = \tilde{L}_0(r_t)dt + \tilde{L}_i(r_t)dB^i_t.$$

The projection of the Milstein scheme for $(\hat{x}_t, \hat{e}_t)$ is

$$\begin{aligned}\hat{x}_{t+h} &= \hat{x}_t + A_0(\hat{x}_t)\Delta t + \hat{e}^l_k(t)A_l(\hat{x}_t)\Delta W^k_t \\ &+\frac{1}{2}\hat{e}(t)^l_k\hat{e}^{l'}_j\left(A_l \circ A_{l'} - \Gamma^i_{l,l'}A_i\right)(\hat{x}_t))\left(\Delta B^k_t \Delta B^j_t - h\delta^k_j\right)\end{aligned}$$

The CMT scheme is equal in law to the result of replacing all the $\hat{e}^i_j$ above with e^i_j; equivalently

$$\begin{aligned}X_t &= X_0 + A_0(X_0)\Delta t + A_i(X_0)\Delta B^i_t + A_i \circ A_j(X_0)\left(\Delta B^i_t \Delta B^j_t - h\delta^i_j\right) \\ &+A_i(X_0)K^i_{jk}(X_0)\left(\Delta B^k_t \Delta B^j_t - h\delta^k_j\right)\end{aligned}$$

Hypoelliptic diffusion

We now consider SDEs on M of the form

$$dX = A_i(X_t) \circ dB_t^i + A_0(X_t)dt$$

where $i = 1, \dots, r$, and $r < n$ where n is the dimension of M. This is hypoelliptic if $A_1, \dots, A_r$ obey a Hörmander condition (their Lie brackets (almost) everywhere span the tangent space of M), and hence admits a smooth density.

Now we can define a cometric $g^{ij} = \sigma^T \sigma$ where $\sigma_{ij} = A_j^i$ as before, but this will not be invertible. Hence we obtain a sub-Riemannian geometry. We would like to associate solutions of this equation to sub-Riemannian Laplacians and sub-Riemannian Brownian motions.

Horizontal frame bundles

We obtain a metric on the horizontal bundle $\mathcal{H}$ which is of constant dimension r; it is therefore possible to define an $O(r)$-bundle of orthogonal horizontal frames. A connection on this bundle allows the definition of basic horizontal horizontal vector fields $\tilde{L}_i$ as before. Then sub-Riemannian Brownian motions can be constructed as projections of solutions to

$$dr_t = \tilde{L}_0(r_t)dt + \tilde{L}_i(r_t)dB^i_t.$$

(Open) problem: torsion? Need a solder form (to replace the canonical solder form θ from before), here this should be equivalent to specifying a projection of TM onto $\mathcal{H}$. Would there then exist a torsion-free connection? If so, how can we guarantee that brackets of vector fields would be vertical vertical, and not merely horizontal vertical?

Elliptic diffusions on homogeneous spaces

A natural problem to explore is the following: suppose we consider equations

$$dX = A_i(X_t) \circ dB_t^i + A_0(X_t)dt$$

where M is a Riemannian homogeneous space - in this case it is possible to adapt Lie group integration techniques to the stochastic setting (see Malham,Wiese).

If we are interested in strong simulations of weak solutions (for instance, Multilevel Monte Carlo simulations), we can adopt the same procedure and lift the vector fields A_i to the orthonormal frame bundle $O(M)$. Can we find in this manner a strong order 1 scheme which does not require simulations of Lévy areas?

Thank you for listening!