Permutation Invariant Matrix Valued Levy Processes and Partition Algebra

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Permutation-invariant Matrix valued Levy processes

$$M_0 = 0\,;$$

$$\mathcal{M}_N(\mathbb{C}) \otimes L^{\infty^-} \ni (M_t)_{t \geq 0}$$

$$\underbrace{M_{t+s} - M_t}_{\substack{\wr\wr \\ M_s}} \perp (M_u)_{u \leq t}$$

For any $S \in \mathfrak{S}(N)$, $S M_t S^{-1} \underset{law}{\simeq} M_t$

Ex: • <u>Hermitian Brownian Motion</u>

$$H_t = \sum_i B_t^i \, h_i$$

iid B. Motion

orthon. basis $\langle X, Y \rangle = N\,\mathrm{Tr}(XY)$

• <u>Unitary Brownian Motion</u> $\approx e^{iH_t}$

$$dU_t = i\,U_t\, dH_t - \frac{1}{2} U_t\, dt$$

• <u>Simple Random Walk on $\mathfrak{S}_N$</u>

S_t on Cayley Graph

Permutation-invariant Matrix valued Levy processes

$$M_0 = Id$$

$$\underbrace{M_{t+s} M_t^{-1}}_{\underset{M_s}{\simeq}} \perp (M_u)_{u \le t}$$

$$\mathcal{M}_N(\mathbb{C}) \otimes L^{\infty-} \ni (M_t)_{t \geq 0}$$

For any $S \in \mathfrak{S}(N)$, $S M_t S^{-1} \underset{law}{\simeq} M_t$

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• Simple Random Walk on $\mathfrak{S}_N$

S_t on Cayley Graph

Question: As $N \to \infty$, does the empirical eigenvalues distribution converge?

$$\underbrace{\text{empirical eigenvalues distribution}}_{\mu_M = \frac{1}{N}\sum_i \delta_{\lambda_i}}$$

eigenvalue

• Hermitian BM (Wigner, ...)

$$\mu_{H_t} \xrightarrow{N \to \infty} \text{Semi circle law} = \mu_{\mathbb{H}}^t$$

$$\int_{\mathbb{R}} x^k \mu_{\mathbb{H}}^t(dx) = \delta_{k=2n}\, t^n \frac{1}{n+1}\binom{2n}{n}$$

• Unitary BM (Xu, Biane, Lévy, ...)

$$\mu_{U_t} \xrightarrow{N \to \infty} \mu_{\mathbb{U}}^t$$

$$\int_{\mathbb{U}} z^k \mu_{\mathbb{U}}^t(dz) = e^{-\frac{nt}{2}} \sum_{k=0}^{n-1} \frac{(-nt)^k}{n\,k!}\binom{n}{k+1}$$

• Simple R.W on $\mathfrak{S}_N$ (... Berestiky, Kozma, ...)

$$\mu_{S_t} \xrightarrow{N \to \infty} \mu_{\mathfrak{S}}^t$$

Formula for moments

Observations:

① Distinct ways to prove these results:

> Wick Formula, Itô Formula, Rep. Theory

Hermitian BM — Unitary B.M. — RW on $\mathfrak{S}_N$

② Limit is deterministic

③ Explicit computations

④ Phase transition (Unitary, RW on $\mathfrak{S}_N$)

$t_c = 4$ $\qquad$ $t_c = \frac{1}{2}$

Questions: Can we prove all this in a unified framework?

Can we extend the result about the simple random walk on $\mathfrak{S}_N$ to a more general setting?

Observations :

① Distinct ways to prove these results :

> Wick Formula, Itô Formula, Rep. Theory

Levy process $\Rightarrow \frac{d}{dt}\Big|_0 \mathbb{E}[f(M_t)]$ exists characterizes (G. Cébron)

② Limit is deterministic

Not always : need a criterion for that.

③ Explicit computations

In our setting : yes

④ Phase transition (Unitary, RW on $\mathfrak{S}_N$)

Will be valid.

Random walks on $\mathfrak{S}_N$

Setting: For any N, we consider

- A permutation s_N and $\lambda_N = \{\sigma s_N \sigma^{-1}; \sigma \in \mathfrak{S}_N\}$
- For any i, $\lambda_N(i)$ = Nb of integers in a cycle of s_N of size i.

Ex: $\lambda_N(1) = \#$ fixed points of s_N.

Hypothesis	
	$\dfrac{N - \lambda_N(1)}{N} \xrightarrow[N\to\infty]{} \alpha$ $\dfrac{\lambda_N(i)}{N - \lambda_N(1)} \xrightarrow[N\to\infty]{} \lambda_i \quad i > 1$

- Random Walk:

$$S_0 = Id \xrightarrow{\mathcal{E}\left(N/N-\lambda_N(1)\right)} S_{T_1} = \sigma S_0 \quad \sigma \sim Unif(\lambda_N) \qquad \text{Wait } \mathcal{E}(N/N$$

$$H_N(f)(\sigma_0) = \frac{N}{N-\lambda_N(1)} \frac{1}{\#\lambda_N} \sum_{\sigma \in \lambda_N} f(\sigma\sigma_0) - f(\sigma_0)$$

Theorem (G.) Under these hypotheses:

- **Convergence:** $\mu_{S_t} \xrightarrow[\text{Law}]{} \mu_t^{\lambda}$ ← possibly random

- **Deterministic limit:**

$$\mu_t^{\lambda} \text{ is determ.} \iff \lim_{N\to\infty} \frac{\lambda_N(1)}{N} = 1$$

propor of fixed pts

In the deterministic case:

- **Explicit formula for μ_t^{λ}:**

$$\mu_t^{\lambda} = \sum_{n\in\mathbb{N}^*} x_{n,t}^{\lambda}\, \lambda_{1^{1/n}} + x_{\infty,t}^{\lambda}\, \lambda_{\mathbb{U}}$$

Uniform on $\{z^n = 1\}$

$$x_{n,t}^{\lambda} = e^{-nt} \sum_{k=0}^{n-1} t^k \frac{n^{k-1}}{n!} \sum_{\substack{\sum_{j=1}^{k} i_j = n-1 \\ i_j > 0}} \prod_{j=0}^{k} \lambda(i_j+1)$$

- **Phase transition (large cycles)**

Generalization of N. Berestycki

$$t_c = \delta_{\sum \lambda_j = 1} \left(\left(\sum_j j \lambda_j \right) - 1 \right)^{-1}$$

$$\underbrace{\delta_1 \longrightarrow}_{\text{atomic}} \underbrace{\longrightarrow \text{Haar}}_{\text{Lebesgue part}}$$

Outline :

1/ Combinatorial transformation

- The polynomial observables
- Schur-Weyl-Jones duality
- Cumulants for $\mathfrak{S}_N$-inv. matrices

2/ Convergence for Levy processes

- $+$ and $\times$ of indep. matrices ($\triangle$, $\boxdot$)
- General theorem for Levy Processes
- Exemples H_t, U_t, S_t

3/ Deterministic limit ?

- Characters
- Infinitesimal characters
- Exemples

4/ General RW on $\mathfrak{S}_N$ (overview)

Part One

Combinatorial transformation of the problem

Polynomial observables:

$$\mu_{M_N} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}$$

measure on $\mathbb{R}$, $\mathbb{U}$, characterised by its <u>moments</u>

$$\int z^k \mu_{M_N}(dz) = \frac{1}{N}\sum_{i=1}^{N} \lambda_i^k$$

$$= \underbrace{\frac{1}{N}\mathrm{Tr}(M_N^k)}_{\text{Polynomial observables}}$$

$\downarrow$ expectaᵒ encoded in:

"Signature": $\mathcal{M}_N(L^{\infty^-}) \longrightarrow \bigoplus_k \mathrm{End}((\mathbb{C}^N)^{\otimes k})$

$$M_N \longmapsto \left(\mathbb{E}[M_N^{\otimes k}]\right)_k$$

(T. Lévy)

- $$\mathbb{E}\left[M^{\otimes k}\right]^{j_1,\dots,j_k}_{i_1,\dots,i_k} = \mathbb{E}\left[M^{j_1}_{i_1}\cdots M^{j_k}_{i_k}\right]$$

$\sum k_i = K$ $\qquad \in \mathfrak{S}_K$

- $$\mathbb{E}\left[\prod_{i=1}^{n}\int z^{k_i}\,\mu_{M_N}(dz)\right] = \frac{1}{N^{n}}\,\mathrm{Tr}\left[\mathbb{E}\left(M_N^{\otimes K}\right)\circ\rho(\sigma)\right]$$

nb of cycles of σ

$$\rho(\sigma)(x_1\otimes\cdots\otimes x_K) = x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(K)}$$

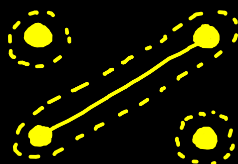
σ = n cycles of size $k_1,\dots,k_n$.

Symmetry : $\underbrace{\sigma M_N \sigma^{-1} \underset{\text{law}}{\simeq} M_N}$

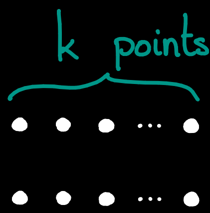
$$\boxed{\sigma^{\otimes k}\,\mathbb{E}[M_N^{\otimes k}]\,(\sigma^{-1})^{\otimes k} = \mathbb{E}[M_N^{\otimes k}]}$$

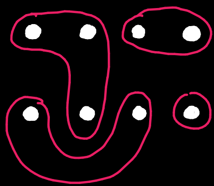
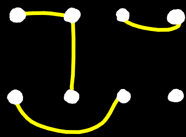
$\mathrm{End}\left[(\mathbb{C}^N)^{\otimes k}\right]^{\mathfrak{S}_k}$

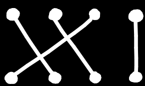
$$\sum_{\sigma} \sigma^{\otimes 2}\, E_1^2 \otimes E_3^1\, (\sigma^{-1})^{\otimes 2} \propto \sum_{\substack{i,j,\ell=1\\ \text{distinct}}}^{N} E_i^j \otimes E_\ell^i$$

$\mathcal{P}_2 \ni$ 

• Combinatorial Set:

$\mathcal{P}_k \ni$ partitions of 

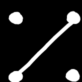
Ex:  = 

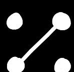
 $\in \mathfrak{S}_4$

• Seen as endomorphism on $(\mathbb{C}^N)^{\otimes k}$

$$e(p) = \sum_{\underbrace{\operatorname{Ker}\begin{pmatrix} j_1 & \cdots & j_k \\ i_1 & \cdots & i_k \end{pmatrix}}_{\in \mathcal{P}_k} \geq p} E^{j_1}_{i_1} \otimes E^{j_2}_{i_2} \cdots \otimes E^{j_k}_{i_k}$$

Coarser

Ex: $\operatorname{Ker}\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} =$ 

$e($$) = \sum_{i,j,\ell=1}^{N} E^{j}_{i} \otimes E^{i}_{\ell}$

Theorem (Schur-Weyl-Jones):

$$\boxed{\mathbb{E}[M_N^{\otimes k}] = \sum_{p\in\mathcal{P}_k} \underbrace{p^*(M_N)}_{\in\mathbb{C}}\, e(p)}$$

$$\begin{tikzcd}
\mathcal{M}_N(L^{\infty-}) \arrow[r] & \bigoplus_k \mathrm{End}\left((\mathbb{C}^N)^{\otimes k}\right)^{\mathfrak{S}_N} \\
M_N \arrow[r, mapsto] \arrow[dr, squiggly] & \left(\mathbb{E}[M_N^{\otimes k}]\right)_k \arrow[d] \\
 & \bigoplus_k \mathbb{C}[\mathcal{P}_k]^*
\end{tikzcd}$$

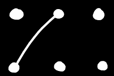
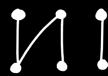
$$p \longmapsto p^*(M_N)$$

Questions:

- Generalization of the polynomial obs. ?
- Can we see the convergence of the obs. using $p^*(M_N)$?

Generalisation of $\frac{1}{N^{n}}\mathrm{Tr}[\mathbb{E}(M^{\otimes k})\,e(\sigma)]$ number of cycles

$$m_p(M_N) = \frac{1}{N^{c(p)}}\mathrm{Tr}[\mathbb{E}(M^{\otimes k})\,e(p)]$$

Ex: $c\Big($  $\Big) = \#$  $= \#$  $= 2$

Pointwise pd

m  $(M) = \frac{1}{N}\mathbb{E}(\mathrm{Tr}(M^tM))$ $\quad m$  $(M) = \frac{1}{N}\mathbb{E}(\mathrm{Tr}(M\circ M))$

$$\mathrm{Tr}[\mathbb{E}(M^{\otimes k})\,e(p)] = \sum_{\mathrm{Ker}\binom{j_1 \dots j_k}{i_1 \dots i_k} \geq p} \mathbb{E}\left(M_{i_1}^{j_1} M_{i_2}^{j_2} \cdots M_{i_k}^{j_k}\right)$$

(C. Male)

Theorem (G.) Given $M_N \in \mathcal{M}_N(L^{\infty -})^{a_N}$ the following assertions are equivalent:

① For any partition p, $m_p(M_N)$ converges $\longrightarrow m_p(M)$

② For any partition p,

$$N^{\#p - c(p)}\, p^*(M_N) \text{ converges} \longrightarrow K_p(M)$$

Remarks:

① Can be generalised to a family of matrices

② If M_N is invariant by conjugation by $U(N) \Longrightarrow \forall p \notin \sigma_k,\ p^*(M_N) = 0$.

③ (Theorem, G.) If M_N and L_N converge are independent and one is invariant by σ_N,

a/ (M_N, L_N) converge

b/ Mixed K_p vanish: ex: $K_{\cdot/\cdot}(M,L) = 0$

$\hookrightarrow$ K_p are cumulants for Male's trafic freeness

Proof: $\mathbb{E}[M \otimes L] \overset{\text{indep.}}{=} \mathbb{E}[M] \otimes \mathbb{E}[L]$

$\overset{\text{Schur Weyl. Jones}}{=} \sum c_{p_1 \cdot p_2} \underbrace{p_1 \otimes p_2}_{\neq \cdot/\cdot}$

Question: How can we relate m_p and K_p?

New order on $\mathcal{P}_k$:

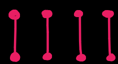
Theorem: (G.) The formula:

$$d(p,q) = \frac{\#p + \#q}{2} - \#(p \vee q)$$

supremum

defines a distance on $\mathcal{P}_k$. The following assertions define the same order on $\mathcal{P}_k$:

① $p \leq q$ i.if $d(\mathrm{id},q) = d(\mathrm{id},p) + d(p,q)$



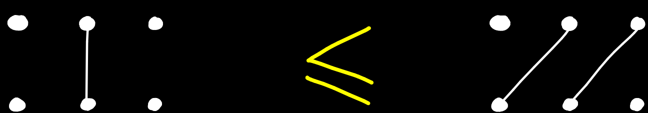
↳ Restric° to σ_k: geo. order Cayley graph.

② $p \leq q$ i.if one can go from q to p by

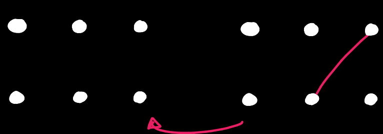
- gluing two blocks without gluing two cycles
- cutting a block in 2 & because of this creating a new cycle.

↳ Easy to compute the Mobius f°.

Ex:



glue two blocks
same nb of cycles



split a block
split a cycle

This order allows to relate moments and cumulants:

<u>Theorem</u> (G.) If M_N converges:

$$m_p(M) = \sum_{p' \leq p} \underbrace{\kappa_{p'}(M)}_{\text{Cumulants}}$$

R-transform: $\underbrace{\mathcal{R}_M(p)}_{\in \left(\bigoplus_k \mathbb{C}[\mathcal{P}_k]\right)^*} = \kappa_p(M)$

<u>Question</u>: Lévy processes $\simeq$ ($\times$ or $+$) of infinitesimal independant random matrices. Cumulants behave well for independant matrices.

$\hookrightarrow$ Are they formulae for

$$\mathcal{R}_{ML} \text{ and } \mathcal{R}_{M+L}$$

when M_N, L_N are independent, invariant by $\mathcal{O}_N$ and converge?

Part Two

Convergence of Lévy Processes

Answer: We will introduce 2 coproducts on

$$\bigoplus_K \mathbb{C}[\mathcal{P}_k] =: \mathbb{C}[\mathcal{P}]$$

$$\Delta_{\boxdot} : \mathbb{C}[\mathcal{P}] \longrightarrow \mathbb{C}[\mathcal{P}] \otimes \mathbb{C}[\mathcal{P}]$$

$\boxplus$ or $\boxtimes$

Which define two convolutions

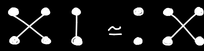
$$\boxdot : \mathbb{C}[\mathcal{P}]^* \otimes \mathbb{C}[\mathcal{P}]^* \longrightarrow \mathbb{C}[\mathcal{P}]^*$$

$$f \boxdot g(p) = (f \otimes g)\, \Delta_{\boxdot}(p)$$

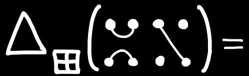
Theorem: (G.) If M_N, L_N are indep., invar. by $\mathcal{O}_N$, and converge,

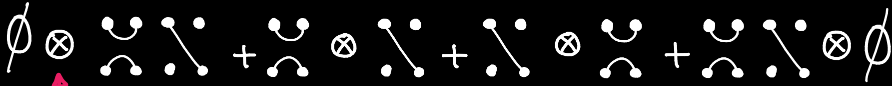
$$\mathcal{R}_{M+L} = \mathcal{R}_M \boxplus \mathcal{R}_L$$

$$\mathcal{R}_{ML} = \mathcal{R}_M \boxtimes \mathcal{R}_L$$

Addition, $\Delta_{\boxplus}$ (under hyp: )

- $\Delta_{\boxplus}(p) \longrightarrow$ all possibilities to partition the cycles of p in 2 bloks.

Ex: 



external $\otimes$

internal $\otimes$

- $p \otimes q = \boxed{\;p\;}\boxed{\;q\;}$

- $\varepsilon_{\boxplus}(p) \longrightarrow \delta_{p=\emptyset}$

Proposition

$(\mathbb{C}[\mathcal{P}], \otimes, \emptyset, \Delta_{\boxplus}, \varepsilon_{\boxplus})$ is a graded con. Hopf algebra.

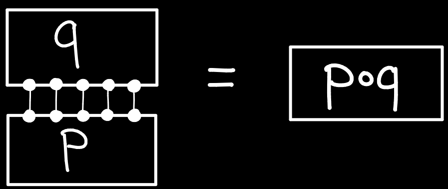
Rq: $\varepsilon_{\boxplus}$ is the neutral element for $\boxplus$

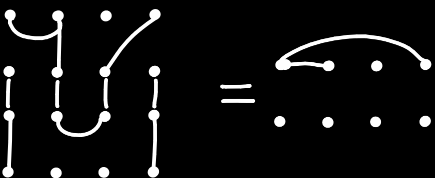
Multiplication, $\Delta_{\boxtimes}$ (under hyp: )

all possibilities to decompose

- $\Delta_{\boxtimes}(p) \longrightarrow p = p_1 \circ p_2 \leftarrow$ Kreweras compl. of p_1 in p

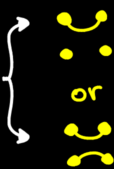
- product

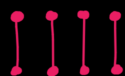


Ex: 

- Kreweras : $p_2 \in K_{p_1}(p)$
  - $p = p_1 \circ p_2$
  - $p_1 \leq p$
  - $p_2 \leq {}^t p_1 \circ p_1 \circ p_2$

Ex: $p_1 =$  $p =$ 

${}^t p_1 \circ p_1 = p_1 \longrightarrow p_2 \leq p$ 

- $\varepsilon_{\boxtimes}(p) \longrightarrow \delta_{p = id_k} \leftarrow$ 

Proposition

$(\bigoplus_k \mathbb{C}[\mathcal{P}_k], \otimes, \phi, \Delta_{\boxtimes}, \varepsilon_{\boxtimes})$ is an associative co-associative bialgebra

Rq: $\varepsilon_{\boxtimes}$ is the neutral element for $\boxtimes$

Levy processes:

Setting: $(M_{N,t})_{t\geq 0}$ σ_N-invariant Levy process

$\mathbb{E}[M_t^{\otimes k}]$, $\frac{d}{dt}\Big|_{t=0}\mathbb{E}[M_t^{\otimes k}] =: G_k$ exist.

Claim 1: $\left(\frac{d}{dt}\Big|_{t=0}\mathbb{E}[M_t^{\otimes k}]\right)_{k>0}$ characterises $\mathbb{E}[M_t^{\otimes k}]$

generator

"semi" group

Proof: $\mathbb{E}[M_{t+s}^{\otimes k}] = \mathbb{E}[\tilde{M}_s^{\otimes k} \cdot M_t^{\otimes k}] = \mathbb{E}[M_s^{\otimes k}]\,\mathbb{E}[M_t^{\otimes k}]$

indep

Claim 2: $G_k \in \mathrm{End}((\mathbb{C}^N)^{\otimes k})^{\sigma_N}$: we can define its "finite dim" cumulants, the notion of convergence...

Consequence: The "finite dim" cumulants of M_t satisfy a system of linear equations, caracterised by the "finite dim" cumulants of $(G_k)_{k>0}$.

Question: Does the convergence of $(G_k)_k$ implies the convergence of $(M_{N,t})_{t>0}$?

Answer: Yes it does.

Theorem (G.) If G_k converges for any k, the Levy process $(M_{N,t})_{t>0}$ converges as $N \to \infty$.
Besides,

$$\frac{d}{ds}\Big|_t \mathcal{R}_{M_s} = \mathcal{R}_G \boxdot \mathcal{R}_{M_t}$$

$$\boxed{\mathcal{R}_{M_t} = e^{\boxdot t \mathcal{R}_G}}$$

Remarks:

1/ To prove the convergence of G_k, compute:
- its cumulants
- its moments
- (or its exclusive moments, but that's an other story)

2/ Generalized easily for the moments of $M_{N,t}$ and $M^*_{N,t}$

Examples:

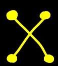
1/ Hermitian Brownian Motion: $H_t = \sum_i B_t^i h_i$

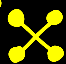
$$dH_t^{\otimes k} = \sum H_t \otimes \cdots \otimes dH_t \otimes \cdots H_t + \sum H_t \otimes \cdots \otimes dH_t \otimes \cdots dH_t \otimes \cdots H_t$$

$$\hookrightarrow G_k = \frac{d}{dt}\Big|_0 \mathbb{E}(H_t^{\otimes k}) = \underbrace{\frac{dH_t \otimes dH_t}{dt}}_{\sum_i h_i \otimes h_i} \text{ if } k=2 \text{; } 0 \text{ if } k \neq 2$$

$c(p) = 1$, $\#p = 2$

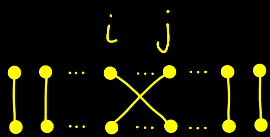
normalization $\to \times N^{2-1}$

$$G_k = \frac{1}{N}$$ 

$\to$ Convergence $+ \ \mathcal{R}_G(p) = \delta$  (p)

2/ Unitary Brownian Motion: "e^{iH_t}"

$\to$ Itô:

$$G_k = \underbrace{\frac{k}{2}\, id_k}_{\times N^0} + \sum_{i,j} \frac{1}{N}$$  N^1

$\to$ Convergence $+$

$$\mathcal{R}_G(p) = \begin{cases} k/2 & \text{if } p = id_k \\ 1 & \text{if } p \simeq (1,2) \\ 0 & \text{else} \end{cases}$$

Part Three

Deterministic limit?

Setting: (only for simplicity, can be generalized)

↳ matrices in $\mathcal{M}_N(\mathbb{R})$

Claim: If $\mathbb{E}(X_N) \longrightarrow a$ and $\underbrace{\mathrm{Var}(X_N) \to 0}_{\mathbb{E}(X_N^2) \simeq \mathbb{E}(X_N)^2}$ then

$X_N \xrightarrow[\mathbb{L}^2]{} a$ ← deterministic limit

Observables: $\mathbb{E}\Big(\underbrace{\frac{1}{N^{c(p)}} \mathrm{Tr}(M_N^{\otimes k} \circ e(p))}_{X_N}\Big)$

↳ Need to understand $\mathrm{Var}(X_N)$ hence $\mathbb{E}(X_N^2)$

Claim: $\mathbb{E}(X_N^2) = m_{p \otimes p}(M_N)$

Consequence: To see if the limit of a system of matrices converge towards a deterministic limit, we only need to check:

$$m_{p\otimes q}(M) = m_p(M)\, m_q(M)$$

Characters of $(\mathbb{C}[\mathcal{P}], \otimes)$: $X[\mathcal{P}]$

Definition: $\Phi \in \mathbb{C}[\mathcal{P}]^*$ is a character i.i.f:

$$\Phi(p \otimes q) = \Phi(p)\Phi(q)$$

Question: Deterministic limit $\longleftrightarrow \underbrace{(p \mapsto m_p(M))}_{\mathcal{M}_M} \in X[\mathcal{P}]$

But what about $\mathcal{R}_M$?

Proposition: $\mathcal{R}_M \in X[\mathcal{P}] \Longleftrightarrow \mathcal{M}_M \in X[\mathcal{P}]$

Good news since if $(M_{N,t})_t$ is a convergent $\mathfrak{S}_N$ inv. Levy process, $\mathcal{R}_{M_t} = e^{\boxdot t \mathcal{R}_G}$.

Question: can we characterise the generator of smooth one dim. semigroup of characters for the two convolutions $\boxplus, \boxtimes$?

Infinitesimal characters : $\mathfrak{X}_{\boxplus}[\mathcal{P}]$ and $\mathfrak{X}_{\boxtimes}[\mathcal{P}]$

$$\Phi_t \in X[\mathcal{P}] \longrightarrow \underbrace{\Phi_t(p \otimes q) = \Phi_t(p)\,\Phi_t(q)}_{d/dt|_0}$$

$$\dot{\Phi}_0(p \otimes q) = \Phi_0(p)\,\dot{\Phi}_0(q) + \dot{\Phi}_0(p)\,\Phi_0(q)$$

Definition: $\Psi \in \mathfrak{X}_{\boxdot}(\mathcal{P})$ iif $\qquad$ counit

$$\psi(p \otimes q) = \varepsilon_{\boxdot}(p)\,\Psi(q) + \psi(p)\,\varepsilon_{\boxdot}(q)$$

Ex: $\boxplus$: $\varepsilon_{\boxplus}(p) = \delta_{p=\emptyset}$: $\psi(p) = 0$ if $c(p) > 1$

$c(q)=1$

$\boxtimes$: $\varepsilon_{\boxtimes}(p) = \delta_{p=id_k}$:
- $\psi(p) = 0$ if $p \neq q \otimes id_\ell$
- $\psi(p \otimes id_k) = \psi(p)$
- $\psi(id_k) = k\,\psi(id_1)$

Proposition: Characterisation via moments:

$\psi \in \mathfrak{X}_{\boxdot}(\mathcal{P})$ iif $m_\psi(p) = \sum_{p' \leq p} \psi(p')$ is:

$\boxplus$: a $\boxplus$ infinitesimal character,

$\boxtimes$: an additive character:

$$m_\psi(p \otimes q) = m_\psi(p) + m_\psi(q)$$

Proposition: If $\Phi_t = e^{\boxdot t \psi}$ there is equivalence:

① $\forall t \geqslant 0$, $\Phi_t \in X[\mathcal{P}]$

② $\psi \in x_{\boxdot}(\mathcal{P})$

Recall:

$$\left.\begin{array}{l}\mathcal{R}_{M_t}\\ \mathcal{R}_G\end{array}\right\} \text{encodes the limit of the } \underbrace{\text{cumulants}}_{\text{normalized coordinates}} \text{ of } \left\{\begin{array}{l}\mathbb{E}[M_{N,t}^{\otimes \cdot}]\\ G_{N,\cdot} = \left.\frac{d}{dt}\right|_{t=0} \mathbb{E}[M_{N,t}^{\otimes \cdot}]\end{array}\right.$$

$m_{\cdot}(p)$ encodes the limit of normalized moments.

$$\mathcal{R}_{M_t} = e^{\boxdot t \mathcal{R}_G}$$

Theorem: (G.) If one of these conditions hold:

- cumulants: $\mathcal{R}_G$ is a $\boxdot$-infinitesimal character
- moments: m_G is a $\left\{\begin{array}{l}\boxplus \text{ infinitesimal character, } \boxplus\\ \text{additive character } \boxtimes\end{array}\right.$

Then the limiting objects (ex: normalised moments, eigenvalues emp. distribution) are deterministic.

Examples:

1/ Hermitian Brownian Motion:

$$\mathcal{R}_G(p) = \delta_{\times}(p)$$

one cycle

is a $\boxplus$ infinitesimal character.

$\rightarrow$ deterministic limit

2/ Unitary Brownian Motion:

$$\mathcal{R}_G(\mathrm{id}_k) = \frac{k}{2}$$

$$\mathcal{R}_G((1,2) \underset{\sigma}{\otimes} \mathrm{id}_\ell) = 1$$

$$\mathcal{R}_G(p) = 0$$

is a $\boxtimes$ infinitesimal character.

$\rightarrow$ deterministic limit

Part Four

General Random Walks on the Symmetric group

Overview: Random Walk on $\mathfrak{S}_N$:

$$G_k^N = \frac{N}{N-\lambda_N(1)} \frac{1}{\#\lambda_N} \sum_{\sigma \in \lambda_N} \left(\sigma^{\otimes k} - Id_N^{\otimes k}\right)$$

Compute moments (via exclusive~)

\# fixed pts

$$\underbrace{\text{Moments converge}}_{\text{Convergence of } \mu_{S_t^N}} \text{ and if } \underbrace{\lambda_N(1) \sim N,\ m_G(p \otimes q) = m_G(p) + m_G(q)}_{\text{Towards a deterministic limit}}$$

Convergence of $\mu_{S_t^N}$

empirical eig.val distribution

Towards a deterministic limit

$$\mu_t = \sum_{n \in \mathbb{N} \cup \infty} \chi_n(t)\, \lambda_{U_n}$$

Exact computations:

$$\chi(t,z) = \sum_n e^{nt} \chi_n(t) z^n \quad \text{and} \quad S(z) = \sum_n \lambda_{n+1} z^n$$

$$\boxed{\mathcal{R}_{S_t} = e^{\boxtimes t \mathcal{R}_G}}$$

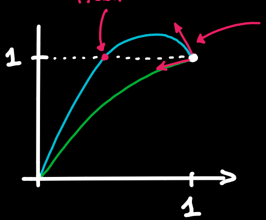
χ solution of

$$Z e^{-tS(Z)} \Big|_{Z=\chi(t,z)} = z$$

Lagrange inversion form.

$1 - \chi_\infty$ solution of

$$z\, e^{-t(S(z)-1)} = 1$$



$1 - tS'(1) < 0$

$$t_c = \frac{1}{S'(1)}$$

Conclusion

> Cumulants def. by S.W.J duality or "à la Speicher"

Can we have a Fock Space formulation ?

> Not only the eigenvalued distribution conv. but also the mixed moments

Levy process ⟶ Trafic free Levy process

for O_N Walks: increments are not free

Can we characterise trafic free Levy processes?

> Fluctuations ?

→ Algebraic : $m_p = \sum \frac{m_p^\ell}{N^\ell} \Longrightarrow$ Sim. results.

Can we relate these to prob. fluctua° ?

→ Probabilistic : Can be reformulated using

(A.Dahlqvist)

$$M_N \longrightarrow \mathbb{E}\left(\begin{pmatrix} M_N^{(1)} & & 0 \\ & \ddots & \\ 0 & & M_N^{(\ell)} \end{pmatrix}^{\otimes k}\right)$$

+ Version Schur Weyl: Partitionned partitions

⇝ Other "cumulants".

What is missing → Observables conv → Cum. conv.

Tusen Takk !