Fredholm Grassmannian flows and their applications to nonlinear PDEs and SPDEs

Margaret Beck, Anastasia Doikou, Simon J.A. Malham, Ioannis Stylianidis and Anke Wiese

NTNU Trondheim 10th May 2019

Outline

1. Motivation.
2. Canonical linear system.
3. Integrable nonlinear PDEs.
4. Smoluchowski-type equations.

Motivation: Integrable systems and Fredholm determinants

Dyson; Miura; Ablowitz, Ramani & Segur; Pöppe; Sato; Segal & Wilson; Tracy & Widom. . . . Quoting Pöppe:

> *"For every soliton equation, there exists a linear PDE (called a base equation) such that a map can be defined mapping a solution p of the base equation to a solution g_* of the soliton equation. The properties of the soliton equation may be deduced from the corresponding properties of the base equation which in turn are quite simple due to linearity. The map $p \to g_*$ essentially consists of constructing a set of linear integral operators using p and computing their Fredholm determinants."*

Motivation: Marchenko equation

Ablowitz, Ramani & Segur: Marchenko equation, $y \geqslant x$:

$$p(x,y) = g(x,y) + \int_x^\infty g(x,z)q(z,y;x)\,\mathrm{d}z,$$

1. Scattering data: $p = p(x+y)$ and q.
2. KdV $q = -p$:

$$\begin{aligned} \text{Suppose} \quad & \partial_t p + \partial_x^3 p = 0, \\ \text{then} \quad & \gamma(x) := -2(\mathrm{d}/\mathrm{d}x)g(x,x) \\ \text{satisfies} \quad & \partial_t \gamma + \partial_x^3 \gamma = 6\gamma\partial_x\gamma. \end{aligned}$$

3. Pöppe: elevated argument to operator level.

Example PDEs with local/nonlocal nonlinearities

1. Nonlinear Schrödinger equation, local nonlinearity:

$$\mathrm{i}\partial_t\gamma(x;t)=\partial_x^2\gamma(x;t)-\gamma(x;t)\big|\gamma(x;t)\big|^2.$$

2. PDE with quadratic nonlocal nonlinearity:

$$\partial_t g(x,y;t)=d(\partial_x)\,g(x,y;t)-\int_{\mathbb{R}}g(x,z;t)\,g(z,y;t)\,\mathrm{d}z.$$

Canonical linear system

$$
\begin{aligned}
\partial_t Q &= AQ + BP, && (Q := \mathrm{id} + \hat{Q}) \\
\partial_t P &= CQ + DP, && (Q_0 = \mathrm{id}, P_0 = G_0) \\
P &= G\,Q.
\end{aligned}
$$

$$
\Rightarrow \qquad \begin{aligned}
(\partial_t G)\,Q &= \partial_t P - G\,\partial_t Q \\
&= CQ + DP - G\,(AQ + BP) \\
&= (C + DG)\,Q - G\,(A + BG)\,Q
\end{aligned}
$$

$$
\Rightarrow \qquad \partial_t G = C + DG - G\,(A + BG).
$$

"Big matrix" PDEs

$$\partial_t G = DG - G\,BG$$

$$\Leftrightarrow \quad \partial_t g(x,y;t) = d(\partial_x)\,g(x,y;t) - \int_{\mathbb{R}} g(x,z;t)\,b(z)\,g(z,y;t)\,\mathrm{d}z.$$

In practice: Quadratic “Big matrix” PDE

Suppose we wish to solve the PDE

$$\partial_t g(x,y;t) = d(\partial_x)\, g(x,y;t) - \int_{\mathbb{R}} g(x,z;t)\, b(z)\, g(z,y;t)\,\mathrm{d}z.$$

Then our prescription says set up:

$$\begin{aligned}
\partial_t p(x,y;t) &= d(\partial_x)\, p(x,y;t), \\
\partial_t q(x,y;t) &= b(x)\, p(x,y;t). \\
p(x,y;t) &= \int_{\mathbb{R}} g(x,z;t)\, q(z,y;t)\,\mathrm{d}z.
\end{aligned}$$

Here we set $q(x,y;t) = \delta(x-y) + \hat{q}(x,y;t)$.

In practice: What have we gained?

$$\partial_t p(x,y;t) = d(\partial_x)\, p(x,y;t),$$
$$\partial_t q(x,y;t) = b(x)\, p(x,y;t).$$

$$\mathfrak{p}(k,y;t) = \mathrm{e}^{d(2\pi\mathrm{i}k)\,t}\,\mathfrak{p}_0(k,y),$$
$$\mathfrak{q}(k,y;t) = \mathrm{e}^{2\pi\mathrm{i}ky} + \int_{\mathbb{R}} \mathfrak{b}(k-\kappa)\,\mathfrak{I}(\kappa,t)\,\mathfrak{p}_0(\kappa,y)\,\mathrm{d}\kappa,$$
$$\mathfrak{I}(k,t) := \big(\mathrm{e}^{d(2\pi\mathrm{i}k)\,t} - 1\big)/d(2\pi\mathrm{i}k).$$

- $\Rightarrow$ Solve the *linear* equations for p and q explicitly.
- $\Rightarrow$ Evaluate them for any given time $t > 0$.
- $\Rightarrow$ Solve *linear* Fredholm equation for g.
- $\Rightarrow$ Solution g to the PDE at that time.

Integrable systems: Nonlinear Schrödinger equation

$$
\begin{aligned}
\mathrm{i}\partial_t P &= \partial_x^2 P, \\
\hat{Q} &= P^\dagger P, \\
P &= G(\mathrm{id} + \hat{Q}).
\end{aligned}
$$

$$
\begin{aligned}
(P\psi)(y;x) &:= \int_{-\infty}^{0} p(y+z+x)\psi(z)\,\mathrm{d}z, \\
\hat{q}(y,z;x,t) &= \int_{-\infty}^{0} p^*(y+\xi+x;t)p(\xi+z+x;t)\,\mathrm{d}\xi, \\
p(y+z+x;t) &= g(y,z;x,t) + \int_{-\infty}^{0} g(y,\xi;x,t)\hat{q}(\xi,z;x,t)\,\mathrm{d}\xi.
\end{aligned}
$$

Product rule

Assume R additive with kernel $r = r(y + z + x)$.

1. Define:

$$\langle G\rangle(x;t) := g(0,0;x,t);$$

2. Product rules:
   1. Primitive:

$$[F\partial_x(RR')F'](y,z) \equiv \big([FR](y,0)\big)\big([R'F'](0,z)\big)$$

   2. Complete:

$$\langle F\partial_x(RR')F'\rangle \equiv \langle FR\rangle\langle R'F'\rangle.$$

Product rule: proof

Proof.

$$\int_{\mathbb{R}_-^3} f(y,\xi_1)\partial_x\big(r(\xi_1+\xi_2+x)r'(\xi_2+\xi_3+x)\big)f'(\xi_3,z)\,\mathrm{d}\xi_3\,\mathrm{d}\xi_2\,\mathrm{d}\xi_1$$

$$=\int_{\mathbb{R}_-^3} f(y,\xi_1)\partial_{\xi_2}\big(r(\xi_1+\xi_2+x)r'(\xi_2+\xi_3+x)\big)f'(\xi_3,z)\,\mathrm{d}\xi_3\,\mathrm{d}\xi_2\,\mathrm{d}\xi_1$$

$$=\int_{\mathbb{R}_-^2} f(y,\xi_1)r(\xi_1+x)r'(\xi_3+x)f'(\xi_3,z)\,\mathrm{d}\xi_3\,\mathrm{d}\xi_1$$

$$=\int_{\mathbb{R}_-} f(y,\xi_1)r(\xi_1+x)\,\mathrm{d}\xi_1\cdot\int_{\mathbb{R}_-} r'(\xi_3+x)f'(\xi_3,z)\,\mathrm{d}\xi_3$$

$$=\big([FR](y,0)\big)\big([R'F'](0,z)\big).$$

□

Nonlinear Schrödinger equation

Proof.

Set $U := (\mathrm{id} + \hat{Q})^{-1}$, so $G = PU$, and $\hat{Q} = PP^\dagger \Rightarrow$:

$$\mathrm{i}\partial_t G - \partial_x^2 G = 2\big(PU(P_x^\dagger P)_x U + P_x U(P^\dagger P)_x U + PU_x(P^\dagger P)_x U\big).$$

$$\begin{aligned}
\mathrm{i}\partial_t g(y,z) - \partial_x^2 g(y,z) &= 2\big([(PUP^\dagger)_x](y,0)\big)\big([PU](0,z)\big)\\
&= 2\big([V(PP^\dagger)_x V](y,0)\big)\big([PU](0,z)\big)\\
&= 2\Big(\big[VP](y,0)\big)\big([P^\dagger V](0,0)\big)\Big)\big([G](0,z)\big)\\
&= 2\Big(\big[G](y,0)\big)\big([G^\dagger](0,0)\big)\Big)\big([G](0,z)\big)\\
&= 2g(y,0)g^*(0,0)g(0,z).
\end{aligned}$$

□

Examples

- Integrable hierarchy (algebra?);
- Nonlocal reaction-diffusion (classes of);
- Higher degree nonlocal nonlinearities;
- Nonlinear elliptic systems (classes of);
- SPDEs (classes of);
- Smoluchowski coagulation (?).

SPDEs with nonlocal nonlinearities

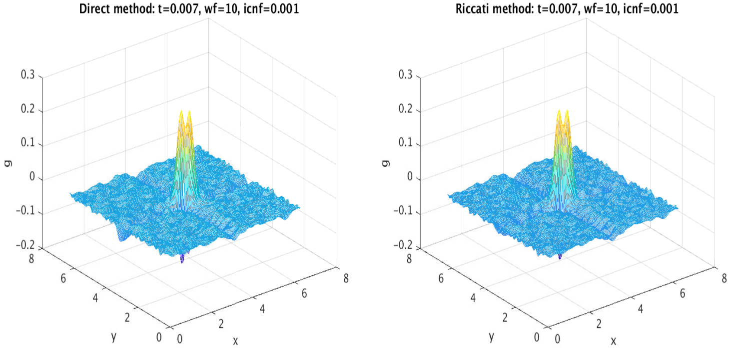
On $\mathbb{T} = [0, 2\pi]^2$ with $q = q(x, y; t)$, $p = p(x, y; t)$, $g = g(x, y; t)$:

$$\begin{aligned} \partial_t p &= \partial_1^2 p + \gamma \dot{W} * p, \\ \partial_t q &= \epsilon p, \\ p &= g \star q. \end{aligned}$$

$$\Rightarrow \quad \partial_t g = \partial_1^2 g + \gamma \dot{W} * g - \epsilon g \star g.$$

Here $W = W(x; t)$ is a Wiener field.

SPDEs with nonlocal nonlinearities: figure



Smoluchowski's coagulation equation

$$\partial_t g(x;t) = \underbrace{\tfrac{1}{2}\int_0^x K(y,x-y)\,g(y;t)g(x-y;t)\,\mathrm{d}y}_{\text{coagulation gain}}$$

$$\underbrace{-\,g(x;t)\int_0^\infty K(x,y)\,g(y;t)\,\mathrm{d}y}_{\text{coagulation loss}}\,.$$

- $g(x,t) =$ density of clusters of mass x;
- Applications: polymerisation, aerosols, clouds/smog, clustering stars/galaxies, schooling/flocking.
- Solvable cases: (i) $K = 2$; (ii) $K = x + y$ and (iii) $K = xy$.

Smoluchowski's coagulation equation: $K = 1$

Consider the case $K = 1$ (many other cases by rescaling).

$$\partial_t g(x;t) = \tfrac{1}{2}\int_0^x g(y;t)g(x-y;t)\,\mathrm{d}y - g(x;t)\underbrace{\int_0^\infty g(y;t)\,\mathrm{d}y}_{m_0(t)}.$$

Direct integration $\Rightarrow \dot{m}_0 = -\tfrac{1}{2}m_0^2 \Rightarrow m_0$ known.

$$\begin{aligned}\partial_t \mathfrak{p} &= -m_0 \mathfrak{p},\\ \partial_t \mathfrak{q} &= -\tfrac{1}{2}\mathfrak{p},\\ \mathfrak{p} &= \mathfrak{g}\mathfrak{q}.\end{aligned}$$

$$\Rightarrow \quad \partial_t \mathfrak{g} = \tfrac{1}{2}\mathfrak{g}^2 - m_0\mathfrak{g}.$$

Smoluchowski: more generally

$$\partial_t g(x;t) = \int_0^x g(x-y;t)b(\partial_y)g(y;t)\,\mathrm{d}y - \int_0^x a(x-y;t)g(y;t)\,\mathrm{d}y + d(\partial_x)g(x;t) - \int_0^x \int_0^y b_0(y-z;t)g(z;t)\,\mathrm{d}z\, g(x-y;t)\,\mathrm{d}y.$$

Desingularised Laplace Transf.: $\pi(s,t) = \int_0^\infty (1-\mathrm{e}^{-sx})\, g(x,t)\,\mathrm{d}x.$

Menon & Pego (2003) $\Rightarrow$

$$\begin{aligned} K = 1: &\qquad \partial_t\pi + \tfrac{1}{2}\pi^2 = 0; \\ K = x+y: &\qquad \partial_t\pi + \pi\partial_s\pi = -\pi; \\ K = xy: &\qquad \partial_t\tilde{\pi} + \tilde{\pi}\partial_s\tilde{\pi} = 0. \end{aligned}$$

Optimal nonlinear control: Riccati PDEs

Inspired by Byrnes (1998) and Byrnes & Jhemi (1992) $\Rightarrow$

$$\begin{aligned}\dot{q} &= aq + bp,\\ \dot{p} &= cq + dp,\\ p &= \pi(q,t).\end{aligned}$$

$$\Rightarrow \qquad \partial_t \pi = cq + d\pi - (\nabla\pi)(aq + b\pi)$$

Ex.
$$\begin{aligned}\dot{q} &= p,\\ \dot{p} &= 0,\end{aligned}$$

$$\Rightarrow \qquad \partial_t \pi = (\nabla\pi)\pi.$$

Inviscid Burgers solution

To find $\pi = \pi(x,t)$ solve by "Characteristics":

$$\dot{p} = 0 \quad \Leftrightarrow \quad p(a,t) = \pi(q(a,t),t) = \pi_0(a)$$

$$\dot{q} = p \quad \Leftrightarrow \quad q(a,t) = a + t\pi_0(a).$$

$$q(a,t) = x \quad \Leftrightarrow \quad x = a + t\pi_0(a) = (\mathrm{id} + t\pi_0) \circ a$$

$$\Leftrightarrow \quad a = (\mathrm{id} + t\pi_0)^{-1} \circ x.$$

$$\Rightarrow \quad \pi(x,t) = \pi_0 \circ (\mathrm{id} + t\pi_0)^{-1} \circ x.$$

$\Rightarrow$ Smoluchowski additive and multiplicative cases.

Burgers flow

$$
\begin{aligned}
Q_t(a) &= a + \int_0^t P_s(a)\,\mathrm{d}s + \sqrt{2\nu}B_t,\\
\partial_t\pi &+ (\nabla\pi)\pi + \nu\Delta\pi = 0,\\
P_t(a) &= \pi\big(Q_t(a), t\big).
\end{aligned}
$$

Itô:

$$
\begin{aligned}
\pi\big(Q_t(a), t\big) = \pi_0(a) &+ \int_0^t \big(\partial_s\pi + (\nabla\pi)\pi + \nu\Delta\pi\big)\big(Q_s(a), s\big)\,\mathrm{d}s\\
&+ \sqrt{2\nu}\int_0^t (\nabla\pi)(Q_s(a), s)\,\mathrm{d}B_s.
\end{aligned}
$$

$$
\Rightarrow \quad \pi(x,t) = \mathbb{E}\left[\pi_0\big(Q_t^{-1}(x)\big)\right]. \qquad \text{(Constantin \& Iyer 2008)}
$$

Stochastic Burgers flow

$$\begin{aligned} Q_t(a) &= a + \int_0^t P_s(a)\,\mathrm{d}s + \sqrt{2\nu}B_t, \\ P_t(a) &= P_0(a), \\ P_t(a) &= \pi_t\big(Q_t(a), t\big). \end{aligned}$$

Generalised Itô:

$$\begin{aligned} \pi_t\big(Q_t(a), t\big) = \pi_0(a) &+ \int_0^t \big((\nabla\pi_s)\pi_s - \nu\Delta\pi_s\big)\big(Q_s(a), s\big)\,\mathrm{d}s \\ &+ \sqrt{2\nu}\int_0^t (\nabla\pi_s)(Q_s(a), s)\,\mathrm{d}B_s + \int_0^t \pi_s\big(Q_s(a), \mathrm{d}s\big). \end{aligned}$$

$$\Rightarrow \quad \mathrm{d}\pi_t + \big((\nabla\pi_t)\pi_t - \nu\Delta\pi_t\big)\,\mathrm{d}t + \sqrt{2\nu}(\nabla\pi_t)\,\mathrm{d}B_t = 0.$$

Looking forward I

McKean (1975) $\Rightarrow$ for FKPP equation

$$\partial_t w = \partial_x^2 w + w^2 - w,$$

$$w(x,t) = \mathbb{E}\Big[\prod w_0(Y_t^i)\Big],$$

1. Any initial data $0 \leqslant w_0 \leqslant 1$, $x \in \mathbb{R}^n$;
2. Y_t is a Branched Brownian Motion (BBM);
3. Product over all individual particles alive at time t.

Looking forward II

Coordinate patches, we chose $p = \pi_t \circ q \Rightarrow$

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} q \\ \pi_t \circ q \end{pmatrix} = \begin{pmatrix} \mathrm{id} \\ \pi_t \end{pmatrix} \circ q.$$

Eg. instead choose $q = \pi'_t \circ p$ (of course $\pi' = \pi^{-1}$) $\Rightarrow$

$$\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \pi'_t \circ p \\ p \end{pmatrix} = \begin{pmatrix} \pi'_t \\ \mathrm{id} \end{pmatrix} \circ p.$$

Same base eqns $\dot{q} = p$ and $\dot{p} = 0 \Rightarrow$

$$p_0 = p(t) = \dot{q} = (\partial_t \pi'_t) \circ p(t) = (\partial_t \pi'_t) \circ p_0.$$

Setting $y = p_0 \Rightarrow \pi'_t \circ y = \pi'_0 \circ y + ty.$

Thank you for listening!

Fisher–Kolmogorov–Petrovskii–Piskunov equation (FKPP)

Consider nonlocal FKPP (see Britton or Bian, Chen & Latos):

$$\partial_t g(x;t) = d(\partial_x) g(x;t) - g(x;t) \int_{\mathbb{R}} b(z,\partial_z)\, g(z;t)\, \mathrm{d}z.$$

1. $\partial_t p(x) = d(\partial_x)\, p(x),$
2. $\partial_t q(x) = b(x,\partial_x)\, p(x),$
3. $p(x) = g(x) \int_{\mathbb{R}} q(z)\, \mathrm{d}z. \qquad \big(= g(x)\overline{q}\big).$

$$\begin{aligned}\big(\partial_t g(x;t)\big)\, \overline{q}(t) &= \partial_t p(x;t) - g(x;t)\, \partial_t \overline{q}(t)\\ &= d(\partial_x) p(x;t) - g(x;t) \int_{\mathbb{R}} b(z,\partial_z)\, p(z;t)\, \mathrm{d}z\\ &= d(\partial_x) g(x;t)\, \overline{q}(t) - g(x;t) \int_{\mathbb{R}} b(z,\partial_z)\, g(z;t)\, \mathrm{d}z\, \overline{q}(t).\end{aligned}$$

FKPP II

Given data g_0, set $\overline{q}(0) = 1$ and $p(x;0) = g_0(x) \Rightarrow$

$$g(x;t) = \frac{p(x;t)}{\overline{q}(t)}$$

Consider the case $b = 1$:

$$\begin{aligned}\widehat{p}(k;t) &= \exp\big(d(2\pi\mathrm{i}k)\,t\big)\,\widehat{g}_0(k),\\ \overline{q}(t) &= 1 + \left(\frac{\exp(t\,d(0)) - 1}{d(0)}\right)\widehat{g}_0(0).\end{aligned}$$

$d(0) = 0 \Rightarrow \overline{q}(t) = 1 + t\,\widehat{g}_0(0)$.

$\Rightarrow$ explicit solution for any diffusive or dispersive $d = d(\partial_x)$.

Nonlocal reaction-diffusion system

With $d_{11} = \partial_1^2 + 1$, $d_{12} = -1/2$, $b_{12} = 0$ and $b_{11} = N(x, \sigma)$:

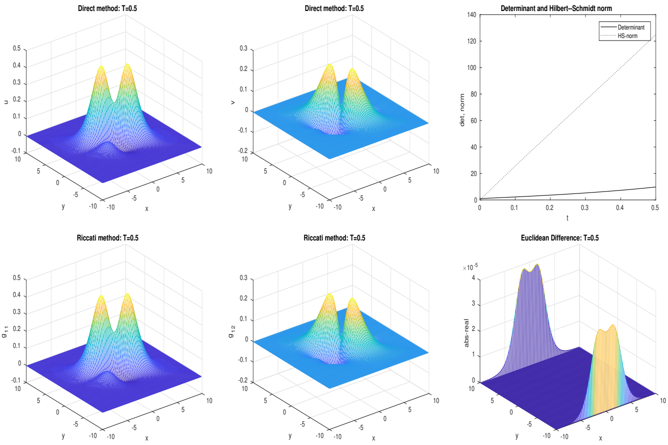
$$\partial_t u = d_{11} u + d_{12} v - u \star (b_{11} u) - u \star (b_{12} v) - v \star (b_{12} u) - v \star (b_{11} v),$$
$$\partial_t v = d_{11} v + d_{12} u - u \star (b_{11} v) - u \star (b_{12} v) - v \star (b_{12} v) - v \star (b_{11} u),$$

$$u_0(x, y) := \mathrm{sech}(x+y)\,\mathrm{sech}(y) \quad \text{and} \quad v_0(x, y) := \mathrm{sech}(x+y)\,\mathrm{sech}(x).$$

$$p = \begin{pmatrix} p_{11} & p_{12} \\ p_{12} & p_{11} \end{pmatrix}, \qquad q = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{11} \end{pmatrix} \qquad \text{and} \qquad g = \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{11} \end{pmatrix}.$$

Similar forms for d and b. Riccati equation: $\partial_t G = dG - G\,(bG)$

Nonlocal reaction-diffusion system II



Stochastic PDEs with nonlocal nonlinearities II

$$s(x) = \begin{pmatrix} \sin(x) \\ \sin(2x) \\ \vdots \end{pmatrix} \quad \text{and} \quad c(x) = \begin{pmatrix} c_o \\ \cos(x) \\ \cos(2x) \\ \vdots \end{pmatrix}$$

$$p_t(x,y) = \begin{pmatrix} s^{\mathrm{T}}(x) & c^{\mathrm{T}}(x) \end{pmatrix} \begin{pmatrix} p^{ss} & p^{sc} \\ p^{cs} & p^{cc} \end{pmatrix} \begin{pmatrix} s(y) \\ c(y) \end{pmatrix}$$

$$= \begin{pmatrix} s^{\mathrm{T}}(x) & c^{\mathrm{T}}(x) \end{pmatrix} P \begin{pmatrix} s(y) \\ c(y) \end{pmatrix}$$

Stochastic PDEs with nonlocal nonlinearities III

$$\Xi := \frac{1}{\sqrt{\pi}}\mathrm{diag}\{W_t^1, \frac{1}{2}W_t^2, \ldots, 0, W_t^1, \frac{1}{2}W_t^2, \ldots\}$$
$$D := -\mathrm{diag}\{1, 2^2, 3^2, \ldots, 0, 1, 2^2, 3^2, \ldots\}$$

$$\Rightarrow \partial_t P = DP + \dot{\Xi}P$$
$$\Rightarrow \partial_t p_{nm} = -n^2 p_{nm} + \frac{1}{n}\dot{W}_t^n p_{nm}$$
$$\Rightarrow p_{nm} = \exp\Big(-n^2 t - \tfrac{\sqrt{\pi}}{n} W_t^n - \tfrac{1}{2}\tfrac{\pi}{n^2} t\Big) p_{nm}(0)$$
$$\Rightarrow p_{0m} = p_{0m}(0)$$
$$\Rightarrow q'_{nm} = \int_0^t p_{nm}(\tau)\,\mathrm{d}\tau.$$