Integration with respect to the non-commutative fractional Brownian motion

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Outline

1. **Non-commutative processes**
   - Motivation: large random matrices
   - The non-commutative probability setting
   - The non-commutative fractional Brownian motion (NC-fBm)
2. **Integration with respect to NC-fBm**
   - The Young case: $H > \frac{1}{2}$
   - The rough case: $H \leq \frac{1}{2}$
   - Further results when $H = \frac{1}{2}$

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Motivation: large random matrices

Consider two independent families $(x(i,j))_{j\geq i\geq 1}$ and $(\tilde{x}(i,j))_{j\geq i\geq 1}$ of independent Brownian motions.

Then, for every fixed dimension $d \geq 1$, we define the (d-dimensional) **Hermitian Brownian motion** as the process $X^{(d)}$ with values in the space of the $(d \times d)$-Hermitian matrices and with upper-diagonal entries given for every $t \geq 0$ by

$$X_t^{(d)}(i,j) := \frac{1}{\sqrt{2d}}\big(x_t(i,j) + \imath\, \tilde{x}_t(i,j)\big) \quad \text{for } 1 \leq i < j \leq d \ ,$$

$$X_t^{(d)}(i,i) := \frac{x_t(i,i)}{\sqrt{d}} \quad \text{for } 1 \leq i \leq d \ .$$

Objective: to catch the behaviour, as $d \to \infty$, of the mean spectral dynamics of the process $X^{(d)}$.

Motivation: large random matrices

Observation: let A be a $(d \times d)$-matrix with complex random entries admitting finite moments of all orders. Denote the (random) eigenvalues of A by $\{\lambda_i(A)\}_{1\leq i\leq d}$, and set $\mu_A := \frac{1}{d}\sum_{i=1}^{d}\delta_{\lambda_i(A)}$. Then it is readily checked that

$$\mathbb{E}\Big[\int_{\mathbb{C}} z^r\,\mu_A(\mathrm{d}z)\Big] = \varphi_d\big(A^r\big)\ ,$$

where $\varphi_d(A) := \frac{1}{d}\mathbb{E}\big[\mathrm{Tr}(A)\big]$, $\mathrm{Tr}(A) := \sum_{i=1}^{d} A(i,i)$.

Based on this observation, a natural way to reach our objective is to study (the asymptotic behaviour of) the quantities

$$\varphi_d\big(X^{(d)}_{t_1}\cdots X^{(d)}_{t_r}\big)\ ,\ \text{for all times } t_1,\ldots,t_r \geq 0\ .$$

Motivation: large random matrices

Theorem (Voiculescu, *Invent. Math.*, 91'): For all $r \geq 1$ and $t_1, \ldots, t_r \geq 0$, it holds that

$$\varphi_d(X^{(d)}_{t_1} \cdots X^{(d)}_{t_r}) \xrightarrow{d\to\infty} \varphi(X_{t_1} \cdots X_{t_k}) \ ,$$

for a certain path $X : \mathbb{R}_+ \to \mathcal{A}$, where $(\mathcal{A}, \varphi)$ is a **non-commutative probability space**. This path is called a **non-commutative Brownian motion**.

Remark: this result can be extended to a more general class of Gaussian matrices.

Let us briefly recall the specific definition of the above elements $(\mathcal{A}, \varphi)$.

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The non-commutative probability setting

Definition: We call a **non-commutative (NC) probability space** any pair $(\mathcal{A}, \varphi)$ where:

(i) $\mathcal{A}$ is a unital algebra over $\mathbb{C}$ endowed with an antilinear $*$-operation $X \mapsto X^*$ such that $(X^*)^* = X$ and $(XY)^* = Y^*X^*$ for all $X, Y \in \mathcal{A}$.

In addition, there exists a norm $\|.\| : \mathcal{A} \to [0, \infty[$ which makes $\mathcal{A}$ a Banach space, and such that for all $X, Y \in \mathcal{A}$, $\|XY\| \leq \|X\|\|Y\|$ and $\|X^*X\| = \|X\|^2$.

(ii) $\varphi : \mathcal{A} \to \mathbb{C}$ is a linear functional such that $\varphi(1) = 1$, $\varphi(XY) = \varphi(YX)$, $\varphi(X^*X) \geq 0$ for all $X, Y \in \mathcal{A}$, and $\varphi(X^*X) = 0 \Leftrightarrow X = 0$. We call φ the trace of the space ("analog of the expectation").

We call a **non-commutative process** any path with values in a non-commutative probability space $\mathcal{A}$.

Outline

Reminder: the (commutative) fractional Brownian motion

Definition: In a (classical) probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and for $H \in (0,1)$, we call a **fractional Brownian motion** of Hurst index H any centered *gaussian* process $X : \Omega \times [0,\infty) \to \mathbb{R}$ with covariance function

$$\mathbb{E}[X_s X_t] = c_H(s,t) := \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) \; .$$

When $H = \frac{1}{2}$, we recover the definition of the standard Brownian motion. Thus, the fractional Brownian motion is a natural (and extensively studied!) generalization of the Brownian motion.

Reminder: the (commutative) fractional Brownian motion

Due to Wick formula, the joint moments of the fractional Brownian motion (of Hurst index H) are given, for all $r \geq 1$ and $t_1, \ldots, t_r \geq 0$, by the expression

$$\mathbb{E}[X_{t_1} \cdots X_{t_r}] = \sum_{\pi \in \mathcal{P}_2(r)} \prod_{\{p,q\} \in \pi} c_H(t_p, t_q) \ ,$$

where $\mathcal{P}_2(r)$ the set of the pairings of $\{1, \ldots, r\}$.

The NC-fractional Brownian motion

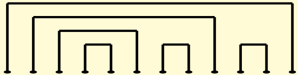
Definition: In a NC-probability space $(\mathcal{A}, \varphi)$, and for $H \in (0,1)$, we call a **non-commutative (NC) fractional Brownian motion** of Hurst index H any path $X : \mathbb{R}_+ \to \mathcal{A}$ such that, for all $r \geq 1$ and $t_1, \ldots, t_r \geq 0$,

$$\varphi(X_{t_1} \cdots X_{t_r}) = \sum_{\pi \in NC_2(r)} \prod_{\{p,q\} \in \pi} c_H(t_p, t_q) \ ,$$

with

$$c_H(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) \ .$$

The above notation $NC_2(r)$ refers to the set of the *non-crossing pairings* of $\{1, \ldots, r\}$: for instance,



The NC-fractional Brownian motion

This (family of) process(es) was first considered in

I. Nourdin and M.S. Taqqu: Central and non-central limit theorems in a free probability setting. J. Theoret. Probab. (2011),

and then further studied in

I. Nourdin: Selected Aspects of Fractional Brownian Motion. Springer, New York, 2012.

Classical approach to non-commutative integration, as developed in

P. Biane and R. Speicher: Stochastic calculus with respect to free Brownian motion and analysis on Wigner space. PTRF (1998)

cannot be applied as soon as $H \neq \frac{1}{2}$.

The NC-fractional Brownian motion

Proposition: For every fixed $H \in (0,1)$, there exists a NC-fractional Brownian motion of Hurst index H. (In other words, there exists a NC-probability space $(\mathcal{A}, \varphi)$ and a NC-fBm $X : [0, T] \to \mathcal{A}$.)

A NC-fractional Brownian motion of Hurst index $H = \frac{1}{2}$ is called a **NC Brownian motion**.

Proposition. For every fixed $H \in (0,1)$, it holds that

$$\|X_t - X_s\| \lesssim |t-s|^H \ .$$

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Objective: given a NC fractional Brownian motion X (of Hurst index H) in $(\mathcal{A}, \varphi)$, provide a natural interpretation of the integral

$$\int Y_t dX_t Z_t \ ,$$

for $Y, Z : [0, T] \to \mathcal{A}$ in a suitable class of integrands. At least

$$\int P(X_t) dX_t Q(X_t) \quad \text{for all polynomials } P, Q \ .$$

Related questions:

- Itô formula, Wong-Zakaï approximation.
- Differential equation $dY_t = P(Y_t) dX_t Q(Y_t)$.

Outline

Prop.: Let H, γ be such that $H + \gamma > 1$. Then, for all γ-Hölder paths $Y, Z : [0, T] \to \mathcal{A}$, all times $0 \leq s \leq t$ and every subdivision $\Delta_{st} = \{t_0 = s < t_1 < \ldots < t_\ell = t\}$ of $[s, t]$ with mesh $|\Delta_{st}|$ tending to 0, the Riemann sum

$$\sum_{t_i \in \Delta_{st}} Y_{t_i}(X_{t_{i+1}} - X_{t_i})Z_{t_i}$$

converges in $\mathcal{A}$ as $|\Delta_{st}| \to 0$. Denoting the limit by $\int_s^t Y_u dX_u Z_u$, one has, if $H > \frac{1}{2}$,

$$\int_s^t P(X_u^{(n)})dX_u^{(n)}Q(X_u^{(n)}) \xrightarrow{n\to\infty} \int_s^t P(X_u)dX_uQ(X_u) \quad \text{in } \mathcal{A} \ ,$$

where $X^{(n)}$ is the linear interpolation of X in $\mathcal{A}$. As a result,

$$P(X_t) - P(X_s) = \int_s^t \partial P(X_u)\sharp dX_u \ .$$

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$H \leq \frac{1}{2}$

For simplicity, let us first consider the integral $\int P(X_t)dX_t$, for some polynomial P.

Idea: corrected Riemann sums

$$\int P(X_t)dX_t := \lim \sum_{t_i} \{P(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + C_{t_i,t_{i+1}}\}.$$

When $H \in (\frac{1}{3}, \frac{1}{2}]$, a natural (potential!) definition:

$$\int P(X_t)dX_t \text{ " := "}$$

$$\lim \sum_{t_k} \left\{P(X_{t_k})(X_{t_{k+1}} - X_{t_k}) + \int_{t_k}^{t_{k+1}} \nabla P(X_{t_k})(X_u - X_{t_k})dX_u\right\}.$$

But: Remember that in the classical finite-dimensional situation,

$$\int_s^t \nabla f(x_s)(x_u - x_s)dx_u = \partial_i f_j(x_s) \int_s^t \int_s^u dx_v^{(i)} dx_u^{(j)}.$$

This separation is no longer possible for $\int_s^t \nabla P(X_s)(X_u - X_s)dX_u$...

For instance, when $P(x) = x^p$,

$$\begin{aligned}&\int_s^t \nabla P(X_s)(X_u - X_s)dX_u \\ &= \sum_{i=0}^{p-1} X_s^i \int_s^t (X_u - X_s)X_s^{p-1-i} dX_u \\ &= \sum_{i=0}^{p-1} X_s^i \int_s^t \int_s^u dX_v X_s^{p-1-i} dX_u .\end{aligned}$$

Let $\mathcal{A}_s$ be the algebra generated by $\{X_u :\ 0 \leq u \leq s\}$.

We would like to construct a **Lévy-area-operator**, along the formal expression: for all $s \leq t$ and $U \in \mathcal{A}_s$,

$$\mathbb{X}^2_{s,t}[U] := \int_s^t \int_s^u dX_v \, U dX_u \ .$$

If we can exhibit such an element $\mathbb{X}^2_{s,t}$ (with suitable roughness properties), then we will be in a position to define, if $P(x) = x^p$,

$$\int P(X_t) dX_t \ " := "$$

$$\lim \sum_{t_k} \Big\{ P(X_{t_k})(X_{t_{k+1}} - X_{t_k}) + X^i_{t_k} \, \mathbb{X}^2_{t_k, t_{k+1}}[X^{p-1-i}_{t_k}] \Big\} \ .$$

Construction of the Lévy area

$$X_t^{(n)} := X_{t_i^n} + 2^n(t - t_i^n)\{X_{t_{i+1}^n} - X_{t_i^n}\} \quad \text{for } n \geq 1 \text{ and } t \in [t_i^n, t_{i+1}^n] \ .$$

Then we set

$$\mathbb{X}_{st}^{2,(n)}[U] := \int_s^t \int_s^u \mathrm{d}X_v^{(n)} U \mathrm{d}X_u^{(n)} \ , \quad 0 \leq s \leq t \ .$$

Construction of the Lévy area

Proposition (Deya-S.): Assume that $H > \frac{1}{4}$. Then, for all $0 \leq s \leq t$ and $U \in \mathcal{A}_s$, the sequence $\mathbb{X}^{2,(n)}_{st}[U]$ converges in $\mathcal{A}$ as $n \to \infty$. The limit, that we denote by $\mathbb{X}^2_{st}[U]$, satisfies the following properties:

(i) For all $0 \leq s \leq u \leq t \leq 1$ and $U \in \mathcal{A}_s$,

$$\mathbb{X}^2_{st}[U] - \mathbb{X}^2_{su}[U] - \mathbb{X}^2_{ut}[U] = (X_u - X_s)U(X_t - X_u) \ .$$

(ii) There exists a constant $c_H > 0$ such that for all $0 \leq s \leq t \leq 1$, $m \geq 0$ and $0 \leq u_j \leq v_j \leq s$ ($j = 1, \dots, m$),

$$\left\| \mathbb{X}^2_{st}\left[(X_{v_1} - X_{u_1}) \cdots (X_{v_m} - X_{u_m})\right] \right\| \leq c_H^m \, |t-s|^{2H} \prod_{j=1,\dots,m} |u_j - v_j|^H \ .$$

Prop.: Fix $H \in (\frac{1}{3}, \frac{1}{2}]$, and let P be a polynomial. For all $0 \leq s \leq t$ and every subdivision $\Delta_{st} = \{t_0 = s < t_1 < \ldots < t_\ell = t\}$ of $[s, t]$ with mesh $|\Delta_{st}|$ tending to 0, the corrected Riemann sum

$$\sum_{t_i \in \Delta_{st}} \Big\{ P(X_{t_i})(X_{t_{i+1}} - X_{t_i}) + (\partial P(X_{t_i}) \sharp \mathbb{X}^2_{t_i t_{i+1}}) \Big\}$$

converges in $\mathcal{A}$ as $|\Delta_{st}| \to 0$. The limit, that we denote by $\int_s^t P(X_u) dX_u$, is such that

$$\int_s^t P(X_u^{(n)}) dX_u^{(n)} \xrightarrow{n \to \infty} \int_s^t P(X_u) dX_u \quad \text{in } \mathcal{A} .$$

We can extend this construction to define $\int P(X_t) dX_t Q(X_t)$, for all polynomials P, Q. Then, if $P(x) = x^p$, one has

$$P(X_t) - P(X_s) = \sum_{i=0}^{p-1} \int_s^t X_u^i dX_u X_u^{p-1-i} .$$

$H \leq \frac{1}{3}$

When $H \in (\frac{1}{4}, \frac{1}{3}]$, we can (certainly) extend the previous considerations through the involvement of some third-order object, morally

$$\mathbb{X}^3_{s,t}[U,V] := \int_s^t \int_s^u \int_s^v dX_w \, U \, dX_v \, V \, dX_u \ , \ \ U, V \in \mathcal{A}_s \ .$$

Proposition: In a NC probability space $(\mathcal{A}, \varphi)$, consider a NC-fractional Brownian motion $\{X_t\}_{t \geq 0}$ of Hurst index $H \leq \frac{1}{4}$. Then

$$\|\mathbb{X}^{2,(n)}_{01}[1]\| \xrightarrow{n \to \infty} \infty \ .$$

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(i) For all $0 \leq s \leq u \leq t \leq 1$ and $U \in \mathcal{A}_s$,

$$\mathbb{X}^2_{st}[U] - \mathbb{X}^2_{su}[U] - \mathbb{X}^2_{ut}[U] = (X_u - X_s)U(X_t - X_u) \ .$$

(ii) There exists a constant $c_H > 0$ such that for all $0 \leq s \leq t \leq 1$ and $U \in \mathcal{A}_s$,

$$\left\| \mathbb{X}^2_{st}[U] \right\| \leq c_H |t - s|^{2H} \| U \| \ .$$

Consequences

When $H = \frac{1}{2}$, we can extend the previous construction and define the more general integral

$$\int f(Y_t) dX_t g(Y_t)$$

for a large class of functions $f, g : \mathbb{C} \to \mathbb{C}$ and for a suitable class of **controlled paths** $Y : [0, T] \to \mathcal{A}$.

With this definition in hand, we can solve the equation

$$dY_t = f(Y_t) \cdot dX_t \cdot g(Y_t).$$

Continuity: $Y = \Phi(X, \mathbb{X}^{\mathbf{2}})$, with Φ continuous.

Related publications

A. Deya and R. S.
On the rough-paths approach to non-commutative stochastic calculus.
Journal of Functional Analysis, Vol 265, Issue 4, 594-628, 2013.

A. Deya and R. S.
Integration with respect to non-commutative fractional Brownian motion.
To appear in Bernoulli.

Thanks!