Lectures on the Homological Conjectures

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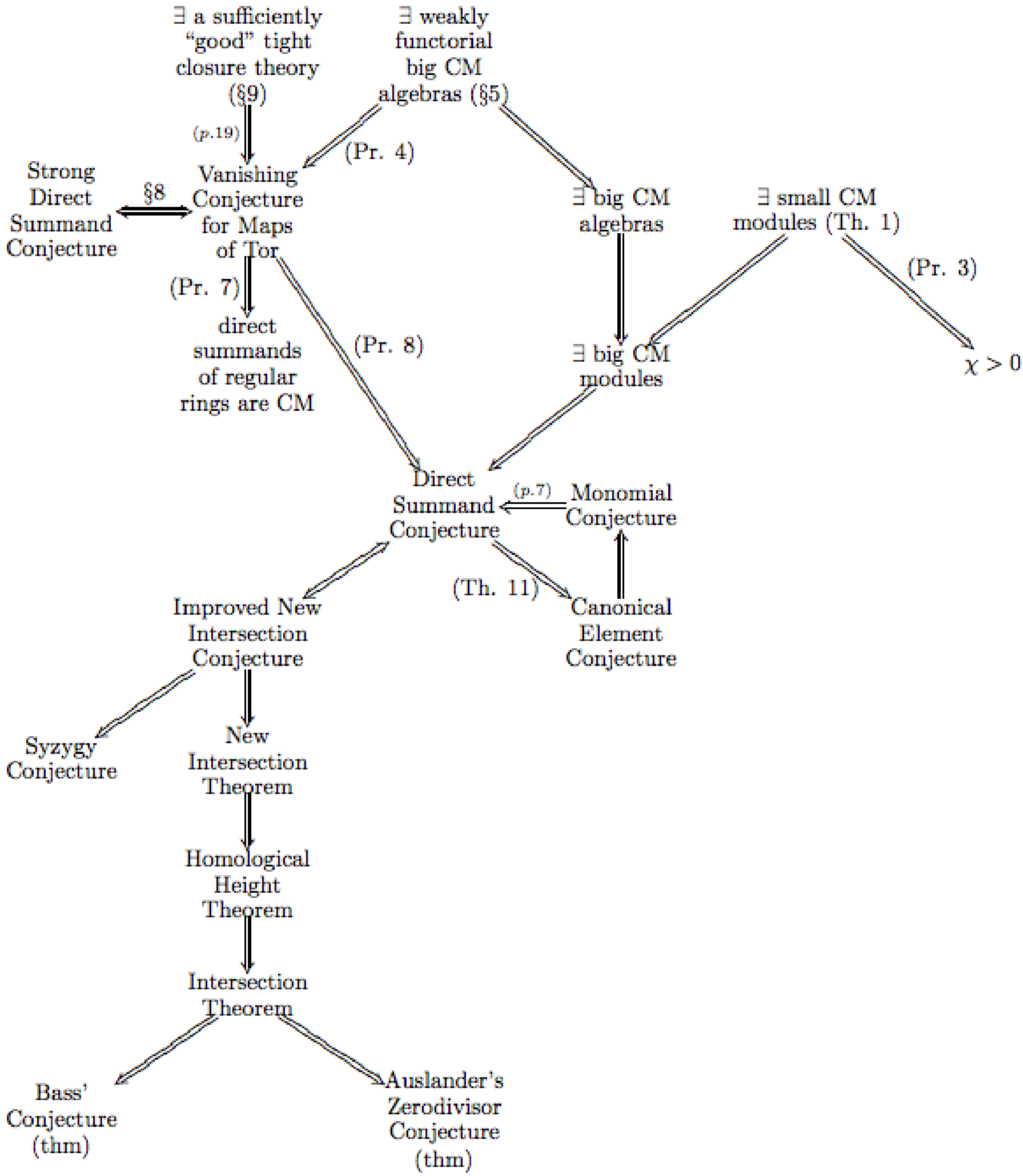
Many thanks to the organizers for the honor of giving this series of lectures, in this lovely place.

My goal in my talks is to explain most of the nodes in the diagram on the next page, and some of the arrows. Many of the nodes in this diagram are now (unconditional) theorems, some of them very recently (August 2016). I will not try to bring the story up to the present day. I think the techniques used around these conjectures in the 50 years 1965-2015 are worth knowing, even if they are not the ones that ended up solving the problems in general.

Caution: these notes contain errors. Some were fixed during the lectures, others surely remain.



(From Hochster, "Current state of the homological conjectures", 2004.)

Lecture 1: Primary decomposition, prime avoidance, and depth via Ext

Throughout, we work with Noetherian rings R. A local ring $(R, \mathfrak{m}, k)$ is a Noetherian ring R with unique maximal ideal $\mathfrak{m}$ and residue field $k = R/\mathfrak{m}$.

This lecture is meant to be a high-speed review of primary decomposition, associated primes, and prime avoidance, with an eye toward the connections with depth.

Definition. An ideal I of R is primary if whenever $ab \in I$, we have either $a \in I$ or $b^n \in I$ for some n. Equivalently, whenever $ab \in I$, either $a \in I$ or $b \in \sqrt{I}$.

Observation. 1. Prime ideals are obviously primary.

2. It's easy to see that if $\mathfrak{p}$ is a prime ideal then $\mathfrak{p}^n$ is primary for any n. However, this is not the only source of primary ideals, as we will see in a moment.

3. If I is primary then $\sqrt{I}$ is a prime ideal. When I is a primary ideal with $\sqrt{I} = \mathfrak{p}$,, we say I is $\mathfrak{p}$-primary. The converse is not true (consider $I = (x^2, xy) \subset k[x, y]$; its radical is (x), but it is not primary).

4. However, if $\sqrt{I}$ is a maximal ideal then I is primary. So an ideal I is $\mathfrak{m}$-primary iff $\sqrt{I} = \mathfrak{m}$. For example, $I = (x^2, y^5) \subset k[x, y]$ is (x, y)-primary. (One can also prove directly that this ideal is primary.) Notice that this is not a power of a prime ideal.

Exercise. The primary ideals of $\mathbb{Z}$ are precisely the ideals (p^m), where p is a prime integer and $m \geqslant 1$. The radical of such an ideal is just (p).

Consequence. *Any ideal $I \subset \mathbb{Z}$ is uniquely an intersection of primary ideals with distinct radicals: $I = (p_1^{m_1}) \cap \cdots \cap (p_r^{m_r})$ for distinct primes $p_1, \ldots, p_r$.*

Primary decomposition is a sweeping generalization of this example.

Theorem (Noether). *Let R be Noetherian and I an ideal. Then I is a finite intersection of primary ideals.*

Sketch of Proof. Say that an ideal Q is irreducible if it cannot be written as a proper intersection of two other ideals. Note that irreducible ideals are primary. Let

$$\Gamma = \{I \subset R \mid I \text{ is } \underline{\text{not}} \text{ a finite intersection of irred ideals}\} .$$

If $\Gamma \neq \emptyset$, then it has a maximal element I, which must be reducible, but then it is an intersection of larger ideals, which must be irreducible, contradiction. $\square$

Definition. A primary decomposition of I is $I = Q_1 \cap \cdots \cap Q_r$, where each Q_i is a primary ideal. Say the decomposition is irredundant if no Q_i can be omitted.

Theorem/Definition (Noether). The prime ideals $\sqrt{Q_i}$ are uniquely determined by I, though the Q_i themselves need not be. These prime ideals are called the associated primes of I, written $\operatorname{Ass}_R(R/I)$. The associated primes are exactly those of the form $\mathfrak{p} = \operatorname{Ann}_R(\overline{x})$ for some $\overline{x} \in R/I$; equivalently they are the ones for which we have $R/\mathfrak{p} \hookrightarrow R/I$.

The Q_i corresponding to the minimal elements of $\operatorname{Ass}_R(R/I)$ are uniquely determined, and appear in every primary decomposition of I (called the isolated components).

Example. $(x^2, xy) = (x) \cap (x^2, y) = (x) \cap (x^2, xy, y^2)$ are two primary decompositions of $I = (x^2, xy)$. The associated primes are (x) and (x, y).

Fact. All this can be generalized to submodules N of a Noetherian module M. Every such N can be written as an intersection of primary submodules Q_i, and the associated primes of M/N are $\sqrt{\operatorname{Ann}_R(M/Q_i)}$. As before, they are precisely the primes $\mathfrak{p}$ such that $R/\mathfrak{p} \hookrightarrow M$.

Corollary. *Let M be a finitely generated module over a Noetherian ring R. The set of zerodivisors on M is the union of the associated primes of M:*

$$\mathcal{Z}(M) = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p}\,.$$

Special case: when $(R, \mathfrak{m})$ is a local ring, the module M has depth 0 if and only if $\mathfrak{m} \subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R(M)} \mathfrak{p}$. In fact this is equivalent to $\mathfrak{m} \in \operatorname{Ass}_R(M)$, by the next result.

Theorem (Prime Avoidance)**.** *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be prime ideals of a Noetherian ring R. (Actually, one can get away with only $n-2$ of them being prime.) If I is an ideal such that*

$$I \subseteq \bigcup_{i=1}^{n} \mathfrak{p}_i\,,$$

then $I \subseteq \mathfrak{p}_i$ for some i.

Support: For any ideal I of a ring R, let $V(I)$ be the set of prime ideals containing I.

The support of a module M is the set of prime ideals $\mathfrak{p}$ so that $M_{\mathfrak{p}} \neq 0$. We always have $V(\operatorname{Ann}_R M) \subseteq \operatorname{Supp} M$, that is, if a prime $\mathfrak{p}$ contains $\operatorname{Ann}_R M$ then $M_{\mathfrak{p}} \neq 0$. If M is finitely generated, then equality holds.

Special case: If $(R, \mathfrak{m})$ is a local ring and M is a finitely generated module, then $\operatorname{Supp}_R M = \{\mathfrak{m}\}$ if and only if M has finite length.

From our point of view, the main reason to cover the preceding material is the Ext-criterion for computing depth.

First, a turbo review of Ext and Tor, with no proofs. (I will skip most of this in lecture, but include it here in these notes.) Fix a ring R and two R-modules M and N.

Ext: The functor $\mathrm{Hom}_R(M, -)$ is a covariant left-exact functor, meaning that if

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is a short exact sequence of R-modules, then

$$0 \longrightarrow \mathrm{Hom}_R(M, A) \xrightarrow{f_*} \mathrm{Hom}_R(M, B) \xrightarrow{g_*} \mathrm{Hom}_R(M, C)$$

is also an exact sequence of R-modules. (But we cannot always put a 0 at the far right.) Here the R-module structure on, for example, $\mathrm{Hom}_R(M, A)$ is defined by

$$(r\varphi)(m) = \varphi(rm) = r(\varphi(m))$$

and the induced maps are defined by, for example,

$$f_*(\varphi\colon M \longrightarrow A) = f \circ \varphi\colon M \longrightarrow B\,.$$

Therefore, by general categorical nonsense, there exist right derived functors $\mathrm{Ext}^i_R(M, -)$ for all $i = 0, 1, \ldots,$ which are defined on a module N by applying $\mathrm{Hom}_R(M, -)$ to an injective resolution of N, and taking homology of the resulting complex.

If M is projective, then in fact $\mathrm{Hom}_R(M, -)$ is an exact functor (we can put in the 0 on the right). (This is basically the definition of projectivity.) It follows that $\mathrm{Ext}^i_R(M, N) = 0$ for all $i > 0$ and all N.

OTOH, $\mathrm{Hom}_R(-, N)$ is a contravariant left-exact functor, so transforms a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

into

$$0 \longrightarrow \mathrm{Hom}_R(C, N) \xrightarrow{g^*} \mathrm{Hom}_R(B, N) \xrightarrow{f^*} \mathrm{Hom}_R(A, N)$$

where, for example, $g^*(\varphi\colon C \longrightarrow N) = \varphi \circ g\colon B \longrightarrow N$.

Therefore, the same categorical nonsense gives us right derived functors $\mathrm{Ext}^i_R(-, N)$ for all $i = 0, 1, \dots$. They are defined on a module M by applying $\mathrm{Hom}_R(-, N)$ to a projective resolution of M and taking homology of the resulting complex.

If N is injective, then $\mathrm{Hom}_R(-, N)$ is exact, so $\mathrm{Ext}^i_R(M, N) = 0$ for $i > 0$ and all M.

Fact. $\mathrm{Ext}^i_R(M, N) = \mathrm{Ext}^i_R(M, N)$. That is, you can compute Ext via either a projective resolution of M or an injective resolution of N.

Properties.

- $\mathrm{Ext}^0_R(M, N) = \mathrm{Hom}_R(M, N)$.
- If M is projective, or N is injective, then $\mathrm{Ext}_R(M, N) = 0$ for all $i > 0$. More generally, if M has projective dimension $\leqslant n$ (resp., N has injective dimension $\leqslant n$) then $\mathrm{Ext}^i_R(M, N) = 0$ for $i > n$.
- elements of $\mathrm{Ext}^1_R(M, N)$ are in 1-1 correspondence with equivalence classes of "extensions" (short exact sequences)

  $$0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0$$

  under a certain equivalence relation (which is stronger than isomorphism of exact sequences $\Longrightarrow$ smaller equivalence classes than isomorphism classes).
- (most important one) There is a long exact sequence of Ext: for any short exact sequence

  $$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

of R-modules, there is an exact sequence

$$
\begin{aligned}
0 \longrightarrow \mathrm{Hom}_R(M,A) &\longrightarrow \mathrm{Hom}_R(M,B) \longrightarrow \mathrm{Hom}_R(M,C) \longrightarrow \\
\longrightarrow \mathrm{Ext}^1_R(M,A) &\longrightarrow \mathrm{Ext}^1_R(M,B) \longrightarrow \mathrm{Ext}^1_R(M,C) \longrightarrow \\
\longrightarrow \cdots \qquad\qquad &\qquad\qquad \cdots \longrightarrow \mathrm{Ext}^i(M,C) \longrightarrow \\
\longrightarrow \mathrm{Ext}^{i+1}_R(M,A) &\longrightarrow \mathrm{Ext}^{i+1}_R(M,B) \longrightarrow \cdots
\end{aligned}
$$

and another one

$$
\begin{aligned}
\cdots \longrightarrow \mathrm{Ext}^i_R(B,N) &\longrightarrow \mathrm{Ext}^i(A,N) \longrightarrow \\
\longrightarrow \mathrm{Ext}^{i+1}_R(C,N) &\longrightarrow \mathrm{Ext}^{i+1}_R(B,N) \longrightarrow \cdots .
\end{aligned}
$$

Tor: The functor $M \otimes_R -$ is right-exact and covariant. If M is projective, it is exact. (If it is exact, we define M to be flat.)

The usual abstract nonsense therefore provides us with left derived functors $\mathrm{Tor}_i^R(M,-)$ for all $i = 0, 1, \ldots$. We compute $\mathrm{Tor}^i_R(M,N)$ by applying $M \otimes_R -$ to a projective resolution of N and taking homology of the resulting complex. (Actually, we can take any flat resolution of N.)

Properties.

- $\mathrm{Tor}_0^R(M,N) = M \otimes_R N$.
- $\mathrm{Tor}_i^R(M,N) = \mathrm{Tor}_i^R(N,M)$, that is, we can compute it via projectively resolving either M or N. This is surprisingly difficult to prove.
- If M has projective dimension $\leqslant n$ (or even flat dimension $\leqslant n$), then $\mathrm{Tor}_i^R(M,N) = 0$ for all $i > n$ and every N.

- There is once again a long exact sequence (love typesetting these) associated to every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$:

$$\cdots \longrightarrow \mathrm{Tor}^R_{i+1}(M,B) \longrightarrow \mathrm{Tor}^R_{i+1}(M,C) \longrightarrow \mathrm{Tor}^R_i(M,A) \longrightarrow \cdots$$

$$\cdots \longrightarrow \mathrm{Tor}^R_1(M,A) \longrightarrow \mathrm{Tor}^R_1(M,B) \longrightarrow \mathrm{Tor}^R_1(M,C) \longrightarrow M \otimes_R A \longrightarrow M \otimes_R B \longrightarrow M \otimes_R C \longrightarrow 0$$

One more thing.

Definition. A functor F from R-modules to R-modules is called *multiplicative* if for any $x \in R$,

$$F(M \xrightarrow{x} M) \quad = \quad F(M) \xrightarrow{x} F(M)\,.$$

In other words, F applied to multiplication by x is again multiplication by x.

Examples.

1. $M \otimes_R -$ is multiplicative.
2. $\mathrm{Hom}_R(M,-)$ and $\mathrm{Hom}_R(-,N)$ are both multiplicative.

Fact. If F is multiplicative and has left or right derived functors, then those are multiplicative as well.

So, applying $\mathrm{Ext}^i_R(M,-)$ to $N \xrightarrow{x} N$ gives

$$\mathrm{Ext}^i_R(M,N) \xrightarrow{x} \mathrm{Ext}^i_R(M,N)\,,$$

and similarly for the other variable, as well as Tor.

Consequence. $\operatorname{Ann}_R(\operatorname{Ext}^i_R(M,N)) \supseteq \operatorname{Ann}_R M + \operatorname{Ann}_R N$ *and* $\operatorname{Ann}_R(\operatorname{Tor}^R_i(M,N)) \supseteq \operatorname{Ann}_R M + \operatorname{Ann}_R N$.

Theorem. *Let R be a Noetherian ring, I an ideal, and M a finitely generated module such that $IM \neq M$. Then the following are equivalent.*

1. $\operatorname{Ext}^i_R(R/I, M) \neq 0$ *for* $i = 0, 1, \dots, n-1$
2. *There is an M-regular sequence $x_1, \dots, x_n$ in I.*

Sketch of Proof. For $n = 1$, use associated primes and prime avoidance to show that $\operatorname{Hom}_R(R/I, M) \neq 0$ iff every element of I is a zerodivisor on M. One direction is straightforward; for the other, if $I \subseteq \mathcal{Z}(M)$ then $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R(M)$, so $R/I \twoheadrightarrow R/\mathfrak{p} \hookrightarrow M$ is a nonzero homomorphism.

For the inductive step, use the long exact sequence of Ext arising from a short exact sequence $0 \longrightarrow M \xrightarrow{x_1} M \longrightarrow M/x_1M \longrightarrow 0$. $\square$

Corollary. *If $(R, \mathfrak{m})$ is a local ring, then the depth of a finitely generated module M is*

$$\operatorname{depth} M = \inf \left\{ i \geqslant 0 \;\middle|\; \operatorname{Ext}^i_R(R/\mathfrak{m}, M) \neq 0 \right\} .$$

Quick aside on the Koszul complex

Throughout R is a ring.

Definition. Let $x \in R$. The Koszul complex on x is

$$K_\bullet(x; R) : 0 \longrightarrow R \xrightarrow{x} R \longrightarrow 0\,,$$

indexed homologically (decreasing indices left to right) in degrees 1 and 0.

Notice two things immediately:

1. $H_0(K_\bullet(x;R)) = R/(x)$;
2. $H_1(K_\bullet(x;R)) = 0$ if and only if x is a nonzerodivisor in R.

To extend the construction to more than one element, a quick reminder on tensor products of complexes. To ease the notation a bit, we suppress the ring on all $\otimes$s

Definition. Let $C_\bullet : \cdots \longrightarrow C_n \xrightarrow{d_C} C_{n-1} \longrightarrow \cdots$ and $D_\bullet : \cdots \longrightarrow D_n \xrightarrow{d_D} D_{n-1} \longrightarrow \cdots$ be two complexes of R-modules. Define a complex $C_\bullet \otimes D_\bullet$ by

$$(C_\bullet \otimes D_\bullet)_n = \bigoplus_{p+q=n} C_p \otimes D_q$$

and, if $x \in C_p$, $y \in D_q$, then the differential is defined by

$$d_{C\otimes D}(x \otimes y) = d_C(x) \otimes y + (-1)^q x \otimes d_D(y)\,.$$

Exercise. $d^2_{C\otimes D} = 0$.

Definition. For a sequence of elements $x_1, \ldots, x_n \in R$, the Koszul complex is defined inductively:

$$K_\bullet(x_1,\ldots,x_n;R) = K_\bullet(x_1,\ldots,x_{n-1};R) \otimes K_\bullet(x_n;R)\,;$$

equivalently,

$$K_\bullet(x_1,\ldots,x_n;R) = \bigotimes_{i=1}^{n} K_\bullet(x_i;R)\,.$$

If M is any R-module, then we set $K_\bullet(x_1,\ldots,x_n;M) = K_\bullet(x_1,\ldots,x_n;R) \otimes M$.

Easy computation: the module in degree i in the Koszul complex $K_\bullet(x_1,\ldots,x_n;R)$ is free of rank $\binom{n}{i}$.

Write $H_i(x_1, \ldots, x_n; M)$ for the homology of $K_\bullet(x_1, \ldots, x_n; M)$, and call it the Koszul homology. The main result we need here is the depth-sensitivity of Koszul homology:

Proposition. *Let R be a Noetherian ring, M a finitely generated module, and $\underline{x} = x_1, \ldots, x_n$ a sequence of elements of R.*

1. $H_0(\underline{x}; M) = M/\underline{x}M$;
2. $H_n(\underline{x}; M) = \operatorname{Ann}_R(\underline{x})$;
3. *If $\underline{x}$ is an M-regular sequence, then $H_i(\underline{x}; M) = 0$ for all $i \geqslant 1$.*
4. *Assuming that $\underline{x}$ is contained in $\operatorname{rad}(R)$, the converse holds.*

The ideal $(\underline{x})$ contains an M-regular sequence of length g if and only if $H_i(\underline{x}; M) = 0$ for $i > n - g$.

Buchsbaum-Eisenbud Acyclicity Criterion

Later we will need a very general form of the Acyclicity theorem, but here is the basic version. Let R be a ring. A complex

$$G_\bullet : \cdots \longrightarrow G_m \xrightarrow{d_m} G_{m-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{d_1} G_0 \longrightarrow 0$$

is called acyclic if $H_i(G_\bullet) = 0$ for all $i > 0$. It is split acyclic if the image of each d_m is a direct summand of G_{m-1}.

Theorem. *Let R be a Noetherian ring and*

$$F_\bullet : 0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

a complex of finite free R-modules. Set $r_i = \sum_{ji}^{s} (-1)^{j-i} \operatorname{rank} F_j$. (This is called the expected rank of the map φ_i.) The following are equivalent.

1. $F_\bullet$ is acyclic.

2. For each $i = 1, \ldots, s$, the ideal $I_{r_i}(\varphi_i)$ contains a regular sequence of length i. (Here, if X is a matrix, $I_t(X)$ is the ideal generated by $t \times t$ minors of X.)

Lectures 2-3: Basics of Algebraic Geometry and Intersection Theorems

In this lecture, we work over a field k.

Definition. Affine n-space over k is $\mathbb{A}^n_k = k^n$, the n-dimensional k-vector space, viewed as a geometric space. Points of this space have coordinates: $p = (a_1, \ldots, a_n)$.

Let S be any subset of the polynomial ring $k[x_1, \ldots, x_n]$. The vanishing set of S is

$$V(S) = \{p = (a_1, \ldots, a_n) \in \mathbb{A}^n_k \mid f(a_1, \ldots, a_n) = 0 \text{ for all } p \in S\} .$$

On the other hand, for any subset $U \subseteq \mathbb{A}^n_k$, the ideal of U is

$$I(U) = \{\text{polynomials } f \in k[x_1, \ldots, x_n] \mid f(a_1, \ldots, a_n) = 0 \text{ for all } (a_1, \ldots, a_n) \in U\} .$$

An (affine) variety is any subset of $\mathbb{A}^n_k$ of the form $V(S)$.

Observation.

1. One checks easily that $I(U)$ is always an ideal. In fact $I(U)$ is always a radical ideal (since k has no nilpotent elements).
2. $V(S) = V(\langle S \rangle)$ is determined by the ideal generated by S. By Hilbert's Basis Theorem, every ideal of the polynomial ring is finitely generated. It follows that an affine variety is the set of common zeros of a finite set of polynomials.
3. The functions $V(-)$ and $I(-)$ are order-reversing: if $S \subseteq S'$ then $V(S) \supseteq V(S')$, and if $U \subseteq U'$ then $I(U) \supseteq I(U')$.
4. The compositions IV and VI act like "closures": $I(V(S)) \supseteq S$ and $V(I(U)) \supseteq U$. (It follows formally that $I(V(I(U))) = I(U)$ and $V(I(V(S))) = V(S)$, so doing either IV or VI twice is the same as doing it once.)

5. $V(\sum_\alpha I_\alpha) = \bigcap_\alpha V(I_\alpha)$

6. $V(IJ) = V(I) \cup V(J)$.

Theorem (Nullstellensatz)**.** *Assume k is algebraically closed. Then $I(V(I)) = \sqrt{I}$ for every ideal I. Hence there is an order-reversing bijection*

$$\{\textit{algebraic varieties in } \mathbb{A}_k^n\} \longleftrightarrow \{\textit{radical ideals of } k[x_1, \ldots, x_n]\} \ .$$

In particular the maximal ideals of $k[x_1, \ldots, x_n]$ are all of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for a point $(a_1, \ldots, a_n) \in \mathbb{A}_k^n$.

An affine variety V is irreducible if and only if $I(V)$ is a prime ideal. (Exercise.) It follows from primary decomposition (or topology) that any V is a finite union of irreducible varieties, corresponding to the minimal (or associated) primes of its ideal.

Now we can start on one thread of the Homological Conjectures: the Intersection Theorems.

If U and V are two subspaces of a vector space W, then

$$\dim U \cap V \geqslant \dim U + \dim V - \dim W \ .$$

It follows from the Nullstellensatz that an analogue is true for algebraic varieties in $\mathbb{A}_k^n$ (k algebraically closed): loosely speaking, if two algebraic varieties intersect only at the origin, their dimensions cannot be "too large".

Theorem (Serre's Intersection Theorem 1961)**.** *Let $(R, \mathfrak{m})$ be a regular local ring and let $\mathfrak{p}$, $\mathfrak{q}$ be prime ideals. Then*

$$\operatorname{height}(\mathfrak{p} + \mathfrak{q}) \leqslant \operatorname{height} \mathfrak{p} + \operatorname{height} \mathfrak{q} \ .$$

This is not true for prime ideals in arbitrary rings. For example, in $k[x,y,s,t]/(xy-st)$, we have prime ideals $\mathfrak{p}=(x,s)$ and $\mathfrak{q}=(y,t)$ of height one, but $\operatorname{height}\mathfrak{p}+\mathfrak{q}=\operatorname{height}(x,y,s,t)=3$.

Corollary. *Let $(R,\mathfrak{m})$ be a regular local ring and M, N finitely generated modules such that $M\otimes_R N$ has finite length. Then $\dim M+\dim N\leqslant \dim R$.*

Proof. Recall that $M\otimes_R N$ has finite length if and only if $\operatorname{Supp}_R(M\otimes_R N)=\{\mathfrak{m}\}$, if and only if $\operatorname{Ann}_R(M\otimes_R N)$ is $\mathfrak{m}$-primary. Also it's not hard (using the multiplicativity of the tensor product functor) to see that $\sqrt{\operatorname{Ann}_R(M\otimes_R N)}=\sqrt{\operatorname{Ann}_R M+\operatorname{Ann}_R N}$. Take primes $\mathfrak{p}$ and $\mathfrak{q}$ minimal over $\operatorname{Ann}_R M$ and $\operatorname{Ann}_R N$, respectively. Then $\mathfrak{p}+\mathfrak{q}$ is $\mathfrak{m}$-primary, so we have

$$\dim R=\operatorname{height}(\mathfrak{p}+\mathfrak{q})\leqslant \operatorname{height}\mathfrak{p}+\operatorname{height}\mathfrak{q}=2\dim R-\dim M-\dim N\,,$$

and the claimed inequality follows. □

Serre's Intersection Theorem is actually one-third of a cluster of statements of Serre, referred to as the "Multiplicity Conjectures". Stating them gives a nice illustration of why Cohen-Macaulay rings are worth considering, so I take a quick side trip.

Let $f,g\in k[x,y]$ be two polynomials without any common factors. Then $V(f)$ and $V(g)$ are plane curves in $\mathbb{A}^2_k$ without common components. Suppose that these curves intersect at the origin. We want to compute the intersection multiplicity of the curves at this point.

If $k=\mathbb{C}$, we can talk about small complex numbers ϵ, and count the number of distinct intersection points of $f=0$ and $g=\epsilon$ that lie in a small neighborhood of the origin.

For arbitrary k, the same value can be computed algebraically by working in the localized polynomial ring $R = k[x, y]_{(x,y)}$ and computing the length of the module

$$R/(f, g) \cong R/(f) \otimes R/(g)\,.$$

(Localizing is necessary so that we don't count any other points of intersection away from the origin.)

For an easy example, if $f = y - x^2$ and $g = y$ defines the x-axis, then the length is 2.

In higher dimensions, we can mimic this construction. Let $X = V(I)$ and $Y = V(J)$ be affine varieties in $k[x_1, \ldots, x_n]$ with an isolated intersection point at the origin. Set $R = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$, and consider the length of the module

$$R/(I + J) = R/I \otimes R/J\,.$$

Unfortunately, this does not give the same answer as perturbing the equations and counting distinct intersection points.

Example. For $I = (x^3 - w^2y, x^2z - wy^2, xy - wz, y^3 - xz^2)$ and $J = (w, z)$ in $\mathbb{C}[x, y, z, w]$, the "correct" (geometric) answer is 4, while the length of the tensor product is 5.

Serre observed that difference can be reconciled by adding (and subtracting) error terms.

Definition. For I and J as above, the (Serre's) intersection multiplicity of I and J is

$$\chi(R/I, R/J) = \sum_{i=0}^{\dim R} (-1)^i \operatorname{length}(\operatorname{Tor}_i^R(R/I, R/J))\,.$$

More generally, we can replace R/I and R/J by any finitely generated modules M and N to define $\chi(M, N)$.

Serre proved that χ has many of the properties we want from an intersection multiplicity; for example, Bézout's theorem holds, and χ is additive on short exact sequences. Also, the need for the correction terms is attributable to the failure of the Cohen-Macaulay property:

Theorem (Serre). *Let $(R, \mathfrak{m})$ be a localized polynomial ring and I, J ideals such that $I + J$ is $\mathfrak{m}$-primary. Then we have*

$$\chi(R/I, R/J) = \text{length}(R/I \otimes_R R/J)$$

if and only if R/I and R/J are Cohen-Macaulay rings. More generally, let M and N be modules over R/I and R/J, respectively; then $\chi(M, N) = \text{length}(M \otimes_R N)$ if and only if M and N are maximal Cohen-Macaulay modules over each ring respectively.

Conjecture (Serre). *Let R be a regular local ring and M, N finitely generated modules. Assume that $M \otimes_R N$ has finite length. Then*

1. $\dim M + \dim N \leqslant \dim R$ *(We saw this already!);*
2. *(Nonnegativity)* $\chi(M, N) \geqslant 0$*;*
3. *(Vanishing) If* $\dim M + \dim N < \dim R$*, then* $\chi(M, N) = 0$*;*
4. *(Positivity) If* $\dim M + \dim N = \dim R$*, then* $\chi(M, N) > 0$*.*

Serre proved the first statement, as we have seen, for all regular local rings. He proved the rest for "unramified" regular local rings (roughly speaking, localized polynomial rings). Vanishing was proved by Gillet-Soulé and Paul Roberts in the 1980s using K-theory, and Nonnegativity by Gabber in the 90s. Positivity remains open in general.

Looking ahead a bit, the recent advances that prove the Direct Summand Conjecture (perfectoid things) do not seem to apply to Serre's Positivity conjecture.

On the other hand, existence of small CM modules does imply Positivity (big CM modules are not enough).

Back to the Intersection Theorem: $\dim M + \dim N \leqslant \dim R$ for finitely generated modules M and N over a regular local ring R.

It is a standard guess that a theorem for regular local rings might remain true over arbitrary local rings as long as the modules have finite projective dimension.

Conjecture (Peskine-Szpiro '74)**.** *Let R be a local ring and M, N finitely generated modules with $\operatorname{pd}_R M < \infty$ and $M \otimes_R N$ has finite length. Then $\dim M + \dim N \leqslant \dim R$.*

This is still open. We can weaken it a bit by remembering that $\operatorname{depth} \leqslant \dim$, and by the Auslander-Buchsbaum formula, if $\operatorname{pd}_R M < \infty$ then $\operatorname{pd}_R M = \operatorname{depth} R - \operatorname{depth} M$. This gives

Conjecture ("Intersection Theorem", Peskine-Szpiro '74)**.** *Let $(R, \mathfrak{m})$ be local and M, N finitely generated modules such that $\operatorname{pd}_R N < \infty$ and $M \otimes_R N$ has finite length. Then*

$$\dim N \leqslant \operatorname{pd}_R M\,.$$

Example. Suppose $x_1, \ldots, x_k$ is a regular sequence in R. Then $M = R/(x_1, \ldots, x_k)$ is resolved by the Koszul complex, so has $\operatorname{pd}_R M = k$. For any N, we have $M \otimes_R N = N/(x_1, \ldots, x_k)N$. For this to have finite length, the ideal $\operatorname{Ann}_R N + (x_1, \ldots, x_k)$ must be $\mathfrak{m}$-primary. Equivalently, in the ring $R/\operatorname{Ann}_R N$, the ideal $(x_1, \ldots, x_k)$ is primary to the maximal ideal. By Krull's Height Theorem, $\operatorname{height}(x_1, \ldots, x_k) \leqslant k$, so we have $\dim N = \dim R/\operatorname{Ann}_R N \leqslant k = \operatorname{pd}_R M$, which is exactly the Intersection Theorem in this case.

Peskine and Szpiro proved the Intersection Theorem when R contains a field of positive characteristic, and in many cases in characteristic zero by reducing to positive characteristic. (More on this in the next lecture.)

They also used the Intersection Theorem to resolve two (at the time) open conjectures:

Conjecture (Auslander's zerodivisor conjecture)**.** *Let $(R, \mathfrak{m})$ be a local ring and M a nonzero finitely generated module of finite projective dimension. Then every zerodivisor in R is a zerodivisor on M.*

Proof sketch, assuming the Intersection Theorem. We want to show that every associated prime $\mathfrak{p}$ of R is contained in an associated prime of M. If M has finite length, we're done, so induct on $\dim M$. There are two cases. If $\mathfrak{p} \subseteq \mathfrak{q} \subsetneq \mathfrak{m}$ for some $\mathfrak{q} \in \operatorname{Supp} M$, then localize at $\mathfrak{q}$ and we're done by induction. If not, then $\mathfrak{p} + \operatorname{Ann}_R M$ is $\mathfrak{m}$-primary, so by the Intersection Theorem $\dim R/\mathfrak{p} \leqslant \operatorname{pd}_R M$. It is known that $\operatorname{depth} R \leqslant \dim R/\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass} R$, and some arithmetic shows that $\operatorname{depth} M \leqslant 0$. Therefore $\mathfrak{m} \in \operatorname{Ass}_R M$ and we're done. □

Conjecture (Bass' Question)**.** *Let $(R, \mathfrak{m})$ be a local ring and suppose there is a finitely generated R-module E of finite injective dimension. Then R is Cohen-Macaulay.*

(The proof of this one is fairly involved.)

Peskine-Szpiro and Roberts, independently, proved a stronger version of the Intersection Theorem.

Theorem (New Intersection Theorem, Peskine-Szpiro, Roberts)**.** *Let R be a local ring and suppose that $F : 0 \longrightarrow F_s \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$ is* *a complex of finitely generated free R-modules. Assume that all the homology modules have finite length and that $H_0(F) \neq 0$. Then $s \geqslant \dim R$.*

Proof of IT, assuming NIT. Let M be an R-module of finite projective dimension; we want to prove that if $M \otimes_R N$ has finite length, then $\dim N \leqslant \operatorname{pd}_R M$. Let G be a minimal free resolution of M, with $s = \operatorname{pd}_R M$. Set $I = \operatorname{Ann}_R N$, and pass to the ring R/I: set $F = G \otimes_R R/I$. Then $H_0(F) = M \otimes_R N \neq 0$. The homology $H_i(F_\bullet) = \operatorname{Tor}_i^R(M, R/I)$ is annihilated by $\operatorname{Ann}_R M + I$, which is $\mathfrak{m}$-primary since $M \otimes_R N$ has finite length, so the homology has finite length too. By NIT, $s \geqslant \dim R/I = \dim N$, and we win. $\square$

Paul Roberts proved the New Intersection Theorem for all local rings in the 80s. In particular, the Intersection Theorem, Auslander's Conjecture, and Bass' Question are all true in full generality.

Lectures 4-5: The Monomial and Direct Summand Conjectures

This lecture is about two conjectures of Hochster that occupy the central position in our diagram. I want to emphasize two things: they are both easy(ish) for rings containing either $\mathbb{Q}$ or $\mathbb{F}_p$, and are both trivial for Cohen-Macaulay rings. Pursuing this second point will lead us to further conjectures about CM modules.

First the Direct Summand Conjecture.

Conjecture (Direct Summand Conjecture, Hochster '69)**.** *Let A be a regular local ring and $A \subseteq R$ a ring extension such that R is a finitely generated A-module. Then A is a direct summand of R as an A-module (i.e. there is an A-linear map $f\colon R \longrightarrow A$ so that $f(a) = a$ for all $a \in A$).*

I will give proofs of several special cases of this conjecture.

Proof when A contains $\mathbb{Q}$. Let K and L be the fraction fields of A and R, respectively; then $K \subseteq L$ is a finite separable field extension. The trace map $\mathrm{Tr}\colon L \longrightarrow K$ sends $\alpha \in L$ to the sum of its Galois conjugates, so sends each $a \in A$ to da, where $d = [L : K]$. Then $f = \frac{1}{d}\mathrm{Tr}$ gives the desired splitting. □

(Actually this argument works even if A is only normal, rather than regular! It also works if A doesn't contain $\mathbb{Q}$ but you know for some reason that d is invertible in A.)

For the proof when A contains a field of characteristic p, we need two slightly advanced ingredients: the Frobenius endomorphism, and the Cohen Structure Theorem.

Frobenius: If R contains $\mathbb{F}_p$, then the map $F\colon R \longrightarrow R$ with $F(r) = r^p$ is a ring homomorphism. The key point is the "Freshman's Dream": $(a+b)^p = a^p + b^p$. (The mixed terms coming from the binomial theorem all have coefficients divisible by p.) It follows that the ring of p^{th} powers $R^p \subseteq R$ is isomorphic as a ring to R (as long as R has no nilpotents).

Cohen Structure Theorem: We need only a very special case right now, which says that if R is a complete local domain containing a field of characteristic p, then R can be written as $(R/\mathfrak{m})[[Z_1, \ldots, Z_m]]/Q$ for some prime ideal Q. In particular, if R is assumed to be a regular local ring, then it is a power series ring over a field.

Proof when A contains $\mathbb{F}_p$ and R is a domain. We assume $A = k[[z_1, \ldots, z_d]]$, and we further assume that k is perfect (of characteristic p).

We can choose an A-linear map $\varphi\colon R \longrightarrow A$ with $\varphi(1) \neq 0$. Indeed, R is a domain, and we know that the map $Q(A) \longrightarrow Q(R)$ on fraction fields splits, so we can multiply by some nonzerodivisor to get a nonzero map $A \longrightarrow R$.

Take e so large that $\varphi(1) \notin \mathfrak{m}^{p^e}$, where $\mathfrak{m} = (z_1, \ldots, z_d)A$. Set $q = p^e$, and let $B = k[[z_1^q, \ldots, z_d^q]] \subset A$. Since k is perfect, in fact $B = A^q$. Exercise: A is a free B-module (for example, on the monomials $z_1^{a_1} \cdots z_d^{a_d}$, where $1 \leqslant a_i < q$ for each i). Since $\varphi(1) \notin (z_1^q, \ldots, z_d^q)A$, we can extend $\varphi(1)$ to a B-basis for A. It follows that there is a B-linear maps $\psi\colon A \longrightarrow B$ with $\psi(\varphi(1)) = 1$.

Thus $\psi\varphi$ is a B-linear retraction of R to B. Its restriction to R^q is a B-linear retraction of R^q to $B = A^q$. But we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow[\cong]{F^e} & R^q \\ \uparrow & & \uparrow \\ A & \xrightarrow[\cong]{F^e} & A^q = B \end{array}$$

So a B-linear retraction of the right-hand side implies an A-linear retraction of the left-hand side, which is what we wanted. $\square$

Proof when R is CM. If R is a CM ring, then $\operatorname{depth}_R R = \dim R = \dim A$ (since the extension is integral). The depth of R is the same whether considered as R-module or A-module (if we have a regular sequence in R, we can take high powers to get it in A). By the Auslander-Buchsbaum Theorem, $\operatorname{pd}_A R + \operatorname{depth} R = \operatorname{depth} A$, which forces R to be a free A-module. It follows that $1 \in R$ can be extended to a basis, and then there is a projection $R \longrightarrow A$ splitting the given inclusion. $\square$

That's not the end of the proofs of DSC. The proofs above, plus some standard reductions, reduce the problem to considering regular local rings A of the form $\hat{\mathbb{Z}}_p[[z_1, \ldots, z_d]]$, where $\hat{\mathbb{Z}}_p$ is a complete DVR with maximal ideal generated by the integer p.

Heitmann proved the case of dimension 3 ($d = 2$) in 2002 [Annals]. It is an extremely difficult proof, aspects of which foreshadow the very recent proof of the full Conjecture by André. More on this later maybe.

Conjecture (Monomial Conjecture, Hochster 1971)**.** *Let $(R, \mathfrak{m})$ be a local ring of dimension d, and let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters. (So $(\underline{x})$ is $\mathfrak{m}$-primary; it contains a power of $\mathfrak{m}$.) Then for every $t \geqslant 1$, we have*

$$x_1^t \cdots x_d^t \notin (x_1^{t+1}, \ldots, x_d^{t+1})\,.$$

Notice that the Conjecture is true if the x_i's are variables in a polynomial or power series ring. (The only way for a monomial to be contained in a monomial ideal is if it is divisible by one of the generators.) So MC asserts that systems of parameters "behave like" variables in this sense.

We should immediately point out that the Monomial Conjecture is equivalent to the Direct Summand Conjecture. Here we can only give one proof in the case of equal characteristic, and a few words about the general case.

Proposition. *Direct Summand implies Monomial Conjecture for complete local rings containing a field.*

Proof. Suppose we have a complete local ring R with a system of parameters $x_1, \dots, x_d$, and that

$$(x_1 \cdots x_d)^k = \sum_{i=1}^{d} r_i x_i^{k+1}$$

for some k and some elements $r_i \in R$. Set $A = k[[x_1, \dots, x_d]]$, a regular local ring over which R is a finitely generated module. By Direct Summand, there is a splitting $\varphi\colon R \longrightarrow A$ (which in particular fixes the x_i), and we get

$$(x_1 \cdots x_d)^k = \sum_{i=1}^{d} \varphi(r_i) x_i^{k+1} \,,$$

a contradiction. $\square$

The key to the proofs in general is the following fact [Bruns-Herzog, Lemma 9.2.2].

Lemma. Let $(A, \mathfrak{m})$ be a regular local ring and $\underline{x} = x_1, \dots, x_n$ a regular system of parameters. Suppose that $R \supseteq A$ is a module-finite A-algebra. Then A is a direct summand of R as A-module if and only if $(x_1 \cdots x_n)^k \notin (x_1^{k+1}, \dots, x_n^{k+1})$ for every $k \geqslant 0$.

Sketch of proof. ($\Longrightarrow$) If $R \subseteq A$ splits, then $IR \cap A = I$ for every ideal I of A. We know that $x_1^k \cdots x_d^k \notin (x_1^{k+1}, \dots, x_d^{k+1})A$, so it must remain so in R.

($\Longleftarrow$) First reduce to the case where A is complete. Let $A_k = A/(\underline{x}^k)$, a zero-dimensional Gorenstein ring, and $R_k = R/(\underline{x}^k)R$. The socle of A_k is generated by the monomial $(x_1 \cdots x_d)^k$, and this doesn't go to zero in R_k by hypothesis, so the maps $A_k \longrightarrow R_k$ are all injective. Since A_k is self-injective, they all split. One would like to take a limit of these splittings to get a splitting for $A \longrightarrow R$; there turn out to be some significant technicalities, omitted here. $\square$

Characteristic p

MC is easy in characteristic p. I learned the proof below from Huneke. It uses the same two basic tools as the proof of DSC in positive characteristic. Specifically, the "Freshman's Dream" implies that if we have an equation

$$x_1^t \cdots x_d^t = a_1 x_1^{t+1} + \cdots + a_d x_d^{t+1},$$

we can raise it to the p^{th} power, repeatedly, and get

$$(x_1^t \cdots x_d^t)^{p^e} \in (x_1^{(t+1)p^e}, \ldots, x_d^{(t+1)p^e})$$

for all $e \geqslant 1$.

Proof of MC when R is a complete domain and contains a field of characteristic p. Assume for a contradiction that $x_1^t \cdots x_d^t \in (x_1^{t+1}, \ldots, x_d^{t+1})$ for some t.

Write $R = k[[Z_1, \ldots, Z_m]]/Q$ for some prime ideal Q. Lift the elements x_i to $y_1, \ldots, y_d \in S = k[[Z_1, \ldots, Z_m]]$. We can find elements $w_1, \ldots, w_g \in Q$ so that

1. $w_1, \ldots, w_g, y_1, \ldots, y_d$ is a s.o.p. for S (so $m = g + d$), and
2. $(w_1, \ldots, w_g)_Q = Q_Q$ since S_Q is a RLR.

Using (2), choose an element $c \notin Q$ such that $cQ \subseteq (w_1, \ldots, w_g)$.

Now the ring $A = k[[w_1, \ldots, w_g, y_1, \ldots, y_d]] \subseteq S$ is isomorphic to a power series ring (those elements are algebraically independent). Furthermore S is a finitely generated module over A. In fact it is a free module: by Auslander-Buchsbaum, $m = \operatorname{depth} A = \operatorname{pd}_A S + \operatorname{depth}_A S = \operatorname{pd}_A S + \operatorname{depth}_S S = \operatorname{pd}_A S + m$, so that $\operatorname{pd}_A S = 0$. In particular, S is a flat A-algebra.

Lift the equation that shows $x_1^t \cdots x_d^t \in (x_1^{t+1}, \ldots, x_d^{t+1})$ up to S:

$$y_1^t \cdots y_d^t = \sum_{i=1}^{d} s_i y_i^{t+1} + \pi \,,$$

where $\pi \in Q$. Apply the Frobenius e times:

$$y_1^{tp^e} \cdots y_d^{tp^e} = \sum_{i=1}^{d} s_i^{p^e} y_i^{(t+1)p^e} + \pi^{p^e} \,.$$

Multiply by c, knocking π^{p^e} into $(w_1, \ldots, w_g)$:

$$c y_1^{tp^e} \cdots y_d^{tp^e} \in (y_1^{(t+1)p^e}, \ldots, y_d^{(t+1)p^e}, w_1, \ldots, w_g) \,.$$

This says that c is in a certain colon ideal:

$$c \in ((y_1^{(t+1)p^e}, \ldots, y_d^{(t+1)p^e}, w_1, \ldots, w_g) :_S y_1^{tp^e} \cdots y_d^{tp^e}) \,.$$

Now, by flatness, $(IS :_S JS) = (I :_A J)S$, so we get

$$c \in ((y_1^{(t+1)p^e}, \ldots, y_d^{(t+1)p^e}, w_1, \ldots, w_g) :_A y_1^{tp^e} \cdots y_d^{tp^e})S \,.$$

But in A, the elements y_i and w_j are just variables, so this implies

$$c \in (y_1^{(t+1)p^e}, \ldots, y_d^{(t+1)p^e}, w_1, \ldots, w_g)S \,.$$

For this to happen for all e, we must have $c \in (w_1, \ldots, w_g)S \subseteq Q$, a contradiction. $\square$

The other main case where MC is easy is when R is a Cohen-Macaulay ring. In that case, the system of parameters $\underline{x}$ is automatically a regular sequence. We will give the proof next time, assuming only that R has a Cohen-Macaulay module.

What is MC good for? (Aside from being equivalent to Direct Summand.) Well, it implies all the Intersection Theorems discussed in previous lectures.

Theorem (Hochster)**.** *The Monomial Conjecture implies the (Improved) New Intersection Conjecture.*

Lecture 6: Cohen-Macaulay modules and algebras

A big Cohen-Macaulay module for a local ring $(R, \mathfrak{m})$ is a (not necessarily finitely generated) module M such that $\mathfrak{m}M \neq M$ and every system of parameters for R is a regular sequence on M. If M is finitely generated, it is enough that $M \neq 0$ and some system of parameters is M-regular; in this case we call M a small Cohen-Macaulay module.

An R-algebra B is a big CM algebra if it is big CM as an R-module.

Existence of CM modules (big or small) implies nearly all of the conjectures we have seen so far. In this lecture I'll illustrate this. First, what is known unconditionally?

1. Small CM modules exist in dimension 0 (any module is small CM), 1 (take $M = R/\mathfrak{p}$ for some prime $\mathfrak{p}$), and for complete local rings in dimension 2 (take the integral closure of $R/\mathfrak{p}$, which is finitely generated since R is complete).

2. Existence of a small CM module forces the ring to be catenary (all saturated chains of primes between two primes have the same length). Complete local rings are catenary, so the usual form of the conjecture is that every complete local ring has a small CM module.

3. Big CM modules and algebras exist for local rings containing a field [Hochster '75, '94], in dimension $\leqslant 3$ [Hochster '02, after Heitman], and explicitly in characteristic p (as the integral closure R^+ of R in an algebraic closure of its fraction field) [Hochster-Huneke '92]. They are now know to exist for arbitrary local rings [André '16].

4. A weakly functorial version of big CM algebras are asserted to exist by André,

but details are not given.

5. Small CM modules exist for complete local domains of dimension $\leqslant 2$ (easy: if R is such a ring then the integral closure of R in its fraction field is a small CM module/algebra). They also exist for graded isolated singularities [Hartshorne; Peskine-Szpiro].

Proposition. *If complete local domains have small CM modules, then Serre's positivity conjecture is true.*

Proof. Recall that we want to prove $\chi(M,N) > 0$ when M and N are finitely generated modules over the regular local ring A such that $\dim M + \dim N = \dim A$. We know that χ is additive on exact sequences, and that M and N have filtrations with successive quotients of the form $A/\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} A$. We therefore reduce to showing $\chi(A/\mathfrak{p}, A/\mathfrak{q}) > 0$ when $\mathfrak{p}$ and $\mathfrak{q}$ are primes such that $\mathfrak{p} + \mathfrak{q}$ is primary to the maximal ideal of A.

Now, the rings $A/\mathfrak{p}$ and $A/\mathfrak{q}$ have small CM modules B and C, respectively, of ranks b and c respectively (we may complete A without changing anything about χ). It follows that there are short exact sequences

$$0 \longrightarrow (A/\mathfrak{p})^b \longrightarrow B \longrightarrow B' \longrightarrow 0$$

and

$$0 \longrightarrow (A/\mathfrak{q})^c \longrightarrow C \longrightarrow C' \longrightarrow 0$$

where B' and C' have dimension strictly smaller than B and C. Now use the Vanishing part of Serre's conjectures (which is known) to get

$$\chi(B,C) = bc\ \chi(A/\mathfrak{p}, A/\mathfrak{q})\,.$$

But since B and C are CM the higher Tors vanish, so $\chi(B,C) = \operatorname{length}(B \otimes_A C) > 0$, and so $\chi(A/\mathfrak{p}, A/\mathfrak{q}) > 0$, as desired. $\square$

Existence of big CM modules (or even algebras) does not seem enough to give Serre's Conjecture. So, as Mel says, small ones are better.

Next, we show that big CM modules imply the Monomial Conjecture.

Proposition. *Let S be a local ring and $y_1, \ldots, y_n$ a system of parameters for S. Suppose there is an S-module E (not necessarily finitely generated) such that $(y_1, \ldots, y_n)E \neq E$ and the first Koszul homology module $H_1(y_1, \ldots, y_n; E)$ vanishes (in particular this is true if $y_1, \ldots, y_n$ is a regular sequence on E). Then for every $k \geqslant 1$,*

$$(y_1 \cdots y_n)^k \notin (y_1^{k+1}, \ldots, y_n^{k+1}) \, .$$

Proof. Let $B = \mathbb{Z}[X_1, \ldots, X_n]$, and make S into a B-algebra by mapping each X_i to y_i. Note that we may think of $\mathbb{Z} = B/(X_1, \ldots, X_n)$ as a B-module, and it is resolved by the Koszul complex. It follows that

$$H_1(y_1, \ldots, y_n; E) \cong \operatorname{Tor}_1^B(\mathbb{Z}, E) \, .$$

Let $I = (X_1^k, \ldots, X_n^k) \subset B$ and $J = (X_1^k, \ldots, X_n^k, (X_1 \cdots X_n)^{k+1}) \subset B$. It's easy to check that $J/I \cong \mathbb{Z}$ as B-modules.

Let I_0 be any ideal of B generated by monomials in the X's which contains a power of each X_i. We will show that B/I_0 has a filtration in which each successive quotient is a direct sum of copies of $\mathbb{Z}$. Indeed, as long as $I_0 \neq B$ there is a monomial $m \notin I_0$ such that $mX_1, \ldots, mX_n \in I_0$, and then $(I_0 + mB)/I_0 \cong \mathbb{Z}$. Continue by Noetherian induction on I_0, since $I_0 + mB$ is another ideal of the same form.

Since by hypothesis $\operatorname{Tor}_1^B(\mathbb{Z}, E) = 0$, we see that also $\operatorname{Tor}_1^B(I_0, E) = 0$ for all ideals I_0 as above. Thus the short exact sequence $0 \longrightarrow I_0 \longrightarrow B \longrightarrow B/I_0 \longrightarrow 0$ induces

$$0 \longrightarrow I_0 \otimes_B E \longrightarrow B \otimes_B E \longrightarrow (B/I_0) \otimes_B E \longrightarrow 0$$

and since $(B/I_0) \otimes_B E \cong E/I_0E$, it follows that $I_0 \otimes_B E \cong I_0E$.

Specifically it follows that $I \otimes_B E = IE$ and $J \otimes_B E = JE$. But we also have the short exact sequence $0 \longrightarrow I \longrightarrow J \longrightarrow J/I \longrightarrow 0$, and since $\mathrm{Tor}_1^B(\mathbb{Z}, E) = 0$ we get

$$0 \longrightarrow IE \longrightarrow JE \longrightarrow \mathbb{Z} \otimes_B E \longrightarrow 0\,.$$

Now $\mathbb{Z} \otimes_B E = E/(X_1, \ldots, X_n)E = E/(y_1, \ldots, y_n)E \neq 0$ by hypothesis, which shows that $IE \subsetneq JE$. In particular $(y_1, \ldots, y_n)^k E \not\subseteq (y_1^{k+1}, \ldots, y_n^{k+1})E$, and therefore $(y_1, \ldots, y_n)^k \notin (y_1^{k+1}, \ldots, y_n^{k+1})$. $\square$

Finally, big CM modules also imply all the Intersection Conjectures. Showing this requires a bit of a digression on the Buchsbaum-Eisenbud Acyclicity Criterion. Recall the basic statement:

Theorem. *Let R be a Noetherian ring and*

$$F_\bullet : 0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

a complex of finite free R-modules. Set $r_i = \sum_{ji}^{s} (-1)^{j-i} \operatorname{rank} F_j$. (This is called the expected rank of the map φ_i.) The following are equivalent.

1. *$F_\bullet$ is acyclic.*
2. *For each $i = 1, \ldots, s$, the ideal $I_{r_i}(\varphi_i)$ contains a regular sequence of length i. (Here, if X is a matrix, $I_t(X)$ is the ideal generated by $t \times t$ minors of X.)*

It't perhaps not surprising that there is a version of this that detects when the complex $F_\bullet \otimes_R M$ is acyclic, for an R-module M. It may be surprising that this version does not require that M has to be finitely generated.

We will have to be careful with the phrase "$I_{r_i}(\varphi_i)$ contains a regular sequence of length i". The correct generalization for big modules is as follows.

Definition. Let R be a ring, $I = (x_1, \ldots, x_n)$ an ideal, and M an R-module. We say $\operatorname{grade}(I, M) \geqslant g$ if the Koszul homology $H_i(x_1, \ldots, x_n; M) = 0$ for $i > n - g$.

In particular we take $\operatorname{grade}(I, M) = \infty$ if all the Koszul homology modules vanish.

Note that when M is finitely generated, we have $\operatorname{grade}(I, M) = g$ if and only if there is an M-regular sequence of length g in I. (We saw this back in the section on the Koszul complex.) Also, $\operatorname{grade}(I, M) \leqslant \operatorname{height} I$ for all I (exercise).

Now here is the jazzed-up version of Buchsbaum-Eisenbud. See [Bruns-Herzog, 9.1.5] for a proof.

Theorem. *Let R be a Noetherian ring and*

$$F_\bullet : 0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

a complex of finite free R-modules. Set $r_i = \sum_{ji}^{s} (-1)^{j-i} \operatorname{rank} F_j$. Let M be an R-module. The following are equivalent.

1. *$F_\bullet \otimes_R M$ is acyclic.*
2. *For each $i = 1, \ldots, s$, we have $\operatorname{grade}(I_{r_i}(\varphi_i), M) \geqslant i$.*

$\square$

Corollary. *Assume R is local and let $F_\bullet$ be as above and assume that*

1. *$s \leqslant \dim R$;*

2. $H_i(F_\bullet)$ has finite length for all $i \geqslant 1$.

Let M be a big CM module. Then $F_\bullet \otimes_R M$ is acyclic.

Proof. Fix i with $1 \leqslant i \leqslant s$. Let $h = \operatorname{height} I_{r_i}(\varphi_i)$, and let $\mathfrak{p}$ be a prime ideal of the same height. If $h = \operatorname{height} \mathfrak{p} = \dim R$, then $h = \dim R \geqslant s \geqslant i$ by hypothesis (1). OTOH, if $h = \operatorname{height} \mathfrak{p} < \dim R$, then $F_\bullet \otimes_R R_\mathfrak{p}$ is acyclic by hypothesis (2), and therefore by the Theorem

$$\operatorname{height} I_{r_i}(\varphi_i) \geqslant \operatorname{grade}(I_{r_i}(\varphi_i), R) \geqslant i\,.$$

So in either case we have $h \geqslant i$. It follows that $I_{r_i}(\varphi_i)$ contains elements $x_1, \ldots, x_i$ which form part of a system of parameters for R. Extend to a full s.o.p. $x_1, \ldots, x_d$.

By hypothesis, this s.o.p. is an M-regular sequence, and by the Theorem again we conclude that $F_\bullet \otimes_R M$ is acyclic. □

This is a hugely useful result. For example, Iyengar and Bridgeland use it to prove the following strengthening of the result of Auslander-Buchsbaum-Serre that a local ring $(R, \mathfrak{m})$ is regular iff $R/\mathfrak{m}$ has finite projective dimension.

Proposition (Iyengar-Bridgeland). *Let $(R, \mathfrak{m})$ be a local ring with a big CM module M. If there is a complex*

$$F_\bullet : 0 \longrightarrow F_{\dim R} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

of finite free modules with homology of finite length, and such that $R/\mathfrak{m}$ is a direct summand of $H_0(F_\bullet)$, then $F_\bullet$ is acyclic and R is regular local. □

Very similar arguments (omitted here) prove the following:

Theorem (Strong Intersection Theorem, AKA Improved New Intersection Theorem)**.** *Let $(R, \mathfrak{m})$ be a local ring and $0 \longrightarrow F_s \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0$ a complex of finitely generated free modules. Assume that $H_0(F_\bullet)$ has a minimal generator killed by a power of $\mathfrak{m}$ and that the homology has finite length for $i \geqslant 1$. Assume R has a big CM module. Then $s \geqslant \dim R$.* □

Theorem (New Intersection Theorem)**.** *Let R and $F_\bullet$ be as above. If all the homology of $F_\bullet$ has finite length and $H_0(F_\bullet)$ is nonzero, then $s \geqslant \dim R$.* □

Lim CM Sequences

A few words about lim Cohen-Macaulay sequences of modules over local rings. These are sequences of finitely generated modules $\{M_n\}_{n\in\mathbb{N}}$ such that for any system of parameters $\underline{x}$, the ratio $\mathrm{length}(H_i(\underline{x}; M_n))/\mu(M_n)$ approaches zero for all $i \geqslant 1$. Here the numerator is the length of Koszul homology.

Notice that if M is a small CM module, then $\{M, M, \dots\}$ is a lim CM sequence.

Theorem (Bhatt-Hochster-Ma)**.** *If $(R, \mathfrak{m})$ is a complete local domain of characteristic $p > 0$ with perfect residue field, then $M_n := R^{1/p^n}$ is a lim CM sequence for R.*

Theorem. *If every complete local domain with perfect residue field has a lim CM sequence, then Serre's Positivity conjecture on intersection multiplicities is true.*

Appendix: A very few words about the recent proof of DSC

These remarks are based on my notes from a talk by Bhatt at Oberwolfach last December. Any inaccuracies are entirely my fault.

The Direct Summand Conjecture has now been proved by André, with additions by Bhatt. The key words are "almost ring theory" and "perfectoid spaces".

Recall that we are given a regular local ring A and a module-finite extension domain R, and we want to split the extension. Some elementary remarks:

Remarks. 1. If we are trying to split the ring extension $f\colon A \longrightarrow R$, we are free to make a further extension of R: if $A \longrightarrow R \longrightarrow S$ splits as A-modules, then so does $A \longrightarrow R$.

2. The given extension $f\colon A \longrightarrow R$ defines an extension

$$0 \longrightarrow A \longrightarrow R \longrightarrow R/A \longrightarrow 0\,,$$

whence an element $\alpha_f \in \mathrm{Ext}^1_A(R/A, A)$. We think of this element as an "obstruction" to the splitting of $f\colon A \longrightarrow R$.

3. Another quantification is the trace ideal $\tau_A(R)$, which is generated by the images of all A-linear maps $R \longrightarrow A$. Since A is local, the map splits if and only if $\tau_A(R) = A$.

4. Since these two things quantify the same thing, it is not hard to see that the ideal $\tau_A(R)$ kills the obstruction α_f.

Here is a result of Hochster in characteristic p that will provide a roadmap for what comes next.

Proposition (Hochster). *Let A be a regular local ring containing $\mathbb{F}_p$, and let $f\colon A \hookrightarrow R$ be an extension, with R a domain. Assume that f induces a separable extension of quotient fields. Then f splits.*

Sketch of Proof. We proceed in two steps: "almost" splitting f, then actually splitting it.

Let $A_{\mathsf{perf}} = \varinjlim A$, where the limit is taken over the direct system $A \xrightarrow{F} A \xrightarrow{F} A \xrightarrow{F} \cdots$. Another way to write A_{perf} is A^{1/p^∞}; it's obtained by "adjoining" all p^{th} power roots of elements of A. Crucial fact: this is a functor. So we get $f_{\mathsf{perf}}\colon A_{\mathsf{perf}} \longrightarrow A_{\mathsf{perf}}$.

Observe that $\tau_A(R)$ is nonzero; indeed, the map on quotient fields $Q(A) \longrightarrow Q(R)$ is separable by assumption, so the trace map $Q(R) \longrightarrow Q(A)$ is nonzero, and we can clear denominators.

Furthermore (I don't completely understand this part) $\tau_{A_{\mathsf{perf}}}(R_{\mathsf{perf}}) = \tau_A(R)_{\mathsf{perf}} \subseteq A_{\mathsf{perf}}$. So we can choose a nonzero element $g \in \tau_A(R)$, and then $g^{1/p^n}\alpha_{f_{\mathsf{perf}}} = 0$ for every $n \geqslant 0$. (We say f_{perf} is "almost split".)

Now, since A is regular, the extension $A \longrightarrow A_{\mathsf{perf}}$ is faithfully flat. (It is a direct limit of $A \longrightarrow A^{1/p^n}$, each of which is finite free.) It follows that $\alpha_f = 0$ iff $\alpha_{f_{\mathsf{perf}}} = 0$. In fact it is enough that $\alpha_{f_{\mathsf{perf}}}$ is almost zero, because if $g^{1/p^n} \in \mathrm{Ann}_A(\alpha_{f_{\mathsf{perf}}})$ for every n, then

$$g \in \mathrm{Ann}_A(\alpha_{f_{\mathsf{perf}}})^{p^n}$$

and by Krull's Intersection Theorem, that means either $g = 0$ or that annihilator is the whole ring. So we're done. $\square$

André and Bhatt follow something like the sketch above in the case of mixed characteristic. The steps are roughly as follows:

1. Pick $g \in A$ so that $A[\frac{1}{g}] \longrightarrow R[\frac{1}{g}]$ is finite étale, that is, $R[\frac{1}{g}]$ is a finitely generated projective module and the extension is separable. This is possible because the induced map on quotient fields is separable, since the quotient fields have characteristic zero.

2. (André) Construct a huge extension $A \longrightarrow A_\infty$, which is almost faithfully flat mod p, and such that g becomes perfect in A_∞, that is, has all p^{th} power roots. Here "almost flat mod p" means that $\mathrm{Tor}^i_A(A_\infty, -)$ is killed by arbitrarily high roots of p, and "almost faithfully flat mod p" means that plus the same thing about the kernels of the natural maps $\mathrm{Hom}_A(N, N') \longrightarrow \mathrm{Hom}_{A_\infty}(N \otimes A_\infty, N' \otimes A_\infty)$.

3. Furthermore, A_∞ will be perfectoid, which means that
   (a) it is p-adically complete;
   (b) it is p-torsion-free;
   (c) the Frobenius induces an isomorphism $A_\infty/(p^{1/p}) \longrightarrow A_\infty/(p)$.

4. Show that the base change $A_\infty \longrightarrow R \otimes_A A_\infty$ is almost split. This is the difficult part, which I cannot say anything intelligent about other than the phrase "Almost Purity Theorem" (Faltings, Scholze, Kedlaya-Liu).

5. Descend the almost splitting over A_∞ to an honest splitting over A, as above.

Remark. Bhatt uses similar techniques to also prove a "Derived Direct Summand". Let A be a regular ring and $X = \mathrm{Spec}\, A$. Let $f\colon X \longrightarrow B$ be a proper surjective map of schemes. Then f splits in the derived category, in the sense that $A \longrightarrow R\Gamma(B, \mathcal{O}_B)$ splits in $\mathcal{D}(A)$. This is apparently new even in very simple settings, like a blowup.

Remark. André's paper includes a proof of the existence of big CM algebras. (It also asserts the existence of a weakly functorial version, which is even better, but does not give details.) So DSC, big CM modules, and big CM algebras are all now theorems in full generality.

What remains open? The following are all strictly stronger than Direct Summand. The first two are equivalent, and each implies the ones above it.

1. Existence of lim CM sequences in mixed char.
2. Ranganathan's strong direct summand conjecture: Let A be a regular local ring and $A \subseteq R$ a module-finite extension which is a domain. Let $x \in \mathfrak{m}_A \setminus \mathfrak{m}_A^2$ and let $\mathfrak{Q}$ be a height-one prime of R containing xA. Then xA is a direct summand of $\mathfrak{Q}$ as an A-module.
3. Vanishing Conjecture for maps of Tor: Let $A \longrightarrow R \longrightarrow S$ be Noetherian rings, where A is a regular domain, S is module-finite and torsion-free over A, and S is regular. Then for every A-module M and integer $i \geqslant 1$, the map $\mathrm{Tor}_i^A(M, R) \longrightarrow \mathrm{Tor}_i^A(M, S)$ is zero.
4. Existence of weakly functorial big CM algebras: If $R \longrightarrow S$ is a local map of complete local domains, there exists a map from a big CM algebra B over R to a big CM algebra C over S such that the diagram

$$\begin{tikzcd} B \arrow[r] & C \\ R \arrow[u] \arrow[r] & S \arrow[u] \end{tikzcd}$$

commutes.