Exercises for Graham's lectures

Lecture 1

1. The primary ideals of $\mathbb{Z}$ are precisely the ideals (p^m), where p is a prime integer and $m \geqslant 1$. The radical of such an ideal is just (p).

2. For a local ring $(R, \mathfrak{m}, k)$ and a module M, elements $a_1, \ldots, a_n \in M$ form a minimal generating set for M if and only if their images in $M/\mathfrak{m}M$ form a vector space basis.

3. Suppose x is R-regular. Prove that $\operatorname{Tor}_1^R(M, R/xR)$ is isomorphic to the submodule of M killed by x.

4. Show that the previous exercise is false if x is not assumed R-regular.

5. Show that a finitely generated module M over a local ring $(R, \mathfrak{m})$ has finite length (i.e., a composition series) if and only if $\operatorname{Supp}_R(M) = \{\mathfrak{m}\}$.

6. Let $(R, \mathfrak{m})$ be local and M, N two nonzero finitely generated modules such that $\operatorname{pd}_R M = n < \infty$. Show that $\operatorname{Ext}_R^n(M, N) \neq 0$.

7. For $n \geqslant 2$, let $R = k[x_1, \ldots, x_n]$ and $I = (x_1x_2, x_2x_3, \ldots, x_nx_1)$ (the edge ideal of the n-cycle). Find all n such that R/I is Cohen-Macaulay. (This is challenging.)

8. Let $(R, \mathfrak{m})$ be local and M, N finitely generated. Suppose that $\operatorname{Tor}_i^R(M, N) = 0$ for all $i > 0$. Prove that $\operatorname{pd}_R(M \otimes_R N) = \operatorname{pd}_R M + \operatorname{pd}_R N$.

9. Give an example to show that the previous exercise is false without the vanishing assumption on Tor. In fact, $M \otimes_R N$ might not even have finite projective dimension.

10. Prove that the tensor product of two complexes is a complex.

Lectures 2-3

1. An affine variety $V(I)$ is irreducible if and only if $I(V(I))$ is a prime ideal.

2. Let R be a local ring of dimension d and I an ideal. Prove that $\operatorname{height} I \geqslant i$ if and only if I contains elements $x_1, \ldots, x_i$ forming part of a system of parameters for R.

3. Let R be Noetherian and M finitely generated. Prove that $x \in R$ is a zerodivisor on M if and only if $x \in \mathfrak{p}$ for some $\mathfrak{p} \in \operatorname{Ass}_R M$.

4. Let R be a ring, I a finitely generated ideal, and M an R-module. Prove that $\operatorname{grade}(I, M) \geqslant 1$ if and only if $\operatorname{Hom}_R(R/I, M) = 0$, if and only if I annihilates some element of M. When R is Noetherian, these are equivalent to existence of some $\mathfrak{p} \in \operatorname{Ass}_R M$ with $I \subseteq \mathfrak{p}$.

5. Use the definition of the Koszul complex to write down the Koszul complex on 2 and 3 elements (in terms of matrices).

6. Let $x, y \in R$. Prove that the homology of $K_\bullet(x, y; R)$ is
   (a) $R/(x, y)$ in degree 0;
   (b) $Ann_R(x, y)$ in degree 2;
   (c) $((y) :_R x)/(y)$ in degree 1 if we assume that y is a nonzerodivisor; in particular this is zero if and only if x is a nonzerodivisor on $R/(y)$.

7. Compute the intersection multiplicity of two randomly chosen plane curves at the origin, for example $y = x^3$ and $y = 0$.

8. Let $I = (x^3 - w^2 y, x^2 z - wy^2, xy - wz, y^3 - xz^2)$ and $J = (w, z)$ in $\mathbb{C}[x, y, z, w]$. Show that the length of the tensor product is 5, but the intersection multiplicity is 4.

Lectures 4-5

1. (Theorem of Rees) If $I = (x_1, \ldots, x_n)$ is generated by a R-regular sequence and $F \in R[X_1, \ldots, X_n]$ is a homogeneous polynomial of degree s such that $F(x_1, \ldots, x_n) \in I^{s+1}$, then the coefficients of F are all in I. (Reduce to the case where F is a monomial.)

2. Let $I = (x_1, \ldots, x_n)$ be generated by a regular sequence. Recall that the associated graded ring of I is $\operatorname{gr}_I(R) = \bigoplus_{i=0}^{\infty} I^i/I^{i+1}$, an $\mathbb{N}$-graded ring with R/I in degree zero. Prove that the map $\varphi\colon (R/I)[X_1, \ldots, X_n] \longrightarrow \operatorname{gr}_I(R)$ is an isomorphism. (It suffices to check surjectivity and injectivity on homogeneous elements; use the previous exercise for injectivity.)

3. Use the previous exercise to prove the Monomial Conjecture for Cohen-Macaulay local rings.

4. Can you state and prove generalizations of the previous three exercises that allow you to prove the Monomial Conjecture if R is only assumed to have a module M such that $x_1, \ldots, x_n$ is M-regular?

5. Let $R = k[x_1, \ldots, x_n]$ with k a field of characteristic p such that $[k : k^p] < \infty$. Prove that R is a finitely generated free module over the subring R^p of p^{th} powers.

6. If $R \subseteq S$ are rings such that R is a direct summand of S, prove that $IS \cap R = I$ for every ideal I of R.

7. If R is a domain which is a direct summand of every module-finite extension, prove that R is integrally closed. (Apply the previous exercise to the extension $R \subset R[\frac{y}{x}]$ where $\frac{y}{x}$ is an integral element.)

8. (For those who know about Gorenstein rings.) Let $(R, \mathfrak{m})$ be a Gorenstein local ring and $\underline{x}$ a system of parameters. Let $\Delta \in R$ be a lift of a generator for the socle of $R/(\underline{x})$. Let M be an R-module and assume $\underline{x}$ is M-regular. Prove that R splits as a direct summand from M if and only if $\Delta M \not\subseteq \underline{x}M$. (The forward direction is easier; for the reverse, use the fact that $\varinjlim R/(x_1^t, \ldots, x_d^t)$, where the maps are all multiplication by $x_1 \cdots x_d$, is isomorphic to the injective hull $E_R(k)$.)