Letterplace and co-letterplace ideals

Exercise set 1, Monday

Exercise 1

a. Compute the resolutions of the ideals:

i. $I_1 = (ab, bc, cd) \subseteq \Bbbk[a, b, c, d] = S_1$
ii. $I_2 = (xy, yz, zx) \subseteq \Bbbk[x, y, z] = S_2$
iii. $I_3 = (x^2, xy, y^2) \subseteq \Bbbk[x, y] = S_3$

b. What are the dimensions of S_k/I_k? (Investigate the zero set of the ideals.)

c. Use the Auslander-Buchsbaum to find $\operatorname{depth} S_k/I_k$ in each case. In case i. or ii. find a regular sequence of this length.

d. Which of these quotient rings S_k/I_k are Cohen-Macaulay?

Exercise 2 Compute the resolution of the ideal

$$I = (x, y)(z, w) = (xz, xw, yz, yw) \subseteq \Bbbk[x, y, z, w] = S.$$

What is $\dim S/I$? What is $\operatorname{depth} S/I$. Is S/I a Cohen-Macaulay ring?

Exercise 3 Let M be a finitely generated graded $S = \Bbbk[x_1, \dots, x_n]$-module with minimal free resolution

$$M \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_\ell.$$

Let $S' = S/x_nS$, and $F'_p = F_p/x_nF_p$. Suppose x_n is regular for M. Show that the minimal free resolution of M/x_nM as module over S' is

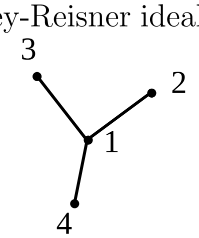
$$M/x_nM \leftarrow F'_0 \leftarrow F'_1 \leftarrow \cdots \leftarrow F'_\ell.$$

Exercise set 2, Tuesday

Recall that a squarefree monomial ideal J in a polyomial ring S is Alexander dual to a squarefree monomial ideal I in S, if J is the set of monomials in S which have non-trivial common divisor with every monomial in I (or equivalently with every generator of I).

Exercise 1

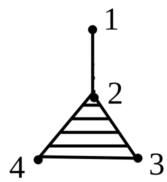
a. What is the Stanley-Reisner ideal I_Δ of the simplicial complex Δ:



b. What is the Alexander dual ideal of the Stanley-Reisner ideal I_Δ? What is the Alexander dual simplicial complex Δ^A?

Exercise 2

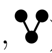
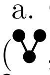
a. What is the Stanley-Reisner ideal I_Δ of the simplicial complex Δ:



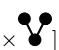
b. Find the Alexander dual ideal of the Stanle-Reisner ideal I_Δ. Find the Alexander dual simplicial complex Δ^A.

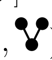
Exercise 3

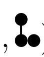
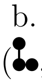
Let  be the three element poset with the lower vertex labelled a and the two upper labelled b and c.

a. Compute the letterplace ideal $L(2,$ $)$ and the co-letterplace ideal $L($$, 2)$. Verify that they are Alexander dual. The projection $[2] \times$  $\to$  has left strict chain fibers. It induces a map of polynomial rings

$$\Bbbk[x_{[2]\times \text{(poset)}}] \to \Bbbk[x_{\text{(poset)}}] = \Bbbk[x_a, x_b, x_c].$$

What is the image of the ideal $L(2,$ $)$ by this map? Which regular sequence have we divided out by?

b. Compute the letterplace ideal $L(2,$$)$ and the co-letterplace ideal $L($$, 2)$. Verify again that they are Alexander dual. The projection $[2] \times ($$) \to ($$)$ has left strict chain fibers. It induces a map of polynomial rings. What is the image of the ideal $L(2,$$)$ by this map? Which regular sequence have we divided out by?

Exercise 4

Write down the letterplace ideal $L([2], [3])$. Find surjective isotone maps:

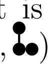
- $[2] \times [3] \to [3]$
- $[2] \times [3] \to [4]$
- $[2] \times [3] \to [5]$

with left stict chain fibers.

What is the image of $L(2, 3)$ by the associated maps of polynomial rings?

Exercise set 3, Thursday

Exercise 1

The letterplace ideal $L(2,$ $)$ defines a simplicial complex $\Delta(2,$ $)$ which is a ball. What is the dimension of this ball? Describe the simplicial complex $\Delta(2,$ $)$.

Exercise 2

The staircase complex $\mathrm{St}(P, n)$ is the simplicial complex defined by the squarefree monomial ideal $B(P, n)$ generated by $x_{p,i}x_{q,j}$ where $p < q$ and $n \geq i > j \geq 1$. Let P be the chain $[3]$ and $n = 2$.

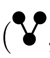
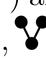
a. Describe the facets of $\mathrm{St}([3], 2)$.
b. Find the Alexander dual $B([3], 2)^A$.
c. Describe the simplicial complex $\Delta(2, [3])$ (which is a ball.
d. Describe the image of the composition

$$\gamma : B([3], 2)^A \to \Bbbk[x_{[3]\times[2]}] \to \Bbbk[\Delta(2, [3])],$$

which is the canonical module $\omega_{\Bbbk[\Delta(2,[3])]}$.

e. The ideal $B([3], 2)^A + L(2, [3])$ defines the boundary $\Sigma(2, [3])$ of $\Delta(2, [3])$. Verify this.

Exercise 3

Take a proper poset ideal $\mathcal{J} \subseteq \mathrm{Hom}($$, [2])$. Describe the co-letterplace ideal $L^{co}($$, 2; \mathcal{J})$ and its Alexander dual $L(2,$ $; \mathcal{J})$. Draw the simplicial complex $\Delta(2,$ $; \mathcal{J})$ defined by the $L(2,$ $; \mathcal{J})$. Is it a ball?