Letterplace and co-letterplace ideals from partially ordered sets
Nordfjordeid, 12-16'th June 2017

Gunnar Fløystad

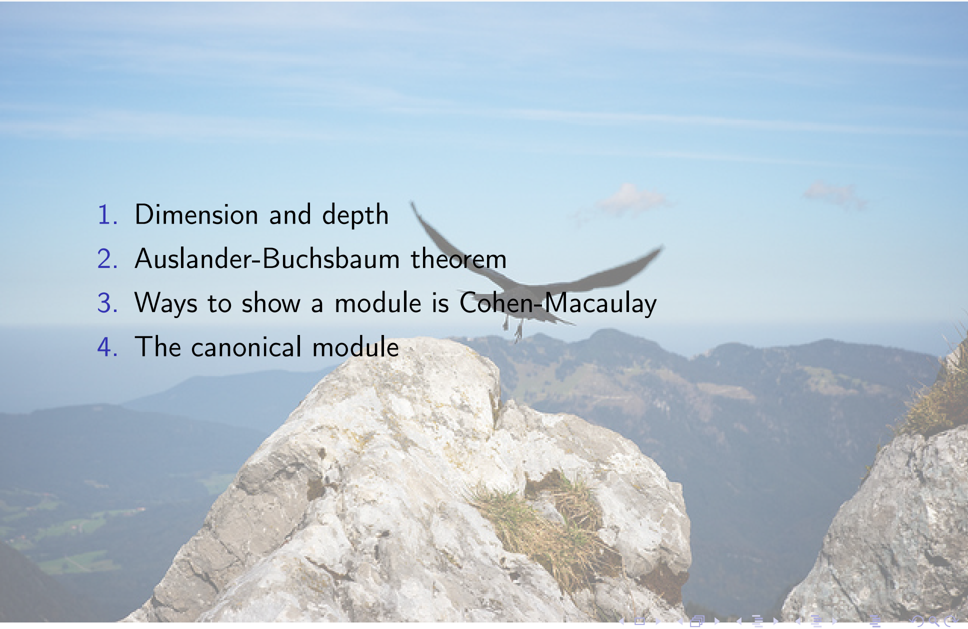
June 15, 2017

Content

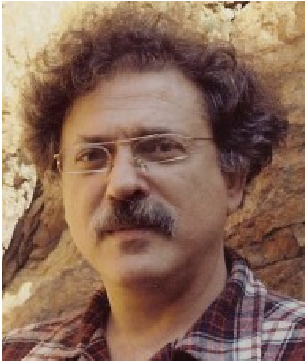
I. Cohen-Macaulay rings and modules
II. Stanley-Reisner rings
III. Letterplace and co-letterplace ideals
IV. Simplicial spheres
V. Generalizations, duality of strongly stable ideals
VI. Deforming letterplace ideals

I. Cohen-Macaulay rings and modules

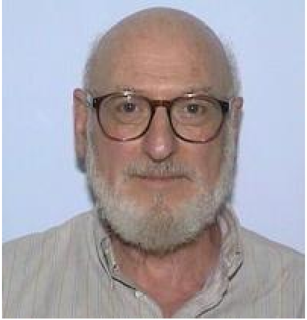
1. Dimension and depth
2. Auslander-Buchsbaum theorem
3. Ways to show a module is Cohen-Macaulay
4. The canonical module



Maurice Auslander 1926-1994, Brandeis

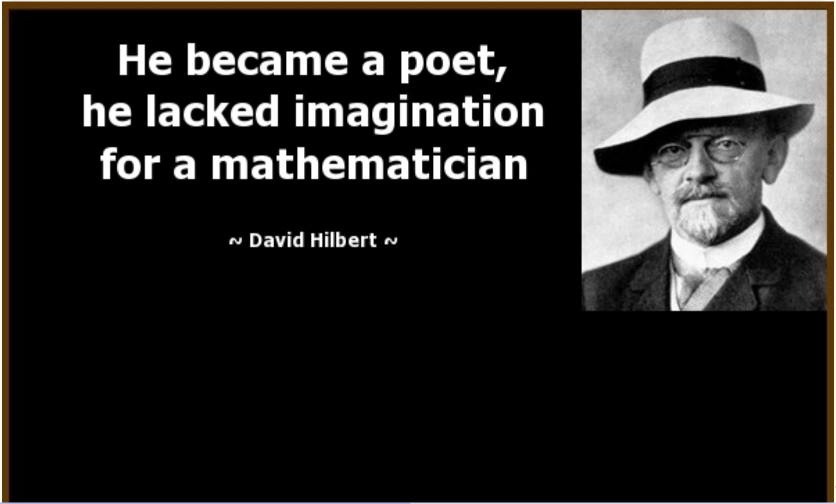


David Buchsbaum, 1929-, Brandeis



David Hilbert
1862-1943

1. Hilbert basis theorem, 2. Hilbert syzygy theorem,
3. Nullstellensatz

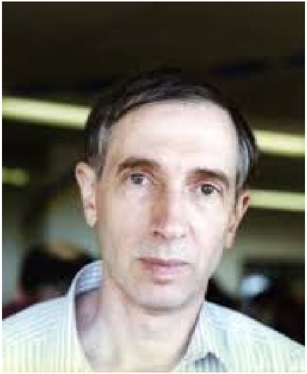


II. Stanley-Reisner rings: Tour d'horizon

1. Term orders and initial ideals
2. Simplicial complexes
3. Shellability
4. Alexander duality
5. Linear resolutions and Eagon-Reiner theorem
6. Squarefree modules

Founding fathers Stanley-Reisner theory

Richard Stanley, MIT



Proof of upper bound conjecture for simplicial spheres, 1975.

Melvin Hochster, U. of Michigan

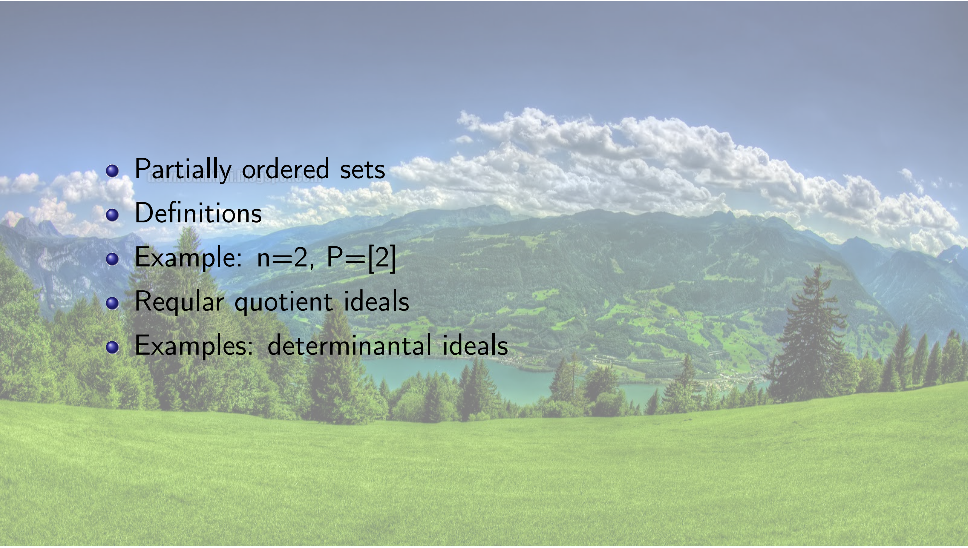
Seminal paper, 1975

Eagon-Reiner theorem 1997

Vic Reiner, Minnesota

Jack Eagon, Michigan

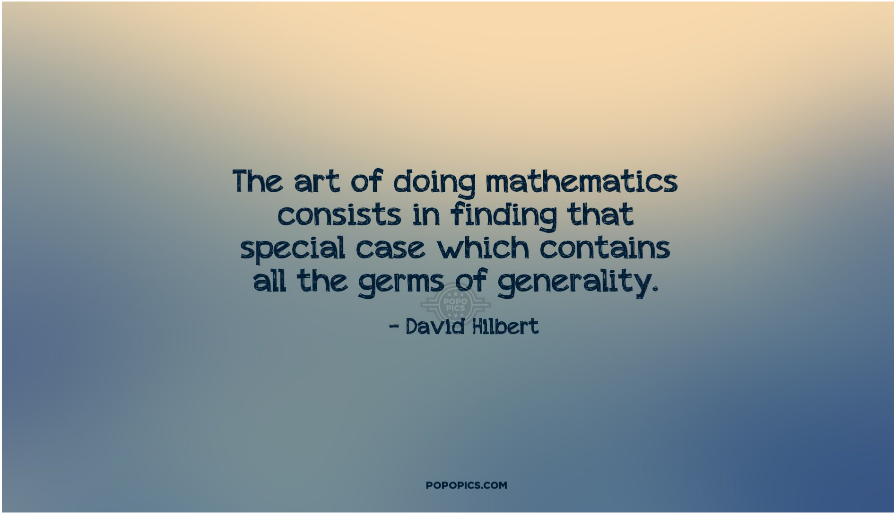
III. Letterplace ideals



- Partially ordered sets
- Definitions
- Example: n=2, P=[2]
- Regular quotient ideals
- Examples: determinantal ideals

V.Ene, J.Herzog, F.Mohammadi: Monomial ideals and toric rings of Hibi type arising from a finite poset, European Journal of Combinatorics, 32 (2011).

G.Fløystad, J.Herzog, B.M.Greve: Letterplace and co-letterplace ideals of posets, Journal of pure and applied algebra (2017).


The art of doing mathematics consists in finding that special case which contains all the germs of generality.
- David Hilbert
POPOPICS.COM

$$A = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,n+m-1} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{n,n+m-1} \end{bmatrix}$$

- I = ideal generated by maximal minors, the n-minors
- Initial ideal $\mathrm{in}(I)$ generated by

$$x_{1i_1}x_{2i_2}\cdots x_{ni_n}, \quad 1 \leq i_1 < i_2 < \cdots < i_n \leq n+m-1.$$

$$A = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,n+m-1} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{n,n+m-1} \end{bmatrix}$$

- I = ideal generated by maximal minors, the n-minors
- Initial ideal $\mathrm{in}(I)$ generated by

$$x_{1i_1}x_{2i_2}\cdots x_{ni_n}, \quad 1 \leq i_1 < i_2 < \cdots < i_n \leq n+m-1.$$

-

$$[n] \times [m] \xrightarrow{\eta} [n] \times [n+m-1]$$
$$(i,j) \mapsto (i, i+j-1)$$

$$A = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,n+m-1} \\ \vdots & & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{n,n+m-1} \end{bmatrix}$$

- I = ideal generated by maximal minors, the n-minors
- Initial ideal in(I) generated by

$$x_{1i_1}x_{2i_2}\cdots x_{ni_n}, \quad 1 \leq i_1 < i_2 < \cdots < i_n \leq n+m-1.$$

- $$[n] \times [m] \xrightarrow{\eta} [n] \times [n+m-1]$$
$$(i,j) \mapsto (i, i+j-1)$$

- Then in(I) is $L^\eta(n,[m]) \subseteq \Bbbk[x_{[n]\times[n+m-1]}]$

$$B = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,n+s-1} \\ \vdots & & & \vdots \\ x_{n+r-1,1} & x_{n+r-1,2} & \cdots & x_{n+r-1,n+s-1} \end{bmatrix}$$

- $I =$ ideal generated by the n-minors
- Initial ideal in(I) generated by $x_{i_1 j_1} x_{i_2 J_2} \cdots x_{i_n j_n}$ where

$$1 \leq i_1 < i_2 < \cdots < i_n \leq n+r-1, \quad 1 \leq j_1 < j_2 \cdots < j_n \leq n+s-1$$

$$B = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,n+s-1} \\ \vdots & & & \vdots \\ x_{n+r-1,1} & x_{n+r-1,2} & \cdots & x_{n+r-1,n+s-1} \end{bmatrix}$$

- I = ideal generated by the n-minors
- Initial ideal in(I) generated by $x_{i_1 j_1} x_{i_2 J_2} \cdots x_{i_n j_n}$ where

  $$1 \leq i_1 < i_2 < \cdots < i_n \leq n+r-1, \quad 1 \leq j_1 < j_2 \cdots < j_n \leq n+s-1$$

- $$[n] \times [r] \times [s] \xrightarrow{\alpha} [n+r-1] \times [n+s-1]$$
  $$(i, a, b) \mapsto (i+a-1, i+b-1)$$

$$B = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,n+s-1} \\ \vdots & & & \vdots \\ x_{n+r-1,1} & x_{n+r-1,2} & \cdots & x_{n+r-1,n+s-1} \end{bmatrix}$$

- I = ideal generated by the n-minors
- Initial ideal in(I) generated by $x_{i_1 j_1} x_{i_2 J_2} \cdots x_{i_n j_n}$ where

  $$1 \leq i_1 < i_2 < \cdots < i_n \leq n+r-1, \quad 1 \leq j_1 < j_2 \cdots < j_n \leq n+s-1$$

- $$[n] \times [r] \times [s] \xrightarrow{\alpha} [n+r-1] \times [n+s-1]$$
  $$(i, a, b) \mapsto (i + a - 1, i + b - 1)$$

- Then in(I) $= L^{\alpha}(n, [r] \times [s])$

Intial ideals of determinantal ideals

Generic matrices, 1992:
Bernd Sturmfels, Max Planck, Leipzig,

Skew-symmetric matrices, 1994:
Jürgen Herzog, Essen,

Ngo Viet Trung, Hanoi,

$$C = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1,n+1} \\ x_{12} & x_{22} & & \vdots \\ \vdots & \vdots & & \\ x_{1,n+1} & x_{2,n+1} & \cdots & x_{n+1,n+1} \end{bmatrix}$$

- I = ideal generated by the 2-minors
- Initial ideal in(I) generated by $x_{i_1 j_1} x_{i_2 j_2}$ where $i_1 < i_2$ and $j_1 < j_2$ (and $i_1 \leq j_1$ and $i_2 \leq j_2$)
- $$[2] \times \mathrm{Hom}([2],[n]) \xrightarrow{\beta} \mathrm{Hom}([2],[n+1])$$
  $$(1, i_1, i_2) \mapsto (i_1, i_2)$$
  $$(2, i_1, i_2) \mapsto (i_1 + 1, i_2 + 1)$$
- $\mathrm{in}(I) = L^{\beta}(2, \mathrm{Hom}([2],[n]))$

$$D = \begin{bmatrix} 0 & x_{12} & \cdots & x_{1,r} \\ -x_{12} & 0 & & x_{2,r} \\ \vdots & & & \vdots \\ -x_{1,r} & -x_{2,r} & \cdots & 0 \end{bmatrix}, \quad r = 2n+1$$

- $I =$ ideal generated by $2n$-Pfaffians

$$D = \begin{bmatrix} 0 & x_{12} & \cdots & x_{1,r} \\ -x_{12} & 0 & & x_{2,r} \\ \vdots & & & \vdots \\ -x_{1,r} & -x_{2,r} & \cdots & 0 \end{bmatrix}, \quad r = 2n+1$$

- I = ideal generated by $2n$-Pfaffians
- $\mathrm{in}(I) = L(n, V)$

IV. Simplicial spheres

- Many more LP and Co-LP ideals from poset ideals in Hom(P,[n])
- Guiding questions
- Staircase complex and the canonical module
- Lots of balls and spheres
- Bier spheres
- Examples

V. Generalizations, duality of strongly stable ideals

- General co-letterplace ideals
- General letterplace ideals
- Regular sequences
- Strongly stable duality
- Principal letterplace ideals

VI. Deforming letterplace ideals

- Flat deformations of ideals
- Deformations of $L(2, P)$
- Gradings on deformed ideals
- Examples

$L(2,$  $) = (a_1 a_2, a_1 b_2, a_1 c_2, b_1 b_2, c_1 c_2)$

Deformations of $L(2,$  $)$

$a \leq b$ and $a \leq c$

$L(2,$  $) = (a_1 a_2, a_1 b_2, a_1 c_2, b_1 b_2, c_1 c_2)$

$J(2,$  $)$:

$$
\begin{aligned}
&a_1 a_2 - u_a b_1 c_1 + u_a u_{cb} u_{bc} && \deg u_a = a_1 + a_2 - b_1 - c_1 \\
&a_1 b_2 - u_a u_{ac} u_{cb} - u_a u_{ab} c_1 && \deg u_{ac} = c_1 + c_2 - a_2 \\
&a_1 c_2 - u_a u_{ab} u_{bc} - u_a u_{ac} b_1 && \deg u_{ab} = b_1 + b_2 - a_2 \\
&c_1 c_2 - a_2 u_{ac} - b_2 u_{bc} && \deg u_{cb} = b_1 + b_2 - c_2 \\
&b_1 b_2 - a_2 u_{ab} - c_2 u_{bc} && \deg u_{bc} = c_1 + c_2 - b_2
\end{aligned}
$$

Deformations of $L(2,$  $)$

$a \leq b$ and $a \leq c$

$L(2,$  $) = (a_1 a_2, a_1 b_2, a_1 c_2, b_1 b_2, c_1 c_2)$

$J(2,$  $)$:

$$
\begin{aligned}
&a_1 a_2 - u_a b_1 c_1 + u_a u_{cb} u_{bc} && \deg u_a = a_1 + a_2 - b_1 - c_1 \\
&a_1 b_2 - u_a u_{ac} u_{cb} - u_a u_{ab} c_1 && \deg u_{ac} = c_1 + c_2 - a_2 \\
&a_1 c_2 - u_a u_{ab} u_{bc} - u_a u_{ac} b_1 && \deg u_{ab} = b_1 + b_2 - a_2 \\
&c_1 c_2 - a_2 u_{ac} - b_2 u_{bc} && \deg u_{cb} = b_1 + b_2 - c_2 \\
&b_1 b_2 - a_2 u_{ab} - c_2 u_{bc} && \deg u_{bc} = c_1 + c_2 - b_2
\end{aligned}
$$

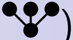
Recall:

$L(2,$  $)$ is the intial ideal of the 4-Pfaffians of generic 5×5 skew-symmetric matrices.

$J(2,$  $)$

The generators of $J(2,$  $)$ are the 4-Pfaffians of:

$$\begin{bmatrix} 0 & u_{bc} & c_2 & b_1 & a_2 \\ -u_{bc} & 0 & u_{ac} & u_a^{-1} & c_1 \\ -c_2 & -u_{ac} & 0 & u_{ab} & b_2 \\ -b_1 & -u_a^{-1}a_1 & -u_{ab} & 0 & u_{cb} \\ -a_2 & -c_1 & -b_2 & -u_{bc} & 0 \end{bmatrix}$$

$L(2,$  $)$
$a \leq b$, $a \leq c$, and $a \leq d$

$$M(a) = \begin{bmatrix} -u_{a,b} & b_1 & -u_{c,b} & -u_{d,b} \\ -u_{a,c} & -u_{b,c} & c_1 & -u_{d,c} \\ -u_{a,d} & -u_{b,d} & -u_{c,d} & d_1 \end{bmatrix},$$

$J(2,$  $)$:

$$b_1 b_2 - a_2 u_{a,b} - c_2 u_{c,b} - d_2 u_{d,b}$$
$$c_1 c_2 - a_2 u_{a,c} - b_2 u_{b,c} - d_2 u_{d,c}$$
$$d_1 d_2 - a_2 u_{a,d} - b_2 u_{b,d} - c_2 u_{c,d}$$
$$a_1 b_2 - u_a D(a)^b$$
$$a_1 c_2 - u_a D(a)^c$$
$$a_1 d_2 - u_a D(a)^d$$
$$a_1 a_2 - u_a D(a)^a$$

$J(2,$  $)$

Question:

Is there a natural *geometric* description of the variety defined by these equations?

The six lectures

I. Cohen-Macaulay rings and modules
II. Stanley-Reisner rings
III. Letterplace and co-letterplace ideals
IV. Simplicial spheres
V. Generalizations, duality of strongly stable ideals
VI. Deforming letterplace ideals

The six lectures

I. Cohen-Macaulay rings and modules
II. Stanley-Reisner rings
III. Letterplace and co-letterplace ideals
IV. Simplicial spheres
V. Generalizations, duality of strongly stable ideals
VI. Deforming letterplace ideals

Thank you!