JASON MCCULLOUGH

STILLMAN'S QUESTION AND THE EISENBUD-GOTO CONJECTURE

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0. About these notes

These notes follow the lectures given by Jason McCullough at the summer school "Homological aspects of commutative algebra" at the Sophus-Lie Conference Center in Nordfjordied, Norway June 12-16, 2017. The workshop was organized by Petter Andreas Bergh (NTNU, Trondheim), Kristian Ranestad (Oslo), Trygve Johnsen(Tromsø) and Gunnar Fløystad (Bergen). The other lecturers were Graham Leuschke (Syracuse) and Gunnar Fløystad.

These notes are not designed to be comprehensive but rather to fill in some of the gaps in the lectures. As such the choice of topics is a bit uneven in scope and difficulty. We assume some familiarity with commutative and homological algebra and topics such as: height of an ideal, colon ideals, dimension of a module, homology of a complex of modules, the correspondence theorem for ideals in a quotient ring, prime avoidance, Zorn's lemma, unique factorization domains, flatness, etc. For background on these topics and more detail on the topics covered, here are a few recommended texts to consult:

1. **Graded Syzygies** by Peeva [1]
2. **Commutative Algebra** by Atiyah and MacDonald [2]
3. **Cohen-Macaulay Rings** by Bruns and Herzog [3]
4. **An Introduction to Homological Algebra** by Rotman [4]
5. **The Geometry of Syzygies** by Eisenbud [5]

There are undoubtedly errors in these notes. If you find any, feel free to notify me at `jmccullo@iastate.edu`.

[1] Irena Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011

[2] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969

[3] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993

[4] Joseph J. Rotman. *An introduction to homological algebra*. Universitext. Springer, New York, second edition, 2009

[5] David Eisenbud. *The geometry of syzygies*, volume 229 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005. A second course in commutative algebra and algebraic geometry

1. Primary Decomposition

In this chapter we review the basics of primary decomposition of ideals and modules over Noetherian rings, focusing on the case of homogeneous ideals in a polynomial ring.

Primary Decomposition and Associated Primes

Definition Let I be an ideal in a ring R. The ideal I is **prime ideal** if $I \neq R$ and $xy \in I \implies x \in I$ or $y \in I$. The **radical** $\sqrt{I}$ of an ideal I is the intersection of all (equivalently, all minimal) prime ideals containing I.

I is prime if and only if R/I is a domain.

Lemma 1.1 *If I is an ideal of R then*

$$\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n > 0\}.$$

Proof ($\supseteq$) Suppose that $x^n \in I$. Then for each prime ideal $\mathfrak{p}$ containing I, $x^n \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $x \in \mathfrak{p}$.

($\subseteq$) By the Correspondence Theorem, the prime ideals of A containing I correspond bijectively to the ideals of A/I, hence we reduce to the case where $I = (0)$.

Suppose that x is not nilpotent. We'll show that

$$x \notin \bigcap_{\mathfrak{p} \supseteq (0)} \mathfrak{p}.$$

Consider the set of ideals of R:

$$\mathcal{S} = \{J : x^i \notin J \text{ for some } i > 0\}$$

Note that $(0) \in \mathcal{S}$ and that $\mathcal{S}$ may be ordered by inclusion. Now let $\mathcal{C}$ be any chain of ideals in $\mathcal{S}$. This chain has an upper bound in $\mathcal{S}$, namely the ideal:

$$\bigcup_{J \in \mathcal{C}} J$$

Hence by Zorn's Lemma, $\mathcal{S}$ has a maximal element, call it $\mathfrak{p}$. We claim that $\mathfrak{p}$ is prime. Suppose that $a, b \notin \mathfrak{p}$. Hence $(a) + \mathfrak{p}$ and $(b) + \mathfrak{p}$ are ideals not contained in $\mathcal{S}$. Thus for some $m, n \in \mathbb{N}$:

$$x^m \in (a) + \mathfrak{p} \qquad \text{and} \qquad x^n \in (b) + \mathfrak{p}$$

Moreover,

$$x^{m+n} \in (ab) + \mathfrak{p}$$

and so we see that $ab \notin \mathfrak{p}$. Hence $\mathfrak{p}$ is prime and $x \notin \mathfrak{p}$. Thus x is not in the intersection of the prime ideals of A. ■

Definition The ideal I is **primary** if $xy \in I \Longrightarrow x \in I$ or $y^n \in I$ for some $n > 0$.

Lemma 1.2 *Let $\mathfrak{q}$ be a primary ideal in a ring R. Then $\sqrt{\mathfrak{q}}$ is the smallest prime ideal containing $\mathfrak{q}$.*

Proof It suffices to show that $\sqrt{\mathfrak{q}}$ is prime. Suppose $xy \in \sqrt{\mathfrak{q}}$. Then $(xy)^n \in \mathfrak{q}$ by the previous lemma. Since $\mathfrak{q}$ is primary, either $x^n \in \mathfrak{q}$ or $y^{mn} \in \mathfrak{q}$ for some $m > 0$. Therefore $x \in \sqrt{q}$ or $y \in \sqrt{q}$ and thus $\sqrt{q}$ is prime. ■

Definition If $\mathfrak{q}$ is primary and $\sqrt{\mathfrak{q}} = \mathfrak{p}$, then $\mathfrak{q}$ is said to be **$\mathfrak{p}$-primary**.

Exercise 1.3 *Let $I = (x^2, xy) \subseteq k[x, y]$. Show that $\sqrt{I} = (x)$ is prime but I is not primary.*

Exercise 1.4 *If R is a UFD, then the principal ideals generated by powers of prime elements are primary ideals.*

Exercise 1.5 *If $\mathfrak{q}_1$ and $\mathfrak{q}_2$ are $\mathfrak{p}$-primary ideals, then so is $\mathfrak{q}_1 \cap \mathfrak{q}_2$.*

Exercise 1.6 *If $\sqrt{I}$ is a maximal ideal, then I is primary. In particular, $\mathfrak{m}^n$ is $\mathfrak{m}$-primary for any maximal ideal $\mathfrak{m}$.*

Exercise 1.7 *Let $\mathfrak{p} = (xz - y^2, x^3 - yz, x^2y - z^2) \subseteq k[x, y, z]$. Show that $\mathfrak{p}$ is prime but $\mathfrak{p}^2$ is not primary. (Hint: $\mathfrak{p}$ is homogeneous if $\deg(x) = 3$, $\deg(y) = 4$ and $\deg(z) = 5$.)*

Definition An ideal I in a ring R is **irreducible** if whenever I can be written as $J_1 \cap J_2$ for ideals J_1, J_2 of R, either $I = J_1$ or $I = J_2$.

For the remainder of this section, we assume R to be a Noetherian ring.

Lemma 1.8 *In a Noetherian ring R, any ideal is a finite intersection of irreducible ideals.*

Proof Let $\mathcal{C}$ be a set of ideals which cannot be written as a finite intersection of irreducible ideals. Since R is Noetherian, $\mathcal{C}$ has a maximal element I. Since $I \in \mathcal{C}$, I is reducible and can be written as $I = I_1 \cap I_2$ for two strictly larger ideals I_1, I_2 of R. Since I is maximal in $\mathcal{C}$, $I_1, I_2 \notin \mathcal{C}$. Therefore, each may be written as a finite intersection of irreducible ideals, whence I can also - a contradiction, unless $\mathcal{C}$ is empty. ■

By the next lemma, any ideal is actually a finite intersection of primary ideals.

Lemma 1.9 *If I is an irreducible ideal in a Noetherian ring R, then I is primary.*

Proof Let I be irreducible and suppose $xy \in I$ with $x \notin I$. Passing to $A = R/I$ we must show that $y^m = 0$ for some $m > 0$. There is an ascending chain of ideals in A:

$$(0) \subseteq \mathrm{Ann}(y) \subseteq \mathrm{Ann}(y^2) \subseteq \cdots .$$

Since A is Noetherian, there exists an n such that

$$\mathrm{Ann}(y^n) = \mathrm{Ann}(y^{n+1}).$$

Since (0) is irreducible in A by assumption and $x \neq 0$, it suffices to show that $(x) \cap (y^n) = (0)$ in A. Let $a \in (x) \cap (y^n)$. Since $a \in (x)$, $a = bx$ for some $b \in A$. Hence $ay = bxy = 0$ in A. Since $a \in (y^n)$, $a = cy^n$ for some $c \in A$. Therefore $cy^n + 1 = ay = 0$. Therefore $c \in \mathrm{Ann}(y^{n+1}) = \mathrm{Ann}(y^n)$ and so $a = cy^n = 0$. Thus $(x) \cap (y^n) = (0)$ in A, and since (0) is irreducible, $y^n = 0$ in A. ■

Definition A **primary decomposition** of an ideal I is an expression realizing I as a finite intersection of primary ideals, that is

$$I = \bigcap_{i=1}^{n} \mathfrak{q}_i,$$

where each $\mathfrak{q}_i$ is primary to some prime ideal $\mathfrak{p}_i$. The decomposition is **irredundant** if for each $j \in \{1, \ldots, n\}$,

$$\bigcap_{i \neq j} \mathfrak{q}_i \neq I.$$

Note that by a previous exercise, we may assume the primes $\mathfrak{p}_i$ are distinct; these are called the **associated primes** of I. An associated prime $\mathfrak{p}_i$ is called a **minimal prime** of I if it does not properly contain any other associated prime; otherwise, it is an **embedded prime**.

Exercise 1.10 *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal and $x \in R$. Then*

1. *If $x \in \mathfrak{q}$, then $\mathfrak{q} : x = R$.*

2. *If $x \notin \mathfrak{q}$, then $\mathfrak{q} : x$ is $\mathfrak{p}$-primary.*

3. *If $x \notin \mathfrak{p}$, then $\mathfrak{q} : x = \mathfrak{q}$.*

Exercise 1.11 1. *If $\mathfrak{p}$ is prime, then $\mathfrak{p}$ is irreducible.*

2. $(I_1 \cap I_2) : x = (I_1 : x) \cap (I_2 : x)$.

3. $\sqrt{I_1 \cap I_2} = \sqrt{I_1} \cap \sqrt{I_2}$.

The following result shows that the definition of associated primes is well-defined and does not depend on the primary decomposition.

Theorem 1.12 *The associated primes of an ideal are unique.*

Proof (sketch) Let I be an ideal of R. Find a primary decomposition $I = \bigcap_{i=1}^{n} \mathfrak{q}_i$ of I. Let $x \in R$. Then

$$\sqrt{I : x} = \sqrt{\bigcap_{i=1}^{n} \mathfrak{q}_i : x} = \bigcap_{j : x \notin \mathfrak{q}_j} \mathfrak{p}_j.$$

By prime avoidance, for each j we can pick $x \in \mathfrak{q}_j \setminus \bigcup_{i \neq j} \mathfrak{q}_i$. Now check that the associated primes are precisely the maximal elements in the set $S = \{\sqrt{I : x} \mid x \in R\}$, and hence do not depend on the decomposition. ■

Example 1.13 *While the associated primes are unique, primary decompositions are not. For example, let $R = k[x, y]$ and consider the ideal $I = (x^2, xy)$. Then*

$$I = (x^2, xy, y^2) \cap (x),$$

is a primary decomposition of I and so the associated primes are (x) and $\sqrt{(x^2, xy, y^2)} = (x, y)$. However, we can also write

$$I = (x^2, y) \cap (x),$$

which is also a primary decomposition of I. Note that the (x, y)-primary components differ.

You can compute a primary decomposition of an ideal in Macaulay2 using the command `primaryDecomposition I`. The associated primes are computed with `ass I`.

Exercise 1.14 *The primary components of an ideal corresponding to a minimal prime are unique.*

We can extend the definition of associated primes (and primary decompositions) to arbitrary modules.

Definition Let M be an R-module. The **associated primes** of M, denotes $\mathrm{Ass}_R(M)$ (or just $\mathrm{Ass}(M)$ when the ring in question is clear), are the primes $\mathfrak{p}$ which are the annihilators of some element $m \in M$.

Exercise 1.15 *Let M be an R-module and set $S = \{I \subseteq R \mid I = \mathrm{Ann}(m)$ for some $m \in M\}$. Show that a maximal element in S is prime. It follows that if $M \neq 0$, then $\mathrm{Ass}_R(M) \neq \varnothing$.*

Note that the two definitions clash if we consider an R-ideal I as a module. For instance, if I is an ideal in a domain, it has no nontrivial annihilators of elements. Thus associated primes of an ideal in the former sense correspond to associated primes of the module R/I.

Exercise 1.16 *Let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of R-modules. Then*

$$\mathrm{Ass}_R(M_1) \subseteq \mathrm{Ass}_R(M_2) \subseteq \mathrm{Ass}_R(M_1) \cup \mathrm{Ass}_R(M_3).$$

Exercise 1.17 *Let* M_1, M_2 *be R-modules. Then*

$$\mathrm{Ass}_R(M_1 \oplus M_2) = \mathrm{Ass}_R(M_1) \cup \mathrm{Ass}_R(M_2).$$

Exercise 1.18 *Let M be an R-module. Then*

$$\{x \in R \mid x \text{ is a zerodivisor on } M\} = \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}.$$

Theorem 1.19 *Let R be a Noetherian ring. For any finitely generated, nonzero R-module M, there exists a filtration of M,*

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = (0),$$

such that for all $i = 1, \ldots, n$, $M_{i-1}/M_i \simeq R/\mathfrak{p}_i$ *where each* $\mathfrak{p}_i$ *is a prime ideal. Moreover, given any such filtration,* $\mathrm{Ass}_R(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.

Proof Let $\mathcal{S}$ be the collection of submodules of M that have a prime filtration as stated above. $\mathcal{S} \neq \varnothing$, since for any $\mathfrak{p} \in \mathrm{Ass}_R(M)$, we have an injection $\iota : R/\mathfrak{p} \hookrightarrow M$ so that $\iota(R/\mathfrak{p}) \in \mathcal{S}$. Since M is Noetherian, $\mathcal{S}$ has a maximal element, say M_0.

We claim $M = M_0$. Suppose not, then we have the exact sequence

$$0 \longrightarrow M_0 \longrightarrow M \xrightarrow{\varphi} M/M_0 \longrightarrow 0.$$

By assumption $M/M_0 \neq (0)$, so there exists a prime $\mathfrak{p}' \in \mathrm{Ass}_R(M/M_0)$. Thus we have an injection $j : R/\mathfrak{p}' \hookrightarrow M/M_0$. Set $T = j(R/\mathfrak{p}')$ and $Q = \varphi^{-1}(T)$. Then we have a new exact sequence

$$0 \longrightarrow M_0 \longrightarrow Q \longrightarrow T \longrightarrow 0.$$

Since M_0 has a prime filtration, and since $Q/M_0 \simeq T \simeq R/\mathfrak{p}'$, Q has a prime filtration. However, $Q \supsetneq M_0$ contradicts that M_0 is maximal in $\mathcal{S}$. Thus, we must have that $M = M_0$.

The second part of the theorem follows from Exercise 1.16. ■

Corollary 1.20 *Let R be a Noetherian ring and M be a finitely generated R-module. Then* $\mathrm{Ass}_R(M)$ *is finite.*

Proposition 1.21 *Let R be a Noetherian ring, M be a finitely generated R-module, and* $\mathfrak{p}$ *be a prime ideal. The following are equivalent:*

1. $M_{\mathfrak{p}} \neq 0$.

2. $\mathfrak{p} \supseteq \mathrm{Ann}_R(M)$.

3. $\mathfrak{p} \supseteq \mathfrak{q}$ *for some prime ideal* $\mathfrak{q} \in \operatorname{Ass}_R(M)$.

Proof Exercise. ■

Definition The set of prime ideals $\mathfrak{p}$ satisfying the four equivalent conditions above are called the **support** of M, denoted $\operatorname{Supp}_R(M)$.

Corollary 1.22 *Let R be a Noetherian ring and M be a finitely generated R-module. The minimal elements of* $\operatorname{Ass}_R(M)$ *are the minimal elements of* $\operatorname{Supp}_R(M)$.

Exercise 1.23 *Let R be a Noetherian ring and*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finitely generated R-modules. Show that

$$\operatorname{Supp}_R(M) = \operatorname{Supp}_R(M') \cup \operatorname{Supp}_R(M'').$$

2. Basic Algebraic Geometry

Here we state the bare minimum facts about algebraic geometry needed for the remainder. This should be seen as a motivation for the primary decomposition results in the previous section.

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k. Let $\mathbb{A}^n_k$ denote the n-dimensional **affine space** over k, i.e $\mathbb{A}^n_k = k^n$. Let $I \subseteq S$ be an ideal. Then the zero set of I defines a subset of $\mathbb{A}^n$ called the (affine) **variety** associated to I, that is

$$V(I) = \{\boldsymbol{a} \in \mathbb{A}^n_k \mid f(\boldsymbol{a}) = 0 \quad \forall f \in I\}.$$

Now let $S = k[x_0, \ldots, x_n]$. Let $\mathbb{P}^n_k$ denote the n-dimensional **projective space** over k, i.e. $\mathbb{P}^n_k = (k^{n+1} - \mathbf{0})/\sim$, where $\sim$ denotes the equivalence relation $\boldsymbol{b} \sim \alpha\boldsymbol{b}$ for all $\alpha \in k - \{0\}$ and all $\boldsymbol{b} \in k^{n+1}$. Let I be a homogeneous ideal in S. Then the projective variety associated to I is

$$V(I) = \{[\boldsymbol{b}] \in \mathbb{P}^n_k \mid f(\boldsymbol{b}) = 0 \quad \forall f \in I\}.$$

Note that since I is homogeneous, this is well-defined. Also since the maximal ideal $\mathfrak{m} = (x_0, \ldots, x_n)$ satisfies $V(\mathfrak{m}) = \varnothing$, we restrict our attention to ideals without $\mathfrak{m}$-primary components.

Similarly, let $X \subseteq \mathbb{A}^n_k$ (resp. $\mathbb{P}^n_k$). Denote by $I(X)$ the ideal of polynomials that vanish on X, that is

$$I(X) = \{f \in S \mid f(\boldsymbol{a}) = 0 \quad \forall \boldsymbol{a} \in X\}.$$

If $X \subseteq \mathbb{P}^n_k$, we use the same notation to denote the homogeneous ideal of X. Example

Note that the following exercise also holds when we consider the affine versions of the statements.

Exercise 2.1 *Let $I_1, I_2 \subseteq S$ be homogeneous ideals and let $V_1, V_2 \subseteq \mathbb{P}^n_k$. Then*

1. $V(I_1 + I_2) = V(I_1) \cap V(I_2)$
2. $V(I_1 \cap I_2) = V(I_1 I_2) = V(I_1) \cup V(I_2)$
3. $V(I) = V(\sqrt{I})$

4. *If* $V_1 \subseteq V_2$*, then* $I(V_1) \supseteq I(V_2)$

5. $I(V_1 \cup V_2) = I(V_1) \cap I(V_2)$

6. $I \subseteq I(V(I))$ *and* $V(I(V(I))) = V(I)$.

Example 2.2 *In general, it is not true that* $I(V(I)) = I$*. Take for instance,* $I = (x^2) \subseteq k[x]$*. Then* $V(I) = \{0\} \subseteq \mathbb{A}^1_k$*. Every polynomial that vanishes at* 0 *is divisible by* x *(and vice versa). Therefore* $I(V(I)) = (x) \supsetneq (x^2)$.

Example 2.3 *Other things can go wrong when we are not working over an algebraically closed field. For example, consider* $I = (x^2 + 1) \subseteq \mathbb{R}[x]$*. Then* $V(I) = \varnothing$*, since* $x^2 + 1 = 0$ *has no solutions over the real numbers.*

Aside from the issues in the previous two examples, the operators $I(-)$ and $V(-)$ satisfy a nice duality summarized by the Nullstellensatz.

Theorem 2.4 (Hilbert's Nullstellensatz) *Let* I *be an ideal of* $S = k[x_1, \ldots, x_n]$*, with* k *algebraically closed. Then* $I(V(I)) = \sqrt{I}$.

The above statement also works in the projective case then when we consider homogeneous ideals.

3. Graded Rings, Hilbert Series, and Resolutions

We primarily consider rings and modules graded over $\mathbb{N}$ or $\mathbb{Z}$, although one can define a grading over any abelian semigroup. A **graded ring** is a ring R together with a direct sum decomposition of R as an additive group $R = \bigoplus_{i\in\mathbb{Z}} R_i$ satisfying $R_iR_j \subseteq R_{i+j}$. Similarly a **graded R-module** is an R-module together with a direct sum decomposition $M = \bigoplus_{i\in\mathbb{Z}} M_i$ satisfying $R_iM_j \subseteq M_{i+j}$. An element $m \in M$ is **homogeneous** if $m \in M_i$ for some i, in which case we say the **degree** of m is i. (Note that the 0 element can have any degree and is sometimes decreed to have no degree.) A submodule N of a graded R-module M is called **graded** if it can be generated by homogeneous elements. Note that N is again a graded R-module by setting $N_i = M_i \cap N$. Similarly an ideal $I \subseteq R$ is called **homogeneous** (or **graded**) if it can be generated by homogeneous elements.

Hilbert Functions and Series

Our standard setting for considering graded rings and modules will a standard graded polynomial ring over a field, say $S = k[x_1, \dots, x_n]$ in which $\deg(x_i) = 1$ for all i. Setting S_i to be the k-vector space of homogeneous polynomials of degree i, we realize S as an $\mathbb{N}$-graded or $\mathbb{Z}$-graded ring. If M is a finitely generated graded S-module, then $\dim_k(M_i) < \infty$. We define the **Hilbert function** of M to be $\dim_k(M_n)$.

For a graded module M, the **Hilbert series** $\mathrm{Hilb}_M(t) = \sum_{i\geq 0}\dim_K(M_i)t^i$ can be written as a rational function of the form $\mathrm{Hilb}_M(t) = \frac{h(t)}{(1-t)^s}$, where $s = \dim(M) - 1$ and h is a polynomial of degree at most n, called the **Euler polynomial** of M. (We will prove this later.) We define the **multiplicity** (sometimes called **degree**) of a graded R-module M to be the value $e(M) = h(1)$. For an artinian module M, the multiplicity is equal to the **length** of the module defined as $\lambda(M) = \sum_{i\geq 0}\dim_K(M_i)$. By convention, for a homogeneous ideal I, we refer to $e(R/I)$ as the multiplicity of I.

That $s = \dim(M) - 1$ follows by an inductive argument. See Theorem 4.1.3 in Bruns-Herzog "Cohen-Macaulay Rings."

Example 3.1 *Let $M = K[x,y]/(x^2, xy^2, y^4)$. M is an artinian module.*

(i.e. M has finite length) and has the following homogeneous basis:

$$\begin{array}{ll} \textit{basis of } M_0: & \{1\} \\ \textit{basis of } M_1: & \{x, y\} \\ \textit{basis of } M_2: & \{xy, y^2\} \\ \textit{basis of } M_3: & \{y^3\}. \end{array}$$

Thus the Hilbert Series of M is

$$\mathrm{Hilb}_M(t) = 1 + 2t + 2t^2 + t^3$$

and the multiplicity of M is:

$$e(M) = \lambda(M) = 1 + 2 + 2 + 1 = 6.$$

Example 3.2 *Let $M = K[x, y, z]/(x^2, xy^2, y^3)$. The module M has the following homogeneous basis:*

$$\begin{array}{lc} \textit{basis of } M_0: & \{1\} \\ \textit{basis of } M_1: & \{x, y, z\} \\ \textit{basis of } M_2: & \{xy, xz, y^2, yz, z^2\} \\ \textit{basis of } M_3: & \{xyz, xz^2, yz^2, y^2z, z^3\} \\ \textit{basis of } M_4: & \{xyz^2, xz^3, yz^3, y^2z^2, z^4\} \\ \textit{basis of } M_5: & \{xyz^3, xz^4, yz^4, y^2z^3, z^5\} \\ \vdots & \vdots \\ \textit{basis of } M_i: & \{xyz^{i-2}, xz^{i-1}, yz^{i-1}, y^2z^{i-2}, z^i\} \quad \textit{for } i \geq 3 \end{array}$$

Thus the Hilbert Series of M is

$$\mathrm{Hilb}_M(t) = 1 + 3t + 5t^2 + 5t^3 + 5t^4 + 5t^5 + 5t^6 + \cdots = \frac{1 + 2t + 2t^2}{1 - t}$$

and the multiplicity of M is:

$$e(M) = 1 + 2 + 2 = 5.$$

Exercise 3.3 *Let $S = K[x_1, \ldots, x_n]$. Show that*

$$\dim_K(S_i) = \binom{n+i-1}{i} = \frac{(n+i-1)!}{i!(n-1)!}$$

Conclude that

$$\mathrm{Hilb}_S(t) = \frac{1}{(1-t)^n}.$$

Graded Free Resolutions

Let k be a field and let $S = k[x_1, \dots, x_n]$ be a polynomial ring over k. We view S as a standard graded ring, that is, we set $\deg(x_i) = 1$ for $i = 1, \dots, n$. Let M be a finitely generated graded S-module. In many cases, M will be a cyclic module, i.e. $M = S/I$ for some homogeneous ideal I. A lot of the structure of M is encoded by its minimal graded free resolution.

Definition A **left complex F of finitely generated free modules over** S is a sequence of homomorphisms of finitely generated free S-modules

The same definitions work over an arbitrary commutative ring.

$$\mathbf{F}_\bullet : \quad \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 ,$$

such that $d_{i-1} \circ d_i = 0$ for all i. The collection of maps $d = \{d_i\}$ is called the **differential** of $\mathbf{F}_\bullet$. The complex is sometimes denoted $(\mathbf{F}_\bullet, d)$. The ith **Betti number** of $\mathbf{F}_\bullet$ is the rank of the module F_i. The **homology** of $\mathbf{F}_\bullet$ is $\mathrm{H}_i(\mathbf{F}_\bullet) = \mathrm{Ker}(d_i)/\mathrm{Im}(d_{i+1})$. The complex is **exact** at F_i, or at **step** i, if $\mathrm{H}_i(\mathbf{F}_\bullet) = 0$. We say that $\mathbf{F}_\bullet$ is *acyclic* if $\mathrm{H}_i(\mathbf{F}_\bullet) = 0$ for all $i > 0$. A **free resolution** of a finitely generated S-module M is an acyclic left complex of finitely generated free S-modules

$$\mathbf{F}_\bullet : \quad \cdots \longrightarrow F_i \xrightarrow{d_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_1} F_0 ,$$

such that $M \cong F_0/\mathrm{Im}(d_1)$.

Before we discuss minimal resolutions, we recall Nakayama's Lemma.

Theorem 3.4 (Nakayama's Lemma) *If I is a proper graded ideal in S and M is a finitely generated graded S-module such that $M = IM$, then $M = 0$.*

There is a corresponding Nakayama's Lemma for local rings. See e.g. ???

Proof Suppose that is $M \neq 0$. We choose a finite minimal system of homogeneous generators of M. Let m be an element of minimal degree in that system. It follows that $M_j = 0$ for $j < \deg(m)$. Since I is a proper ideal, we conclude that every homogeneous element in IM has degree strictly greater than $\deg(m)$. This contradicts to $m \in M = IM$. ■

Theorem 3.5 *Let M be a finitely generated graded S-module. Consider the graded k-vector space $\overline{M} = M/(x_1, \dots, x_n)M$. Homogeneous elements $m_1, \dots, m_r \in M$ form a minimal system of homogeneous generators of M if and only if their images in $\overline{M}$ form a basis. Every minimal system of homogeneous generators of M has $\dim_k(\overline{M})$ elements.*

In particular, Theorem 3.5 shows that every minimal system of generators of M has the same number of elements.

Definition A free resolution $(\mathbf{F}_\bullet, d_\bullet)$ of a finitely generated graded module M is called **graded** if the modules F_i are graded and each d_i is a graded homomorphism of degree 0. The resolution is called **minimal** if

$$d_{i+1}(F_{i+1}) \subseteq (x_1, \ldots, x_n)F_i \quad \text{for all } i \geq 0.$$

This means that no invertible elements (non-zero constants) appear in the matrices defining the differential maps. In this case, we may write

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{ij}(M)} \text{ for each } \ i.$$

The integers $\beta_{ij}(M) \in \mathbb{N}$ are called the **graded Betti numbers** of M.

Example 3.6 *Let $S = k[w, x, y, z]$ and let $I = (wy - x^2, wz - xy, xz - y^2)$. The minimal graded free resolution of S/I is*

I is generated by the 2×2 minors of the matrix $\begin{pmatrix} w & x & y \\ x & y & z \end{pmatrix}$. It defines the image of $\mathbb{P}^1 \to \mathbb{P}^3$ under the 3rd Veronese map, which sends homogeneous coordinate $[s : t]$ to $[s^3 : s^2t : st^2 : t^3]$. The image is a projective curve called the twisted cubic. In particular, this means that I is a prime ideal.

$$0 \to S(-3)^2 \xrightarrow{\begin{pmatrix} y & z \\ -x & -y \\ -w & -x \end{pmatrix}} S(-2)^3 \xrightarrow{\begin{pmatrix} wy - x^2 & wz - xy & xz - y^2 \end{pmatrix}} S$$

So the nonzero graded Betti numbers are

$$\beta_{0,0} = 1$$
$$\beta_{1,2} = 3$$
$$\beta_{2,3} = 2.$$

The word “minimal" refers to the properties in the next two results. On the one hand, Theorem 3.7 shows that minimality means that at each step we make an optimal choice, that is, we choose a minimal system of generators of the kernel in order to construct the next differential. On the other hand, Theorem 3.8 shows that minimality means that we have a smallest resolution which lies (as a direct summand) inside any other resolution of the module.

Theorem 3.7 *The graded free resolution constructed above is minimal if and only if at each step we choose a minimal homogeneous system of generators of the kernel of the differential. In particular, every finitely generated graded S-module has a minimal graded free resolution.*

For a proof, see Theorem 3.4

Irena Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011

Theorem 3.8 *Let M be a finitely generated graded S-module, and $\mathbf{F}_\bullet$ be a minimal graded free resolution of M. If $\mathbf{G}_\bullet$ is any graded free resolution of M, then we have a direct sum of complexes $\mathbf{G}_\bullet \cong \mathbf{F}_\bullet \oplus \mathbf{P}_\bullet$ for some complex $\mathbf{P}_\bullet$, which is a direct sum of short trivial complexes*

$$0 \longrightarrow S(-p) \xrightarrow{1} S(-p) \longrightarrow 0$$

possibly placed in different homological degrees.

For a proof, see Theorem 3.5

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It follows that the minimal graded free resolution of M is unique up to an isomorphism. Hence any invariants, derived from it such as the graded Betti numbers, are also invariants of M. In the next section we study some of these invariants. We'll see in the next chapter that all minimal graded free resolutions are finite.

Check out Hilbert's Syzygy Theorem 4.33.

Projective Dimension, Regularity and Betti Tables

There is a lot of information contained in the minimal graded free resolution. Even ignoring the maps, we can recover many invariants of M from the graded Betti numbers alone. Therefore, it is useful to have a bookkeeping device to keep track of them. We typically write them in a matrix called the **Betti table** of M. In position (i, j) we place $\beta_{i,i+j}(M)$.

Two coarser invariants will be our primary objects of study for the remainder of these notes, namely projective dimension and (Castelnuovo-Mumford) regularity.

Regularity has equivalent definitions in terms of sheaf cohomology (where it was originally defined) and local cohomology. See ???

Definition Let M be a finitely generated graded S-module. The projective dimension M is

$$\operatorname{pd}(M) = \max\{i \mid \beta_{ij}(M) \neq 0\}.$$

Equivalently, $\operatorname{pd}(M)$ is a length of the minimal graded free resolution of M.

Why not free dimension? Over S all finitely generated projective modules are free. The same statement holds over Noetherian local rings. See ???

The regularity of M is

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{ij}(M) \neq 0\}.$$

Example 3.9 *Continuing Example 3.6, we see that the Betti table of S/I is*

You can compute the Betti table of a module `M` in Macaulay2 with the commands `betti res M`.

	0	1	2
0:	1	-	-
1:	-	3	2

Therefore $\operatorname{pd}(S/I) = 2$ *and* $\operatorname{reg}(S/I) = 1$. *Note that you can read the projective dimension of S/I off the Betti table as the index of the last nonzero column. Regularity is then the index of the last nonzero row.*

More on HIlbert Series and Multiplicity

Proposition 3.10 *If $0 \to L \to M \to N \to 0$ is a short exact sequence of finitely generated S-modules, then*

$$\operatorname{Hilb}_M(t) = \operatorname{Hilb}_L(t) + \operatorname{Hilb}_N(t).$$

Proof For each i, the sequence $0 \to L_i \to M_i \to N_i \to 0$ is exact. ■

Theorem 3.11 *Let $\mathcal{F}_\bullet$ be any graded free resolution of M. Write $F = \sum_j S(-j)^{\alpha_{ij}}$. Then*

$$\mathrm{Hilb}_M(t) = \frac{\sum_{i\geq 0}\sum_{j\in\mathbb{Z}}(-1)^i\alpha_{ij}t^j}{(1-t)^n}.$$

In particular

$$\mathrm{Hilb}_M(t) = \frac{\sum_{i\geq 0}\sum_{j\in\mathbb{Z}}(-1)^i\beta_{ij}(M)t^j}{(1-t)^n},$$

where $\beta_{ij}(M)$ are the graded Betti numbers of M.

Proof (sketch) Induct on length$(\mathcal{F}_\bullet)$ and use the previous proposition. ■

Corollary 3.12 *Let M be a finitely generated graded S-module. There exists a polynomial $P_M(x) \in \mathbb{Q}[x]$ of degree at most $n-1$ such that*

$$P_M(i) = \dim_K(M_i) \qquad \text{for } i \gg 0.$$

The polynomial $P_M(x)$ in the previous corollary is called the **Hilbert polynomial** of M.

Example 3.13 *Returning to Example 3.2, we have $M = K[x,y,z]/(x^2, xy^2, y^3)$. The graded free resolution of M has the form:*

$$0 \to S(-4)^2 \to S(-3)^2 \oplus S(-2) \to S.$$

$$\begin{aligned}\mathrm{Hilb}_M(t) &= \tfrac{1-t^2-2t^3+2t^4}{(1-t)^2}\\ &= \tfrac{(1-t)(1+2t+2t^2)}{(1-t)^2}\\ &= \tfrac{1+2t+2t^2}{1-t}.\end{aligned}$$

This seems much easier than finding a basis for M_i for all $i \geq 0$!

Exercise 3.14 *If $\underline{f} = f_1, \ldots, f_m \in S = K[x_1, \ldots, x_n]$ form a regular sequence of forms of degrees $d_1, \ldots, d_m$, then*

$$\mathrm{Hilb}_{S/(\underline{f})}(t) = \frac{\prod_{i=1}^m(1-t^{d_i})}{(1-t)^n}.$$

Theorem 3.15 *Let M be a finitely generated graded S-module. Then*

$$\deg(P_M(x)) = \dim(M) - 1.$$

Proof Exercise. ■

Theorem 3.16 *If $\lambda(M) < \infty$, then $e(M) = \lambda(M)$. If not, then the leading coefficient of $P_M(x)$ is*

$$\frac{e(M)}{(\dim(M)-1)!}.$$

Proof If M is artinian (i.e. $\lambda(M) < \infty$) then $\mathrm{Hilb}_M(t) = h(t)$ and

$$\lambda(M) = \sum_i \dim_K(M_i) = h(1) = e(M).$$

If $\dim(M) > 0$, then

$$\mathrm{Hilb}_M(t) = \frac{h(t)}{(1-t)^s} \qquad \text{where } s = \dim(M) - 1.$$

Write $h(t) = h_r t^r + h_{r-1}t^{r-1} + \cdots + h_0$. Then

$$\begin{aligned}\mathrm{Hilb}_M(t) &= h(t) \cdot \sum_{e \geq 0} \binom{s-1+i}{s-1} t^i \\ &= \sum_{i \geq r} \left(h_r \binom{s-1+i}{s-1} + \cdots + h_0 \binom{s-1+i+r}{s-1} \right) t^i + \text{ terms of degree } < r.\end{aligned}$$

So

$$P_M(x+r) = h_r \binom{s-1+i}{s-1} + \cdots + h_0 \binom{s-1+i+r}{s-1},$$

which is a polynomial of degree $s-1$ in x whose leading coefficient is

$$\frac{h_0 + h_1 + \cdots + h_r}{(s-1)!} = \frac{h(1)}{(\dim(M)-1)!} = \frac{e(M)}{(\dim(M)-1)!}.$$

■

Example 3.17 *Let $S = K[w,x,y,z]$ and $I = (wy - x^2, wz - xy, xz - y^2)$. Then S/I has graded free resolutions:*

$$0 \to S(-3)^2 \to S(-2)^3 \to S.$$

Thus the HIlbert Series of S/I is

$$\begin{aligned}\mathrm{Hilb}_{S/I}(t) &= \frac{1 - 3t^2 + 2t^3}{(1-t)^4} \\ &= \frac{2t+1}{(1-t)^2} \\ &= (2t+1) \cdot \sum_{i \geq 0} (1+i)t^i \\ &= 1 + \sum_{\geq 1} (3i+1)t^i\end{aligned}$$

Therefore the Hilbert polynomial of S/I is

$$\mathrm{Hilb}_{S/I}(x) = 3x + 1$$

and the multiplicity is

$$e(S/I) = 3.$$

Exercise 3.18 *Let $\underline{f} = f_1, \ldots, f_m$ be a regular sequence of forms of degrees $d_1, \ldots, d_m$. Show that*

$$e(S/(\underline{f}) = \prod_{i=1}^{m} d_i.$$

Next, we recall the associativity formula for multiplicity. For an ideal I of S,

$$e(S/I) = \sum_{\substack{\mathfrak{p} \in \operatorname{Spec}(S) \\ \operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(I)}} e(S/\mathfrak{p}) \lambda(S_{\mathfrak{p}}/I_{\mathfrak{p}}).$$

Let I^{un} denote the unmixed part of I, defined as the intersection of minimal primary components of I with height equal to $\operatorname{ht}(I)$. For every $\mathfrak{p} \in \operatorname{Spec}(S)$ with $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(I)$, we have that $I^{un}_{\mathfrak{p}} = I_{\mathfrak{p}}$. Hence

$$e(R/I^{un}) = \sum_{\substack{\mathfrak{p} \in \operatorname{Spec}(R) \\ \dim(R/\mathfrak{p}) = \dim(R/I^{un})}} e(R/\mathfrak{p}) \lambda(R_{\mathfrak{p}}/I^{un}_{\mathfrak{p}}) = \sum_{\substack{\mathfrak{p} \in \operatorname{Spec}(R) \\ \dim(R/\mathfrak{p}) = \dim(R/I)}} e(R/\mathfrak{p}) \lambda(R_{\mathfrak{p}}/I_{\mathfrak{p}}) = e(R/I).$$

Hence only the unmixed part of an ideal contributes to its multiplicity. We will often pass to the unmixed part of I and use the fact that the multiplicity does not change.

We use the following notation to keep track of the possibilities for the associated primes of minimal height of an ideal J.

Definition We say J is of type $\langle e = e_1, e_2, \ldots, e_m | \lambda = \lambda_1, \lambda_2, \ldots, \lambda_m \rangle$ if J has m associated primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of minimal height with $e(R/\mathfrak{p}_i) = e_i$ and with $\lambda(R_{\mathfrak{p}_i}/J_{\mathfrak{p}_i}) = \lambda_i$, for $i = 1, \ldots, m$. (In which case, we have $e(R/J) = \sum_{i=1}^{m} e_i \lambda_i$ by the associativity formula.)

The above notation allows us to systematically study unmixed ideals of a given height and multiplicity by enumerating their types. For instance, an unmixed height two ideal of multiplicity two has one of the following types: $\langle 2; 1 \rangle$, $\langle 1; 2 \rangle$, or $\langle 1, 1; 1, 1 \rangle$. We will use this in our study of Stillman's Question for three quadrics.

4. Regular Sequences, Depth, Ext and Tor, Cohen-Macaulay Modules

In this chapter we summarize basic homological algebra we need.

Ext and Tor

Definition Given a ring R and an R-module N, $\mathbf{Tor}_i^R(-, N)$ is the left derived functor of the right exact covariant functor $- \otimes_R N$.

To be more explicit, consider **any** projective resolution of an R-module M:

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \longrightarrow 0$$

Apply the functor $- \otimes_R N$ and chop off the $M \otimes_R N$ term to get the complex $P_\bullet \otimes_R N$:

$$\cdots \to P_2 \otimes_R N \to P_1 \otimes_R N \to P_0 \otimes_R N \to 0$$

We now define

$$\mathrm{Tor}_i^R(M, N) := H_i(P_\bullet \otimes_R N) = \frac{\mathrm{Ker}(d_i \otimes 1)}{\mathrm{Im}(d_{i+1} \otimes 1)}$$

Note that since

$$P_1 \to P_0 \to M \to 0$$

is exact,

$$P_1 \otimes_R N \to P_0 \otimes_R N \to M \otimes_R N \to 0$$

is also exact. Hence

$$\mathrm{Tor}_0^R(M, N) \simeq M \otimes_R N.$$

Proposition 4.1 $\mathrm{Tor}_i^R(M, N)$ *does not depend on the choice of projective resolution used. Hence it is well-defined.*

Exercise 4.2 *If N is R-flat or if M is R-flat, show that*

$$\mathrm{Tor}_i^R(M, N) = 0$$

for all R-modules M and $i > 0$. Hint: For the second part, first show that if

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is exact and M_2 and M_3 are flat, so is M_1.

Proposition 4.3 *Given an exact sequence of R-modules,*

$$0 \to M' \to M \to M'' \to 0$$

we obtain a long exact sequence of Tor*'s:*

$$\cdots \longrightarrow \mathrm{Tor}^R_{n+1}(M', N) \longrightarrow \mathrm{Tor}^R_{n+1}(M, N) \longrightarrow \mathrm{Tor}^R_{n+1}(M'', N) \xrightarrow{\partial_{n+1}}$$
$$\longrightarrow \mathrm{Tor}^R_{n}(M', N) \longrightarrow \mathrm{Tor}^R_{n}(M, N) \longrightarrow \mathrm{Tor}^R_{n}(M'', N) \xrightarrow{\partial_{n}}$$
$$\longrightarrow \mathrm{Tor}^R_{n-1}(M', N) \longrightarrow \cdots\cdots\cdots\cdots \longrightarrow \mathrm{Tor}^R_{1}(M'', N) \xrightarrow{\partial_{1}}$$
$$\longrightarrow M' \otimes_R N \longrightarrow M \otimes_R N \longrightarrow M'' \otimes_R N \longrightarrow 0$$

Proposition 4.4 *Given a ring R and two R-modules M and N, we then have*

$$\mathrm{Tor}^R_i(M, N) \simeq \mathrm{Tor}^R_i(N, M).$$

Proposition 4.5 *Given an exact sequence of R-modules,*

$$0 \to N' \to N \to N'' \to 0$$

we obtain a long exact sequence of Tor*'s:*

$$\cdots \longrightarrow \mathrm{Tor}^R_{n+1}(M, N') \longrightarrow \mathrm{Tor}^R_{n+1}(M, N) \longrightarrow \mathrm{Tor}^R_{n+1}(M, N'') \xrightarrow{\partial_{n+1}}$$
$$\longrightarrow \mathrm{Tor}^R_{n}(M, N') \longrightarrow \mathrm{Tor}^R_{n}(M, N) \longrightarrow \mathrm{Tor}^R_{n}(M, N'') \xrightarrow{\partial_{n}}$$
$$\longrightarrow \mathrm{Tor}^R_{n-1}(M, N') \longrightarrow \mathrm{Tor}^R_{n-1}(M, N) \longrightarrow \mathrm{Tor}^R_{n-1}(M, N'') \longrightarrow \cdots$$

Definition Given a ring R and an R-module N, $\mathbf{Ext}^{\boldsymbol{i}}_{\boldsymbol{R}}\boldsymbol{(-, N)}$ is the left derived functor of the left exact contravariant functor $\mathrm{Hom}_R(-, N)$.

To be more explicit, consider **any** projective resolution of an R-module M:

$$\cdots \longrightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} M \longrightarrow 0$$

Apply the functor $\mathrm{Hom}_R(-, N)$ and chop off the $\mathrm{Hom}_R(M, N)$ term to get the complex $\mathrm{Hom}_R(P_\bullet, N)$:

$$0 \xrightarrow{d_0^*} \mathrm{Hom}_R(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1, N) \xrightarrow{d_2^*} \mathrm{Hom}_R(P_2, N) \longrightarrow \cdots$$

where $d_0^* := 0$. We now define:

$$\mathrm{Ext}_R^i(M, N) := H^i(\mathrm{Hom}_R(P_\bullet, N)) = \frac{\mathrm{Ker}(d_{i+1}^*)}{\mathrm{Im}(d_i^*)}$$

The shift in degrees of the differentials in the quotient above, compared to the definition of cohomology, is due to the fact that d_i^* is the $(i-1)$th differential in the cocomplex $\mathrm{Hom}_R(P_\bullet, N)$. Since

$$P_1 \to P_0 \to M \to 0$$

is exact,

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1, N)$$

is also exact. Hence

$$\mathrm{Ext}_R^0(M, N) = \frac{\mathrm{Ker}(d_1^*)}{0} \simeq \mathrm{Hom}_R(M, N).$$

Proposition 4.6 $\mathrm{Ext}_R^i(M, N)$ *does not depend on the choice of projective resolution of M used to compute it. Hence it is well-defined.*

Definition Given a ring R and an R-module N, $\mathbf{Ext}_R^i(M, -)$ is the left derived functor of the left exact covariant functor $\mathrm{Hom}_R(M, -)$.

To be more explicit, consider **any** injective resolution of an R-module N:

$$0 \longrightarrow N \xrightarrow{\iota} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \longrightarrow \cdots$$

Apply the functor $\mathrm{Hom}_R(M, -)$ and chop off the $\mathrm{Hom}_R(M, N)$ term to get the complex $\mathrm{Hom}_R(M, E^\bullet)$:

$$0 \xrightarrow{d_*^{-1}} \mathrm{Hom}_R(M, E^0) \xrightarrow{d_*^0} \mathrm{Hom}_R(M, E^1) \xrightarrow{d_*^1} \mathrm{Hom}_R(M, E^2) \longrightarrow \cdots$$

where $d_*^{-1} := 0$. We now define:

$$\mathrm{Ext}_R^i(M, N) := H^i(\mathrm{Hom}_R(M, E^\bullet)) = \frac{\mathrm{Ker}(d_*^i)}{\mathrm{Im}(d_*^{i-1})}$$

Note that since

$$0 \to N \to E^0 \to E^1$$

is exact,

$$0 \longrightarrow \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_R(M, E^0) \xrightarrow{d_*^0} \mathrm{Hom}_R(M, E^1)$$

is also exact. Hence

$$\mathrm{Ext}_R^0(M, N) = \frac{\mathrm{Ker}(d_*^0)}{0} \simeq \mathrm{Hom}_R(M, N).$$

Proposition 4.7 *$\mathrm{Ext}^i_R(M,N)$ does not depend on the choice of injective resolution of N used to compute it. Hence it is well-defined.*

Proposition 4.8 *The two constructions of $\mathrm{Ext}^i_R(M,N)$ given above produce isomorphic modules and hence are equivalent.*

Proposition 4.9 *Given an exact sequence of R-modules,*

$$0 \to M' \to M \to M'' \to 0$$

we obtain a long exact sequence of Ext's:

$$\begin{aligned}
&0 \longrightarrow \mathrm{Hom}_R(M'',N) \longrightarrow \mathrm{Hom}_R(M,N) \longrightarrow \mathrm{Hom}_R(M',N) \xrightarrow{\partial_1} \\
&\longrightarrow \mathrm{Ext}^1_R(M'',N) \longrightarrow \cdots\cdots\cdots\cdots \longrightarrow \mathrm{Ext}^{n-1}_R(M',N) \xrightarrow{\partial_{n-1}} \\
&\longrightarrow \mathrm{Ext}^n_R(M'',N) \longrightarrow \mathrm{Ext}^n_R(M,N) \longrightarrow \mathrm{Ext}^n_R(M',N) \xrightarrow{\partial_n} \\
&\longrightarrow \mathrm{Ext}^{n+1}_R(M'',N) \longrightarrow \mathrm{Ext}^{n+1}_R(M,N) \longrightarrow \mathrm{Ext}^{n+1}_R(M',N) \longrightarrow \cdots
\end{aligned}$$

Proposition 4.10 *Given an exact sequence of R-modules,*

$$0 \to N' \to N \to N'' \to 0$$

we obtain a long exact sequence of Ext's:

$$\begin{aligned}
&\cdots \longrightarrow \mathrm{Ext}^{n-1}_R(M,N') \longrightarrow \mathrm{Ext}^{n-1}_R(M,N) \longrightarrow \mathrm{Ext}^{n-1}_R(M,N'') \xrightarrow{\partial_{n-1}} \\
&\longrightarrow \mathrm{Ext}^n_R(M,N') \longrightarrow \mathrm{Ext}^n_R(M,N) \longrightarrow \mathrm{Ext}^n_R(M,N'') \xrightarrow{\partial_n} \\
&\longrightarrow \mathrm{Ext}^{n+1}_R(M,N') \longrightarrow \mathrm{Ext}^{n+1}_R(M,N) \longrightarrow \mathrm{Ext}^{n+1}_R(M,N'') \longrightarrow \cdots
\end{aligned}$$

Proposition 4.11 *If R is a ring, the following are equivalent:*

1. *M is projective.*
2. $\mathrm{Ext}^i_R(M,N) = 0$ *for all R-modules N and for all $i > 0$.*
3. $\mathrm{Ext}^1_R(M,N) = 0$ *for all R-modules N.*

Proposition 4.12 *If R is a ring, the following are equivalent:*

1. *N is injective.*
2. $\mathrm{Ext}^i_R(M,N) = 0$ *for all R-modules M and for all $i > 0$.*

3. $\operatorname{Ext}^1_R(M, N) = 0$ *for all R-modules M.*

4. $\operatorname{Ext}^1_R(M, N) = 0$ *for all finitely generated R-modules M.*

5. $\operatorname{Ext}^1_R(R/I, N) = 0$ *for all ideals $I \subseteq R$.*

Regular Sequences and Depth

Definition Given a ring R and an R-module M, $x_1, \ldots, x_n \in R$ is called ***M*-regular** or an ***M*-sequence** if the following hold:

1. $(x_1, \ldots, x_n)M \neq M$.

2. For each $i > 0$,
$$\frac{M}{(x_1, \ldots, x_{i-1})M} \xrightarrow{x_i} \frac{M}{(x_1, \ldots, x_{i-1})M}$$
is an injective map; that is, x_i is a nonzerodivisor on $M/(x_1, \ldots, x_{i-1})M$ for $1 \leq i \leq n$.

Lemma 4.13 *If $x_1, \ldots, x_n$ is an M-sequence and*
$$x_1\xi_1 + \cdots + x_n\xi_n = 0,$$
with $\xi_i \in M$ for all i, then $\xi_i \in (x_1, \ldots, x_n)M$ for all i.

Proof Proceed by induction on n. If $n = 1$, then we have $x_1\xi_1 = 0$ which implies that $\xi_1 = 0 \in (x_1)M$.

Now suppose the lemma is true for $1, \ldots, n-1$. Since
$$x_1\xi_1 + \cdots + x_{n-1}\xi_{n-1} + x_n\xi_n = 0,$$
we have that $x_n\xi_n = 0$ in $M/(x_1, \ldots, x_{i-1})$. However, since x_n is a nonzerodivisor on $M/(x_1 \ldots, x_{n-1})$, we see that in fact $\xi_n \in (x_1, \ldots, x_{n-1})M$. Hence for some $m_1, \ldots, m_{n-1}$ we have
$$\xi_n = x_1m_1 + \cdots + x_{n-1}m_{n-1}.$$
Thus
$$x_1\xi_1 + \cdots + x_{n-1}\xi_{n-1} + x_n(x_1m_1 + \cdots + x_{n-1}m_{n-1}) = 0,$$
$$x_1(\xi_1 + x_nm_1) + x_2(\xi_2 + x_nm_2) + \cdots + x_{n-1}(\xi_{n-1} + x_nm_{n-1}) = 0.$$
By induction, $(\xi_i + x_nm_i) \in (x_1, \ldots, x_{n-1})M$ and so we see that
$$\xi_i \in (x_1, \ldots, x_n)M \qquad \text{for } i = 1 \text{ to } n-1$$
and so we are done. ■

Lemma 4.14 *If $x_1, \ldots, x_n$ is an M-sequence, then $x_1^{r_1}, \ldots, x_n^{r_n}$ is an M-sequence for any positive integers $r_1, \ldots, r_n$.*

Proof It is enough to show $x_1^r, x_2, \ldots, x_n$ is an M-sequence as then we may consider $M' = M/x_1^r M$ with M'-sequence $x_2, \ldots, x_n$ and work inductively. Note that it is clear that

$$(x_1^r, \ldots, x_n)M \neq M,$$

since

$$(x_1, \ldots, x_n)M \neq M.$$

Now proceed by induction on r. The case of $r = 1$ is trivial. So let's assume that our theorem is true for $r = 1, \ldots, r-1$. Since

$$\frac{M}{(x_1, \ldots, x_{i-1})M} \xrightarrow{x_1} \frac{M}{(x_1, \ldots, x_{i-1})M}$$

is injective, so is

$$\frac{M}{(x_1, \ldots, x_{i-1})M} \xrightarrow{x_1^r} \frac{M}{(x_1, \ldots, x_{i-1})M}.$$

We must now show that $\mathbf{x}_i$ is a nonzerodivisor on $M/(x_1^r, x_2, \ldots, x_{i-1})M$ for $i = 2, \ldots, n$. Suppose for some $m \in M$ and for $\xi_1, \ldots, \xi_{i-1} \in M$ that

$$x_i m = x_1^r \xi_1 + x_2 \xi_2 + \cdots + x_{i-1}\xi_{i-1}.$$

Since we assume $x_1^{r-1}, \ldots, x_{i-1}$ is an M-sequence by induction, we have

$$m = x_1^{r-1} m_1 + x_2 m_2 + \cdots + x_{i-1} m_{i-1}$$

where $m_1, \ldots, m_{i-1} \in M$. Hence

$$0 = x_1(x_1^{r-1}\xi_1 - x_i m_1) + x_2(\xi_2 - x_i m_2) + \cdots + x_{n-1}(\xi_{i-1} - x_i m_{n-1}).$$

Thus by Lemma 4.13, we see that $(x_1^{r-1}\xi_1 - x_i m_1) \in (x_1, \ldots, x_{n-1})M$. Thus, $(x_i m_1) \in (x_1, \ldots, x_{i-1})M$ and so we see $m_1 \in (x_1, \ldots, x_{i-1})M$, because $x_1, \ldots, x_i$ is an M-sequence. Hence

$$m \in (x_1^r, \ldots, x_{i-1})M$$

and so we see that x_i is an injective map on $M/(x_1, \ldots, x_{i-1})$. Hence $x_1^r, \ldots, x_n$ is an M-sequence. This proves the theorem. ∎

Lemma 4.15 *If R is a Noetherian ring with M a finitely generated R-module and N any R-module, then* $\mathrm{Ass}_R(\mathrm{Hom}_R(M, N)) = \mathrm{Supp}_R(M) \cap \mathrm{Ass}_R(N)$.

Proof ($\subseteq$) Recall that $\mathfrak{p} \in \mathrm{Ass}_R(T)$ if and only if $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_\mathfrak{p}}(T_\mathfrak{p})$. So take $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Hom}_R(M, N))$. Then $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_\mathfrak{p}}(\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}))$. Since $\mathrm{Ass}_{R_\mathfrak{p}}(\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})) \neq \varnothing$, $\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \neq 0$ and hence $M_\mathfrak{p} \neq 0$. Thus $\mathfrak{p} \in \mathrm{Supp}(M)$. Further, we have an injection $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \hookrightarrow \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$. Therefore

$$\mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p}, N_\mathfrak{p}) \simeq \mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})) \neq 0.$$

Since $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p}$ is a vector space over $R_\mathfrak{p}/pR_\mathfrak{p}$, there is a nonzero map from $R_p/\mathfrak{p}R_\mathfrak{p} \to M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$. Therefore $\mathrm{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, N_p) \neq 0$ and so we have an injection $R_p/\mathfrak{p}R_\mathfrak{p} \hookrightarrow N_p$. Therefore $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_p}(N_p)$ and hence $\mathfrak{p} \in \mathrm{Ass}_R(N)$.

($\supseteq$) Working backwards, take $\mathfrak{p} \in \mathrm{Supp}_R(M) \cap \mathrm{Ass}_R(N)$. Then we have that $M_\mathfrak{p} \neq 0$ and there is an injection $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \hookrightarrow N_\mathfrak{p}$. By Nakayama's Lemma,

$$M_p \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \simeq M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p} \neq 0.$$

Since $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a field, $M_p \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$ is a nonzero vector field over $R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Hence there is a nonzero map $M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p} \to R_p/\mathfrak{p}R_p$. Composing with the injection above yields a nonzero map in $\mathrm{Hom}_{R_\mathfrak{p}}(M_p \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, N_\mathfrak{p})$. Therefore

$$\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, \mathrm{Hom}_{R_\mathfrak{p}}(R_p/\mathfrak{p}R_\mathfrak{p}, N_\mathfrak{p})) \simeq \mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p} \otimes_{R_\mathfrak{p}} R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}, N_\mathfrak{p})) \neq 0.$$

Therefore $\mathfrak{p}R_\mathfrak{p} \in \mathrm{Ass}_{R_\mathfrak{p}}(\mathrm{Hom}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}))$, and so $\mathfrak{p} \in \mathrm{Ass}_R(\mathrm{Hom}_R(M, N))$. ■

Theorem 4.16 *If R is a Noetherian ring and M is a finitely generated R-module and I is an ideal of R such that $IM \neq M$. Then the following are equivalent:*

1. $\mathrm{Ext}^i_R(N, M) = 0$ *for all $i < n$ and for all finitely generated R-modules N such that* $\mathrm{Supp}_R(N) \subseteq V(I) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{p} \supseteq I\}$.
2. $\mathrm{Ext}^i_R(R/I, M) = 0$ *for some ideal I of R and for all $i < n$.*
3. $\mathrm{Ext}^i_R(N, M) = 0$ *for all $i < n$ for some finitely generated R-module N where* $\mathrm{Supp}_R(N) = V(I)$.
4. *There exists $x_1, \ldots, x_n \in I$ which form an M-sequence.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (4) First we show that if $\mathrm{Hom}_R(N, M) = 0$, then there exists $a_1 \in I$ a nonzerodivisor on M. By the previous lemma, $\mathrm{Supp}_R(N) \cap \mathrm{Ass}_R(M) = \varnothing$. We prove by induction that there is an M-sequence of length n in I. Let $\mathfrak{p}$ be a prime ideal of R.

$$\begin{aligned} \mathfrak{p} \in \mathrm{Ass}_R(M) &\Rightarrow \mathfrak{p} \notin \mathrm{Supp}_R(N), \\ &\Rightarrow \mathfrak{p} \not\supseteq I. \end{aligned}$$

Hence, by prime avoidance,

$$I \not\subseteq \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(M)} \mathfrak{p}.$$

Since the union of associated primes are exactly the zerodivisors, there exists $a \in I$ such that a is a nonzerodivisor on M.

We now prove (3) $\Rightarrow$ (4) by induction on n. Observe that the $n = 0$ case is vacuous. Above we proved the $n = 1$ case. Now suppose that $Ext^i_R(N, M) = 0$ for $i < n$ for some finitely generated R-module N where $\mathrm{Supp}(R(N) = V(I)$. Since $\mathrm{Hom}_R(N, M) = 0$, there exists $a_1 \in I$ be a nonzerodivisor on M. Then we have an exact sequence

$$0 \to M \xrightarrow{a_1} M \to M/a_1M \to 0.$$

From the long exact sequence for Ext we have $\mathrm{Ext}^i_R(N, M/a_1M) = 0$ for $0 \leq i < n-1$. By induction, there is an M/a_1M-sequence $a_2, \ldots, a_n$. Therefore $a_1, \ldots, a_n$ is an M-sequence.

(4) $\Rightarrow$ (1) Suppose $a_1, \ldots, a_n$ is an M-sequence in I. Let N be a finitely generated R-module with $\mathrm{Supp}_R(N) \subseteq V(I)$. Let $J = \mathrm{Ann}_R(N)$. Then $V(J) \subseteq V(I)$ and $\sqrt{I} \subseteq \sqrt{J}$. Hence there exists $t > 0$ such that $I^t \subseteq J$. Take any $f \in \mathrm{Hom}_R(N, M)$. Fix $n \in N$. Since $a_1^t \in \mathrm{Ann}_R(N)$, we have $a_1^t f(n) = f(a_1^t n) = f(0) = 0$. Since a_1^t is a nonzerodivisor on M, $f(n) = 0$ and so $\mathrm{Hom}_R(N, M) = 0$.

Now consider the exact sequence

$$0 \to M \xrightarrow{a_1^t} M \to M/a_1^tM \to 0.$$

Since $a_2, \ldots, a_n$ is an M/a_1M-sequence, we may assume by induction that $\mathrm{Ext}^i_R(N, M/a_1^tM) = 0$ for $0 \leq i < n-1$. By the long exact sequence for Ext_R the map

$$\mathrm{Ext}^i_R(N, M) \xrightarrow{a_1^t} \mathrm{Ext}^i_R(N, M)$$

is an isomorphism for $0 \leq i < n-1$ and is injective for $i = n-1$. Since $a_1^t \in \mathrm{Ann}_R(N)$, the above map is the zero map for $0 \leq i < n$. Hence $\mathrm{Ext}^i_R(N, M) = 0$ for $0 \leq i < n$. ■

Definition Let R be a Noetherian ring, M a finitely generated R-module, and I an ideal of R such that $IM \neq M$. Then we define

$$\begin{aligned} \mathrm{depth}_I(M) &:= \inf\{i : \mathrm{Ext}^i(R/I, M) \neq 0\}, \\ &= \text{length of maximal } M\text{-sequence in } I. \end{aligned}$$

If R is a local or graded ring with (graded) maximal ideal $\mathfrak{m}$, then $\mathrm{depth}_{\mathfrak{m}}(M)$ will be abbreviated by $\mathrm{depth}(M)$.

Exercise 4.17 *Suppose that R is a local or graded ring and M is a finitely-generated (graded) R-module. Show that* $\mathrm{depth}(M) = 0$ *if and only if* $\mathfrak{m} \in \mathrm{Ass}_R(M)$.

Corollary 4.18 *If*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is a short exact sequence of finitely generated R-modules. Set $d_i = \mathrm{depth}_I(M_i)$. *Then:*

1. $d_1 \geq \min\{d_2, d_3 + 1\}$.
2. $d_2 \geq \min\{d_1, d_3\}$.
3. $d_3 \geq \min\{d_1 - 1, d_2\}$.

Proof Use the long exact sequence for Ext and the previous theorem characterizing depth. ■

Exercise 4.19 *Suppose that R is Noetherian, M and N are R-modules, where $\operatorname{depth}_{\operatorname{Ann}(M)}(N) = n$. Show that*

$$\operatorname{Ext}^i_R(M, N) = \begin{cases} 0 & \text{if } i < n, \\ \operatorname{Hom}_R(M, N/(x_1, \ldots, x_n)N) & \text{if } i = n, \end{cases}$$

where $x_1, \ldots, x_n$ is a maximal N-sequence in $\operatorname{Ann}(M)$.

Exercise 4.20 *Suppose that R is local or graded, M and N are R-modules, with $\operatorname{depth}_{\operatorname{Ann}(M)}(N) = n$. If $x_1, \ldots, x_n$ is both an R-sequence and an N-sequence, then setting $\overline{R} = R/\mathbf{x}R$ we have*

$$\operatorname{Ext}^i_R(M, N) = \operatorname{Ext}^{i-n}_{\overline{R}}(M, N/(x_1, \ldots, x_n)N)$$

for all $i > n$.

Theorem 4.21 *Let $(R, \mathfrak{m})$ be a local or graded ring, and M, N be finitely generated R-modules where $\operatorname{depth}(M) = m$ and $\dim(N) = n$. Then for $i < m - n$*

$$\operatorname{Ext}^i_R(N, M) = 0.$$

Proof If $n = 0$, then $\operatorname{Supp}_R(N) = \{\mathfrak{m}\}$. By assumption there exists an M-sequence of length m in $\mathfrak{m}$. By Theorem 4.16, $\operatorname{Ext}^i_R(N, M) = 0$ for $i < m$.

Now assume $n > 0$. By prime filtration theorem, N has a filtration

$$N = N_1 \supsetneq N_2 \supsetneq \cdots \supsetneq N_t = 0$$

such that $N_j/N_{j+1} \simeq R/\mathfrak{p}_j$ for $j = 1, \ldots, t-1$. To show that $\operatorname{Ext}^i_R(N, M)$ vanishes for $i < m - n$, it is sufficient by the long exact sequence for Ext to show that $\operatorname{Ext}^i_R(R/\mathfrak{p}, M) = 0$ for $i < m - n$. By induction, it suffices to prove this for primes $\mathfrak{p}$ with $\dim(R/\mathfrak{p}) = n$. Pick $x \in \mathfrak{m} - \mathfrak{p}$ and consider the short exact sequence

$$0 \to R/\mathfrak{p} \xrightarrow{x} R/\mathfrak{p} \to R/(\mathfrak{p} + xR) \to 0.$$

Then $\dim(R/(\mathfrak{p} + xR)) = n - 1$. By induction, $\operatorname{Ext}^i_R(R/(\mathfrak{p} + xR), M) = 0$ for $i < m - n + 1$. By the long exact sequence for Ext,

$$\operatorname{Ext}^i_R(R/\mathfrak{p}, M) \xrightarrow{x} \operatorname{Ext}^i_R(R/\mathfrak{p}, M)$$

is an isomorphism for $i < m - n$. Since $x \in \mathfrak{m}$, by Nakayama's Lemma we have $\operatorname{Ext}^i_R(N, M) = 0$ for $i < m - n$. ■

Corollary 4.22 *Let R be a local or graded ring and $N \subseteq M$ finitely generated R-modules. Then* $\dim(N) \geq \operatorname{depth}(M)$.

Proof Since $N \subseteq M$, there is a nonzero map $N \to M$. Thus $\operatorname{Hom}_R(N, M) \neq 0$. The conclusion follows from the previous Theorem. ∎

Corollary 4.23 *Let R be a local or graded ring and M be a finitely generated R-module. Then* $\operatorname{depth}(M) \leq \dim(M)$.

Proof Use the previous Corollary with $N = M$. ∎

Auslander-Buchsbaum Theorem

Lemma 4.24 *Let $(R, \mathfrak{m})$ be local or graded,* $\operatorname{depth}(R) = 0$, *$M$ a finitely generated R-module of finite projective dimension. Then M is free.*

Proof Take a minimal free resolution of M

$$0 \to R^{t_n} \xrightarrow{\phi_n} R^{t_{n-1}} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_1} R^{t_0} \to M \to 0,$$

where $\phi_n = (a_{ij})$ for $a_{ij} \in \mathfrak{m}$. Since $\operatorname{depth}(R) = 0$, for all $x \in \mathfrak{m}$, there exists $a \in R$ with $xa = 0$. Thus $\operatorname{Ker}(\phi_n) \neq 0$ since all entries in (a_{ij}) are in the maximal ideal $\mathfrak{m}$. Yet by exactness of the above sequence, $\operatorname{Ker}(\phi_n) = 0$. Therefore $n = 0$ and M is free. ∎

Lemma 4.25 *Let R be a local or graded ring and let M be a finitely generated R-module with finite projective dimension. Suppose $x \in \mathfrak{m}$ is a nonzerodivisor on R and on M. Then*

$$\operatorname{pd}_R(M) = \operatorname{pd}_{R/xR}(M/xM).$$

Proof Take a minimal free resolution of M

$$0 \to R^{t_n} \to R^{t_{n-1}} \to \cdots \to R^{t_0} \to M \to 0.$$

Observe that since $R \xrightarrow{x} R$ is injective, tensoring by $\overline{R} = R/xR$ yields the complex

$$0 \to \overline{R}^{t_n} \to \overline{R}^{t_{n-1}} \to \cdots \to \overline{R}^{t_0} \to \overline{M} \to 0.$$

The homology of the truncated complex above is clearly $\operatorname{Tor}_i^R(M, \overline{R})$. Since x is a nonzerodivisor on R, $\operatorname{pd}_R(\overline{R}) = 1$, so $\operatorname{Tor}_i^R(M, \overline{R})$ vanish for $i > 1$. Furthermore, $\operatorname{Tor}_1^R(M, \overline{R}) = \operatorname{Ker}(M \xrightarrow{x} M) = (0)$ by assumption. Thus the above complex is exact and so forms a minimal free resolution of $\overline{M}$ over $\overline{R}$. Hence $\operatorname{pd}_R(M) = \operatorname{pd}_{R/xR}(M/xM)$. ∎

Theorem 4.26 (Auslander-Buchsbaum) *Let R be local or graded and M be a finitely generated R-module with finite projective dimension. Then*

$$\operatorname{pd}_R(M) + \operatorname{depth}(M) = \operatorname{depth}(R).$$

Proof We induct on $\operatorname{pd}(M)\operatorname{depth}(R)$. In the base case, $\operatorname{pd}(M)\operatorname{depth}(R) = 0$. If $\operatorname{pd}(M) = 0$, then M is free. Hence $\operatorname{depth}(M) = \operatorname{depth}(R)$ and we are done. Similarly if $\operatorname{depth}(R) = 0$ then by Lemma 4.24 M is free. Hence $\operatorname{pd}(M) = 0$ and $\operatorname{depth}(M) = \operatorname{depth}(R) = 0$.

Now suppose $\operatorname{pd}(M)\operatorname{depth}(R) > 0$.

Case 1: $\operatorname{depth}(M) > 0$.

Since $\operatorname{depth}(M) > 0$ and $\operatorname{depth}(R) > 0$ we have,

$$\mathfrak{m} \not\subseteq \bigcup_{\mathfrak{p}\in \operatorname{Ass}_R(R)} \mathfrak{p} \quad \text{and} \quad \mathfrak{m} \not\subseteq \bigcup_{\mathfrak{q}\in \operatorname{Ass}_R(M)} \mathfrak{q}.$$

So by prime avoidance

$$\mathfrak{m} \not\subseteq \bigcup_{\mathfrak{p}\in \operatorname{Ass}_R(M)\cup \operatorname{Ass}_R(R)} \mathfrak{p}.$$

So there is $x \in \mathfrak{m}$ that is a nonzerodivisor on both R and M. By the previous lemma, $\operatorname{pd}_R(M) = \operatorname{pd}_{R/xR}(M/xM)$. Clearly $\operatorname{depth}(M/xM) = \operatorname{depth}(M) - 1$ and $\operatorname{depth}(R/xR) = \operatorname{depth}(R) - 1$. Since

$$\operatorname{pd}_{R/xR}(M/xM)\operatorname{depth}(R/xR) < \operatorname{pd}(M)\operatorname{depth}(R),$$

by induction we have

$$\operatorname{pd}_{R/xR}(M/xM) + \operatorname{depth}(M/xM) = \operatorname{depth}(R/xR).$$

Therefore

$$\operatorname{pd}_R(M) + \operatorname{depth}(M) = \operatorname{depth}(R).$$

Case 2: $\operatorname{depth}(M) = 0$.

Take an exact sequence

$$0 \to N \to F \to M \to 0,$$

where F is a free R-module. Since $\operatorname{depth}(R) > 0$, we have $\operatorname{depth}(N) = \operatorname{depth}(M) + 1$ by Corollary 4.18. Clearly $\operatorname{pd}(N) = \operatorname{pd}(M) - 1$. Since

$$\operatorname{pd}_R(N)\operatorname{depth}(R) < \operatorname{pd}(M)\operatorname{depth}(R),$$

by induction we have

$$\operatorname{pd}_R(N) + \operatorname{depth}(N) = \operatorname{depth}(R).$$

Hence

$$\operatorname{pd}_R(M) + \operatorname{depth}(M) = \operatorname{depth}(R).$$

■

Koszul Complexes

Given a ring R and $x \in R$, then $K_\bullet(x)$ denotes the **Koszul complex** of R generated by x. We define $K_\bullet(x)$ as follows

$$K_0(x) := R$$
$$K_1(x) := R$$

and we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_1(x) & \xrightarrow{d_1} & K_0(x) & \longrightarrow & 0 \\ & & \| & & \| & & \\ 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & 0 \end{array}$$

where we define d_1 to be multiplication by x. We can now define

$$K_\bullet(x; M) := K_\bullet(x) \otimes_R M.$$

Definition If $X_\bullet$ and $Y_\bullet$ are complexes with

$$X_n \xrightarrow{d_n^X} X_{n-1},$$
$$Y_n \xrightarrow{d_n^Y} Y_{n-1},$$

then we can form a complex $(X_\bullet \otimes Y_\bullet)_\bullet$ defined via

$$(X_\bullet \otimes Y_\bullet)_n := \bigoplus_{p+q=n} X_p \otimes Y_q$$

with the maps

$$(X_\bullet \otimes Y_\bullet)_n \xrightarrow{d_n} (X_\bullet \otimes Y_\bullet)_{n-1}$$

being defined as follows:

$$d_n(x_p \otimes y_q) := d_p^X(x_p) \otimes y_q + (-1)^p x_p \otimes d_q^Y(y_q)$$

where $p + q = n$.

Definition Given a ring R, $x_1, \ldots, x_n \in R$, and M a finitely generated R-module, the **Koszul complex** of R generated by $x_1, \ldots, x_n$ is

$$K_\bullet(x_1, \ldots, x_n) := K_\bullet(x_1) \otimes K_\bullet(x_2) \otimes \cdots \otimes K_\bullet(x_n).$$

Likewise, the Koszul complex of M generated by $x_1, \ldots, x_n$ is defined by

$$K_\bullet(x_1, \ldots, x_n; M) := K_\bullet(x_1, \ldots, x_n) \otimes_R M.$$

From the definition of the Koszul complex we see that $K_p(\mathbf{x})$ is a free R-module. Explicitly, we see that $K_0(x_1, \ldots, x_n) = R$ and for $p \neq 0$, we have

$$K_p(x_1, \ldots, x_n) = \bigoplus_{1 \leq i_1 < \cdots < i_p \leq n} Re_{i_1} \otimes \cdots \otimes Re_{i_p}$$

and moreover that

$$d_p(e_{i_1} \otimes \cdots \otimes e_{i_p}) = \sum_{k=1}^{p} (-1)^{k+1} x_{i_k} e_{i_1} \otimes \cdots \otimes \widehat{e}_{i_k} \otimes \cdots \otimes e_{i_p},$$

where the $\widehat{e}_{i_k}$ denotes that the element e_{i_k} is omitted.

We may also write the differential in the Koszul complex as a matrix. If d_p^n is the pth differential in a Koszul complex of n elements, and I denotes the $\binom{n}{p} \times \binom{n}{p}$ identity matrix, then it is not hard to show:

$$d_{p+1}^{n+1} = \begin{bmatrix} d_{p+1}^n & (-1)^p x_{n+1} I \\ 0 & d_p^n \end{bmatrix}$$

Exercise 4.27 *Let $x_1, \ldots, x_n \in R$. Let us construct the first several Koszul complexes. Starting with $K_\bullet(x_1)$, which we already know has the form:*

$$0 \to R \xrightarrow{\begin{bmatrix} x_1 \end{bmatrix}} R \to 0.$$

Using the notation above, we have

$$d_1^1 = \begin{bmatrix} x_1 \end{bmatrix},$$

and d_i^1 is the empty matrix for $i > 1$.

By above, we have

$$d_1^2 = \begin{bmatrix} d_1^1 & (-1)^0 x_2 I \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

and

$$d_2^2 = \begin{bmatrix} (-1)^1 x_2 I \\ d_1^1 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

So $K_\bullet(x_1, x_2)$ is the complex

$$0 \to R \xrightarrow{\begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}} R^2 \xrightarrow{\begin{bmatrix} x_1 & x_2 \end{bmatrix}} R \to 0.$$

Going one step further we have

$$d_1^3 = \begin{bmatrix} d_1^2 & (-1)^0 x_3 I \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}$$

$$d_2^3 = \begin{bmatrix} d_2^2 & (-1)^1 x_3 I \\ 0 & d_1^2 \end{bmatrix} = \begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix},$$

and

$$d_3^3 = \begin{bmatrix} (-1)^2 x_3 I \\ d_2^2 \end{bmatrix} = \begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}.$$

So $K_\bullet(x_1, x_2, x_3)$ is the complex

$$0 \to R \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}} R \to 0.$$

At this point we may regroup, and note if $F = \bigoplus_{i=1}^n Re_i$, then

$$K_p(x_1, \ldots, x_n) = \bigwedge^p F$$

where $\bigwedge^p F$ denotes the pth exterior power of the free R-module F. Thus we see that:

$$\begin{tikzcd}
0 \arrow[r] & K_n(\mathbf{x}) \arrow[r] \arrow[d, equal] & K_{n-1}(\mathbf{x}) \arrow[r] \arrow[d, equal] & \ldots \arrow[r] & K_1(\mathbf{x}) \arrow[r] \arrow[d, equal] & K_0(\mathbf{x}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & R \arrow[r] & R^{\binom{n}{n-1}} \arrow[r] & \ldots \arrow[r] & R^{\binom{n}{1}} \arrow[r] & R \arrow[r] & 0
\end{tikzcd}$$

A common notational convenience is to set

$$H_p(\mathbf{x}) := H_p(K_\bullet(\mathbf{x})) \qquad \text{and} \qquad H_p(\mathbf{x}; M) := H_p(K_\bullet(\mathbf{x}; M)).$$

From the construction of the Koszul complex we have that:

$$\begin{aligned} H_0(K_\bullet(\mathbf{x})) &= R/\mathbf{x}R, & H_n(K_\bullet(\mathbf{x})) &= (0 :_R \mathbf{x}), \\ H_0(K_\bullet(\mathbf{x}; M)) &= M/\mathbf{x}M, & H_n(K_\bullet(\mathbf{x}; M)) &= (0 :_M \mathbf{x}). \end{aligned}$$

Proposition 4.28 (Serre) *Let $L_\bullet$ be a complex of R-modules and let $x \in R$. Then we have an exact sequence*

$$0 \to \frac{H_p(L_\bullet)}{xH_p(L_\bullet)} \to H_p(L_\bullet \otimes K_\bullet(x)) \to (0 :_{H_{p-1}(L_\bullet)} x) \to 0$$

for all p.

Proof Consider the complex $L_\bullet \otimes K_\bullet(x)$. Let $d_p : L_p \to L_{p-1}$ be the differential map in $L_\bullet$. In $L_\bullet \otimes K_\bullet(x)$, the pth degree part is

$$(L_\bullet \otimes K_\bullet(x))_p = L_p \oplus L_{p-1},$$

and the differential map is

$$\begin{aligned} d'_p : L_p \oplus L_{p-1} &\to L_{p-1} \oplus L_{p-2}, \\ (u, v) &\mapsto (d_p(u) + (-1)^{p-1} xv, d_{p-1}(v)). \end{aligned}$$

We have then the following exact sequence of complexes

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & L_p \arrow[r] \arrow[d, "d_p"] & L_p \oplus L_{p-1} \arrow[r] \arrow[d, "d'_p"] & L_{p-1} \arrow[r] \arrow[d, "d_{p-1}"] & 0 \\
0 \arrow[r] & L_{p-1} \arrow[r] \arrow[d] & L_{p-1} \oplus L_{p-2} \arrow[r] \arrow[d] & L_{p-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

By the corresponding long exact sequence of homology we get that

$$H_p(L_\bullet) \xrightarrow{\pm x} H_p(L_\bullet) \to H_p(L \otimes K_\bullet(x)) \to H_{p-1}(L_\bullet) \xrightarrow{\pm x} H_{p-1}(L_\bullet)$$

is exact for all $p > 0$. Hence we get the

$$0 \to \frac{H_p(L_\bullet)}{xH_p(L_\bullet)} \to H_p(L_\bullet \otimes K_\bullet(x)) \to (0 :_{H_{p-1}(L_\bullet)} x) \to 0$$

is exact for $p > 0$. ■

Corollary 4.29 *If $F_\bullet \to M \to 0$ is a free resolution of M, and x is a nonzerodivisor on M, then*

$$F_\bullet \otimes K_\bullet(x) \to M/xM \to 0$$

is a free resolution of M/xM over R.

Corollary 4.30 *If $x_1, \dots, x_n \in R$, where R is a local or graded ring, and M an R-module, then the following sequence is exact*

$$0 \to \frac{H_p(x_1, \dots, x_{n-1}; M)}{x_n H_p(x_1, \dots, x_{n-1}; M)} \to H_p(x_1, \dots, x_n; M) \to (0 :_{H_{p-1}(x_1, \dots, x_{n-1}; M)} x_n) \to 0.$$

Proof Take $L_\bullet = K_\bullet(x_1, \dots, x_{n-1})$ in the above proposition. ■

Corollary 4.31 *If $x_1, \dots, x_n$ is an M-sequence, then $H_p(\mathbf{x}; M) = 0$ for $p > 0$.*

Proof Suppose $n = 1$. Then x is a nonzerodivisor on M. Hence

$$H_1(x_1; M) = (0 :_M x_1) = 0,$$

and we are done.

Now assume the result is true for M-sequences of length less than n. Then

$$H_p(x_1, \dots, x_{n-1}; M) = 0$$

for $p > 0$ by induction. So by the previous corollary $H_p(x_1, \ldots, x_n) = 0$ for $p > 1$ and

$$H_1(x_1, \ldots, x_n; M) = (0 :_{H_0(x_1,\ldots,x_{n-1};M)} x_n) = (0 :_{M/(x_1,\ldots,x_{n-1})M} x_n) = 0,$$

since x_n is a nonzerodivisor on $M/(x_1, \ldots, x_{n-1})M$. ■

Corollary 4.32 *If $x_1, \ldots, x_n$ is an R-sequence, then $K_\bullet(\mathbf{x})$ is a free resolution of $R/(\mathbf{x})$.*

Finally we can return to prove that all minimal graded free resolutions of finitely generated graded modules over $S = K[x_1, \ldots, x_n]$ are finite.

Theorem 4.33 (Hilbert's Syzygy Theorem) *The minimal graded free resolution of a finitely generated graded S-module is finite and its length is at most n.*

Proof Recall that $\mathrm{pd}_S(M) = \max\{i \mid \beta_{ij}(M) \neq 0\}$. If $\mathbf{F}_\bullet$ is the minimal graded free resolution of M, when we tensor by $K = S/(x_1, \ldots, x_n)$, then all the maps become zero. Thus we can also compute the graded Betti numbers as

$$\beta_{ij}(M) = \dim_K\left((\mathbf{F}_i) \otimes K)_j\right) = \dim_K\left(\mathrm{Tor}_i^S(M, K)_j\right).$$

Recall that we can also compute Tor by taking a resolution of K, which in this case is just a Koszul complex. Since the Koszul complex has length n, the theorem follows. ■

Cohen-Macaulay Modules

Definition If R is a local or graded ring and M finitely generated R-module, then M is called **Cohen-Macaulay** if $\dim(M) = \mathrm{depth}(M)$. R is called **Cohen-Macaulay** if it is Cohen-Macaulay as a module over itself.

Theorem 4.34 (Properties of Cohen-Macaulay Modules) *If $(R, \mathfrak{m})$ is a local or graded ring and M is a finitely generated R-module, then the following are true:*

1. *M is Cohen-Macaulay implies that for every prime ideal $\mathfrak{p} \in \mathrm{Ass}_R(M)$, $\dim(R/\mathfrak{p}) = \mathrm{depth}(M)$. Note that this shows us that R has no embedded primes.*

2. *Given a nonzero $x \in \mathfrak{m}$ such that x is a nonzerodivisor on M, M is Cohen-Macaulay if and only if M/xM is Cohen-Macaulay.*

3. *M is Cohen-Macaulay if and only if $M_\mathfrak{p}$ is Cohen-Macaulay for every $\mathfrak{p} \in \mathrm{Supp}(M)$. In this case, $\mathrm{depth}_\mathfrak{p}(M) = \mathrm{depth}(M_\mathfrak{p})$.*

4. *M is Cohen-Macaulay if and only if every system of parameters of M form a M-sequence if and only if some system of parameters form a M-sequence.*

5. *Let $x_1, \ldots, x_n$ be a system of parameters of M. M is Cohen-Macaulay if and only if*
$$H_1(\mathbf{x}; M) = 0$$
if and only if
$$M/\mathbf{x}M[X_1, \ldots, X_n] \to \bigoplus_{t=0}^{\infty} \frac{I^t M}{I^{t+1} M}$$
is an isomorphism, where $I = (x_1, \ldots, x_n)$, if and only if
$$e_I(M) = \ell(M/IM)$$
where $e_I(M)$ is the Hilbert-Samuel multiplicity of M with respect to I.

6. *If M is Cohen-Macaulay and $\mathfrak{p} \in \operatorname{Supp}(M)$, then*
$$\dim(M_{\mathfrak{p}}) + \dim(R/\mathfrak{p}) = \dim(M).$$

7. *If M is Cohen-Macaulay and $\mathfrak{p}, P \in \operatorname{Supp}(M)$, then any two saturated chains from*
$$\mathfrak{p} \subsetneq \cdots \subsetneq P$$
have the same length. In other words, $R/\operatorname{Ann}_R(M)$ is catenary.

Proof (1) For all $\mathfrak{p} \in \operatorname{Ass}_R(M)$, one has
$$\operatorname{depth}(M) \leq \dim(R/\mathfrak{p}) \leq \dim(M).$$
Since M is Cohen-Macaulay, $\dim(M) = \operatorname{depth}(M)$. Hence, $\dim(R/\mathfrak{p}) = \dim(M)$ for all $\mathfrak{p} \in \operatorname{Ass}_R(M)$ and so M has no embedded primes.
(2) Let x be a nonzerodivisor on M. Then
$$\dim(M/xM) = \dim(M) - 1 \quad \text{and} \quad \operatorname{depth}(M/xM) = \operatorname{depth}(M) - 1.$$
Hence M is Cohen-Macaulay if and only if M/xM is Cohen-Macaulay.
(3) One can show that for any finitely generated module M,
$$\operatorname{depth}_{\mathfrak{p}}(M) \leq \operatorname{depth}(M_{\mathfrak{p}}) \leq \dim(M_{\mathfrak{p}}).$$
We show that $\operatorname{depth}_{\mathfrak{p}}(M) = \dim(M_{\mathfrak{p}})$ if and only if M is Cohen-Macaulay. That this condition is sufficient to force M to be Cohen-Macaulay is clear, so assume M is Cohen-Macaulay. We prove the above equality by induction on $\operatorname{depth}_{\mathfrak{p}}(M)$. If $\operatorname{depth}_{\mathfrak{p}}(M) = 0$, then $\mathfrak{p} \in \operatorname{Ass}_R(M)$. Since M has no embedded primes, $\dim(M_{\mathfrak{p}}) = 0$. Now assume $\operatorname{depth}_{\mathfrak{p}}(M) > 0$. Then let $x \in \mathfrak{p}$ be a nonzerodivisor on M. Then
$$\operatorname{depth}_{\mathfrak{p}}(M/xM) = \operatorname{depth}_p(M) - 1 \quad \text{and} \quad \dim(M_p/xM_p) = \dim(M_p) - 1.$$

Since M/xM is Cohen-Macaulay by (2), we are done by induction.
(4) Suppose M is Cohen-Macaulay and let $x_1, \ldots, x_n$ be a system of parameters for M. Then for $1 \leq i \leq n$,

$$\dim(M/(x_1, \ldots, x_i)M) = \dim(M) - i.$$

Observe that it suffices to show that if $x \in \mathfrak{m}$ and $\dim(M/xM) = \dim(M) - 1$, then x is regular on M. If $x \in \mathfrak{p}$ for some $\mathfrak{p} \in \mathrm{Ass}_R(M)$, then $\dim(M) = \dim(M/xM)$ since M has no embedded primes. Therefore

$$x \notin \bigcup_{\mathfrak{p} \in \mathrm{Ass}_R(m)} \mathfrak{p}.$$

It follows that x is a nonzerodivisor on M and that $M/xM \neq 0$. Hence x is regular on M and we are done. That is, all systems of parameters are M-regular sequences.

Now suppose M is Cohen-Macaulay and that $x_1, \ldots, x_n$ is a system of parameters that is also M-regular. Then

$$\dim(M) = s(M) \leq \mathrm{depth}(M) \leq \dim(M).$$

Hence $\dim(M) = \mathrm{depth}(M)$ and so M is Cohen-Macaulay.
(5) Let $x_1, \ldots, x_n$ be a system of parameters for M. Then $H_1(\mathbf{x}; M) = 0$ implies that $x_1, \ldots, x_n$ forms an M-regular sequence. Thus $\mathrm{depth}(M) = \dim(M)$. In particular, $x_1, \ldots, x_n$ is an M-quasi-regular sequence. Hence

$$M/IM[X_1, \ldots, X_n] \simeq \bigoplus I^t M / I^{t+1} M.$$

By Serre's Theorem on dimension, the previous isomorphism is equivalent to the equality

$$e_I(M) = \ell(M/IM).$$

(6) Suppose M is Cohen Macaulay. If $\dim(M) = 0$, there is nothing to prove. If $\dim(M) > 0$, then if $\mathfrak{p} \in \mathrm{Ass}_R(M)$ we have $\mathfrak{p} \in \mathrm{Supp}(M)$. Hence $\dim(M_\mathfrak{p}) = 0$ and $\dim(R/\mathfrak{p}) = \dim(M)$, since M has no embedded primes. If $\mathfrak{p} \notin \mathrm{Ass}_R(M)$, then there exists a nonzerodivisor $x \in \mathfrak{p}$ on M. Thus

$$\dim(M/xM) = \dim(M) - 1 \quad \text{and} \quad \dim(M_\mathfrak{p}/xM_\mathfrak{p}) = \dim(M_\mathfrak{p}) - 1.$$

The result then follows from induction on $\dim(M)$.
(7) Suppose M is Cohen-Macaulay and let $\mathfrak{p} \subseteq \mathfrak{p}'$ be two primes in $\mathrm{Supp}(M)$. Let

$$\mathfrak{p} = \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r = \mathfrak{p}'$$

be a saturated chain of prime ideals. We'll prove that

$$\dim(R/\mathfrak{p}) - \dim(R/\mathfrak{p}') = r.$$

It is enough to show that

$$\dim(R/\mathfrak{p}_0) - \dim(R/\mathfrak{p}_1) = 1.$$

By (6),

$$\dim(R_{\mathfrak{p}_0}) + \dim(R/\mathfrak{p}_0) = \dim(R) \quad \text{and} \quad \dim(R_{\mathfrak{p}_1}) + \dim(R/\mathfrak{p}_1) = \dim(R).$$

Rlso by (6),

$$\dim(R_{\mathfrak{p}_0}) + \dim(R_{\mathfrak{p}_1}/\mathfrak{p}_0 R_{\mathfrak{p}_1}) = \dim(R_{\mathfrak{p}_1}).$$

Since there are no primes strictly between $\mathfrak{p}_0$ and $\mathfrak{p}_1$, $\dim(R_{\mathfrak{p}_1}/\mathfrak{p}_0 R_{\mathfrak{p}_1}) = 1$. Thus

$$\dim(R/\mathfrak{p}_0) - \dim(R/\mathfrak{p}_1) = \dim(R_{\mathfrak{p}_1}) - \dim(R_{\mathfrak{p}_0}) = \dim(R_{\mathfrak{p}_1}/\mathfrak{p}_0 R_{\mathfrak{p}_1}) = 1.$$

■

A ring is called **Cohen-Macaulay** if it is Cohen-Macaulay as a module over itself.

5. Eisenbud-Goto Conjecture

We previously defined regularity in terms of the nonvanishing of graded Betti numbers of a module. There are parallel definitions in terms of both local cohomology and sheaf cohomology, where the notion was first defined by Mumford. Eisenbud-Goto gave the equivalent definition in terms of Betti numbers. [6] In the same paper, they made a conjecture about the regularity of nondegenerate prime ideals. This question became known as the **Eisenbud-Goto Conjecture** and was very challenging.

[6] David Eisenbud and Shiro Goto. Linear free resolutions and minimal multiplicity. *J. Algebra*, 88(1):89–133, 1984

In this chapter, we first state the Eisenbud-Goto Conjecture and prove it in the case that S/I is Cohen-Macualay. We also show that the proposed upper bound is nonnegative for nondegenerate prime ideals. Then we construct counterexamples due to McCullough-Peeva using step-by-step homogenization and Rees-like algebras.

History of the Conjecture

In Bayer-Stillman gave another characterization of regularity: $\operatorname{reg}(I)$ is equal to the maximal degree of a Gröber basis element of I in the revlex monomial order if we first take a generic change of coordinates. Since Gröbner bases are required for many computational tasks, this means that finding upper bounds on regularity equates to finding bounds on the computational complexity of an ideal. Unfortunately, in the most general setting possible, such upper bounds are quite large. Set $\operatorname{maxdeg}(I)$ to be the maximal degree of a minimal generator of I.

Theorem 5.1 (Bayer-Mumford, Galligo, Giusti, Caviglia-Sbarra) *Let I be a homogeneous ideal in $S = k[x_1, \dots, x_n]$. Then*

$$\operatorname{reg}(I) \leq (2 \operatorname{maxdeg}(I))^{2^{n-2}}.$$

This doubly exponential bound grows quickly with respect to n. Unfortunately, this bound is nearly optimal. In one construction derived from the so-called Mayr-Meyer ideals, Koh proved that for any integer $r \geq 1$ there exists an ideal I_r in $S_r = k[x_1, \dots, x_{22r-1}]$ such that

$$\begin{aligned} \operatorname{maxdeg}(I_r) &= 2 \\ \operatorname{reg}(S_r/I_r) &\geq 2^{2^{r-1}}. \end{aligned}$$

Thus doubly exponential behavior cannot be avoided. We note for later reference that I_r is generated by $22r-3$ quadrics and 1 linear form while the regularity is realized at the first syzygies of I (second syzygies of S/I).

The ideals I_r have many associated primes and embedded primes; in particular, they are far from prime. Better bounds were expected for ideals with some geometric content. This expectation was expressed in the following conjecture:

Conjecture 5.2 (Eisenbud-Goto (1984)) *Let $S = k[x_1, \dots, x_n]$ with $k = \overline{k}$ and suppose $\mathfrak{p} \subseteq (x_1, \dots, x_n)^2$ is a homogeneous prime ideal. Then*

$$\operatorname{reg}(S/\mathfrak{p}) \leq e(S/\mathfrak{p}) - \operatorname{ht}(\mathfrak{p}).$$

The condition $\mathfrak{p} \subseteq (x_1, \dots, x_n)^2$ is equivalent to saying that the projective variety corresponding to $\mathfrak{p}$ is not contained in a hyperplane and thus is optimally embedded. Such ideals are called **nondegenerate**.

It is worth noting that the right-hand side of the inequality above is always positive for nondegenerate prime ideals.

Theorem 5.3 *Let $\mathfrak{p}$ be a nondegenerate prime ideal in $S = k[x_1, \dots, x_n]$. Then*

$$e(S/\mathfrak{p}) \geq \operatorname{ht}(\mathfrak{p}) + 1.$$

For a proof, see Eisenbud's Geometry of Syzygies.

The Eisenbud-Goto Conjecture can be viewed as an expectation that for ideals with more geometric content, regularity is better behaved. Via the Bayer-Stillman result, this would then ensure that computations involving prime ideals are much better behaved that that of arbitrary ideals. Castelnuovo essentially showed in 1893 that smooth curves in $\mathbb{P}^3_{\mathbb{C}}$ satisfy the conjecture. In 1983 Gruson-Lazarsfeld-Peskine proved the bound for all curves (smooth and singular) in any projective space. (Remember that $\mathfrak{p}$ defines a projective curve when $\dim(S/\mathfrak{p}) = 2$.) Pinkham (1986) and Lazarsfeld (1987) proved the bound for smooth projective surfaces over $\mathbb{C}$. Ran (1990) proved the bound for most smooth projective 3-folds over $\mathbb{C}$. Numerous other special cases have also been proved.

Exercise 5.4 *Show that the hypotheses of the Eisenbud-Goto Conjecture are necessary by checking that each of the ideals below fails to satisfy the conjectured regularity bound but also fail to satisfy one of the hypotheses.*

1. $(x, y) \subseteq k[x, y]$

2. $(x^2, xy, y^2, xw^2 + yz^2) \subseteq k[w, x, y, z]$.

3. $(wy, wz, xy, xz) \subseteq k[w, x, y, z]$

4. $(w^2 + x^2, y^2 + z^2, wz - xy, wy - xz) \subseteq \mathbb{Q}[w, x, y, z]$

The Cohen-Macaulay Case

In this section we prove the Eisenbud-Goto conjecture for any nondegenerate Cohen-Macaulay ideal. Note that we do not need to assume primeness.

Lemma 5.5 *Suppose M is an S-module of finite length. Then*

$$\operatorname{reg}(M) = \max\{i \mid M_i \neq 0\}.$$

Proof Consider the Koszul complex resolving $k = S/(x_1, \ldots, x_n)$:

$$0 \to S(-n) \to \cdots \to S(-2)^{\binom{n}{2}} \to S(-1)^n \to S.$$

Tensoring by M, we get the complex

$$0 \to M(-n) \to \cdots \to M(-2)^{\binom{n}{2}} \to M(-1)^n \to M.$$

Let $d = \max\{i \mid M_i \neq 0\}$ and fix $0 \neq m \in M_d$. Then $x_i m \in M_{d+1} = 0$ for all i. By the definition of the Koszul complex, m is in the kernel of $M(-n) \to M(-n+1)^n$ and hence represents a nonzero element of degree $d + n$ in $\operatorname{Tor}_n(M, k)$. Therefore $\operatorname{reg}(M) \geq d + n - n = d$. Since $M_i = 0$ for $i > d$, $M(-j)_i = 0$ for $i > d + j$ and thus $\operatorname{Tor}_j(M, k)_i = 0$ for $i > d + j$. Hence $\operatorname{reg}(M) = d$. ∎

Exercise 5.6 *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of finitely generated graded S-modules. Then*

1. $\operatorname{reg}(M_1) \leq \max\{\operatorname{reg}(M_2), \operatorname{reg}(M_3) + 1\}$

2. $\operatorname{reg}(M_2) \leq \max\{\operatorname{reg}(M_1), \operatorname{reg}(M_3)\}$

3. $\operatorname{reg}(M_3) \leq \max\{\operatorname{reg}(M_1) - 1, \operatorname{reg}(M_2)\}$

Lemma 5.7 *Let M be a finitely generated S-module and let ℓ be a linear form which is regular on M. Then*

$$\operatorname{reg}(M) = \operatorname{reg}(M/\ell M).$$

Proof Consider the short exact sequence

$$0 \to M(-1) \xrightarrow{\ell} M \to M/\ell M \to 0.$$

Note that $\operatorname{reg}(M(-1)) = \operatorname{reg}(M) + 1$. Now use the previous exercise.

∎

Lemma 5.8 *Let M be a finitely generated S-module and let ℓ be a linear form which is regular on M. Then*

$$e(M) = e(M/\ell M).$$

Proof Again consider the short exact sequence

$$0 \to M(-1) \xrightarrow{\ell} M \to M/\ell M \to 0.$$

Let $P_M(i)$ denote the HIlbert polynomial of M and set $d = \dim(M)$. By assumption $P_M(i)$ has the form:

$$\frac{e(M)}{(d-1)!} i^{d-1} + q i^{d-2} + \cdots,$$

for some $q \in \mathbb{Q}$. Therefore

$$\begin{aligned} P_{M(-1)}(i) &= P_M(i-1) \\ &= \frac{e(M)}{(d-1)!}(i-1)^{d-1} + q(i-1)^{d-2} + \cdots \\ &= \frac{e(M)}{(d-1)!} i^{d-1} + \left(q - \frac{e(M)}{(d-2)!}\right) i^{d-2} + \cdots. \end{aligned}$$

Since length is additive in short exact sequences we have

$$\lambda\left((M/\ell M)_i\right) = \lambda(M_i) - \lambda(M_{i-1}),$$

for all $i \geq 0$. But for $i \gg 0$, $\lambda(M_i) = P_M(i)$. Therefore

$$P_{M/\ell M}(i) = P_M(i) - P_M(i-1) = \frac{e(M)}{(d-2)!} i^{d-2} + \cdots.$$

In particular, $e(M) = e(M/\ell M)$. ■

Theorem 5.9 (Eisenbud-Goto) *If I is a nondegenerate, homogeneous ideal of S such that S/I is Cohen-Macaulay, then*

$$\operatorname{reg}(S/I) \leq e(S/I) - \operatorname{ht}(I).$$

Proof We may always assume the basefield is infinite. Set $h = \operatorname{ht}(I)$. Since S/I is Cohen-Macaulay of dimension $n-h$, we can find linear forms $\ell_1, \ldots, \ell_{n-h}$ that form a regular sequence on S/I. Set $\overline{S} = S/(\ell_1, \ldots, \ell_h)$ and $\overline{I} = I\overline{S}$. Then $\overline{S}$ is a polynomial ring in h variables and by the previous lemmas:

$$\begin{aligned} \operatorname{reg}_S(S/I) &= \operatorname{reg}_{\overline{S}}(\overline{S}/\overline{I}) \\ e(S/I) &= e(\overline{S}/\overline{I}) \\ \operatorname{ht}(I) &= \operatorname{ht}(\overline{I}). \end{aligned}$$

Since $\dim(\overline{S}/\overline{I}) = 0$, we have $\lambda(\overline{S}/\overline{I}) < \infty$. Let $d = \max\{i \mid (\overline{S}/\overline{I})_i \neq 0)$. Then $(\overline{S}/\overline{I})_i \neq 0$ for $0 \leq i \leq d$. Moreover, since I is nondegenerate, so is $\overline{I}$; hence $\dim_k(\overline{S}/\overline{I}) = \dim_k(\overline{S}) = h$. Therefore

$$\begin{aligned} e(S/I) &= e(\overline{S}/\overline{I}) \\ &= \lambda(\overline{S}/\overline{I}) \\ &\geq d + h \\ &= \operatorname{reg}_{\overline{S}}(\overline{S}/\overline{I}) + \operatorname{ht}(\overline{I}) \\ &= \operatorname{reg}_S(S/I) + \operatorname{ht}(I) \end{aligned}$$

■

Rees-Like Algebras

Before defining Rees-like algebras, let's briefly recall Rees algebras. Fix an ideal $I = (f_1, \ldots, f_m) \subseteq S$. Let t be a new variable and consider the algebra $S[It] \subseteq S[t]$ generated over S by the elements $f_i t$. $S[It]$ is called the **Rees algebra** of I. It defines the projective coordinate ring of the blow-up of projective space along the variety defined by I. As such it is important in the resolution of singularities in algebraic geometry and therefore its defining equations are of great interest. In general however, this is a very difficult question and one of active research.

Here we focus on the closely-related **Rees-like algebra** of I, defined as $S[It, t^2] \subseteq S[t]$. To calculate its defining equations, we define a new polynomial ring $T = S[y_1, \ldots, y_m, z]$, where $\deg(y_i) = \deg(f_i) + 1$ and $\deg(z) = 2$. Note that T is a positively graded polynomial ring. Let $\varphi : T \twoheadrightarrow S[It, t^2]$ be the surjective S-algebra homomorphism:

$$\begin{aligned} \varphi : T &\to S[It, t^2] \\ y_i &\mapsto f_i t \\ z &\mapsto t^2. \end{aligned}$$

Since $S[It, t^2]$ is a domain, $Q = \operatorname{Ker}(\varphi)$ is a prime ideal. The first surprising fact about Rees-like algebras is that, unlike the usual Rees algebras, the defining equations are easy to describe in all situations.

Proposition 5.10 *In the notation above, the ideal Q is generated by the elements*

$$\mathcal{A} = \left\{ y_i y_j - z f_i f_j \;\middle|\; 1 \leq i, j \leq m \right\}$$

and

$$\mathcal{B} = \left\{ \sum_{i=1}^{m} c_{ij} y_i \;\middle|\; \sum_{i=1}^{m} c_{ij} f_i = 0 \right\}.$$

Proof First note that the elements in $\mathcal{A}$ and $\mathcal{B}$ are in $Q = \operatorname{Ker}(\varphi)$ since

$$\varphi\left(y_i y_j - z f_i f_j\right) = f_i t f_j t - t^2 f_i f_j = 0$$
$$\varphi\left(\sum_{i=1}^m c_{ij} y_i\right) = t \sum_{i=1}^m c_{ij} f_i = 0.$$

Let $e \in Q$. We may write $e = f + g$, where $f \in (y_1, \dots, y_m)^2$ and $g \in S[z]\mathrm{Span}_k\{1, y_1, \dots, y_m\}$. Using elements in $\mathcal{A}$ we reduce to the case when $f = 0$, so $e = h(z) + \sum_{i=1}^m h_i(z) y_i$ with $h(z), h_1(z), \dots, h_m(z) \in S[z]$. Then

$$0 = \varphi(e) = h(t^2) + \sum_{i=1}^m h_i(t^2) t f_i \in S[t]$$

implies that $h(z) = 0$ since $h(t^2)$ contains only even powers of t while $\sum_{i=1}^m h_i(t^2) t f_i$ contains only odd powers of t. Thus $e \in (y_1, \dots, y_m)$ and we may write

$$e = z^p \sum_{i=1}^m g_i y_i + \text{ (terms in which } z \text{ has degree} < p)$$

for some $p \geq 0$ and $g_1, \dots, g_m \in S$. We will argue by induction on p that e is in the ideal generated by the elements in $\mathcal{B}$. Suppose $e \neq 0$. We consider

$$0 = \varphi(e) = t^{2p} t \sum_{i=1}^m g_i f_i + \text{ (terms in which } t \text{ has degree} \leq 2p - 1),$$

and conclude that $\sum_{i=1}^m g_i f_i = 0$. It follows that $\sum_{i=1}^m g_i y_i$ is in the ideal generated by the elements in $\mathcal{B}$. The element

$$e - z^p \sum_{i=1}^m g_i y_i \in \operatorname{Ker}(\varphi)$$

has smaller degree with respect to the variable z. The base of the induction is $e = 0$. ∎

Perhaps more surprising is that the resolution of T/Q has a well-defined structure. We will not need this to prove that the defining primes ideals of Rees-like algebras provide the counterexamples we are looking for. We simply remark here that it is clear that z is a nonzerodivisor on T/Q above. Hence the graded Betti numbers of $\overline{T}/\overline{Q}$ are the same as that of T/Q, where $\overline{T} = T/(z)$ and $\overline{Q} = Q\overline{T}$. It turns out that the minimal resolution of $\overline{T}/\overline{Q}$ can be viewed as a mapping cone of two resolutions associated to the following short exact sequence

$$0 \to (\overline{\mathcal{B}})/(\overline{\mathcal{A}}) \to \overline{T}/(\overline{\mathcal{A}}) \to \overline{T}/\overline{Q} \to 0.$$

For details, see .

Jason McCullough and Irena Peeva. Counterexmaples to the eisenbud-goto regularity conjecture. preprint, 2016

Example 5.11 *Let $S = k[x_1, x_2, x_3]$ and $I = (x_1x_2, x_1x_3, x_2x_3)$. Then S/I has the following minimal free resolution*

$$0 \to S(-3)^2 \to S(-2)^3 \to S.$$

By the previous proposition, the prime ideal $Q \subseteq S[y_1, y_2, y_3, z]$ associated to the Rees-like algebra of I is generated by:

$$\begin{array}{r|lll} \mathcal{A}: & y_1^2 - z(x_1x_2)^2, & y_1y_2 - z(x_1x_2)(x_1x_3), & y_1y_3 - z(x_1x_2)(x_2x_3), \\ & y_2^2 - z(x_1x_3)^2, & y_2y_3 - z(x_1x_3)(x_2x_3), & y_3^2 - z(x_2x_3)^2, \\ \mathcal{B}: & x_3y_1 - x_2y_2, & x_2y_2 - x_1y_3 & \end{array}$$

Step-by-Step Homogenization

Now that we have a construction for a wide class of positively graded prime ideals, we would like to convert these into standard-graded counterparts. Unfortunately the usual ways of homogenizing ideals are not so helpful here.

Example 5.12 *Consider the ideal $Q = (x^2 - y, x^3 - z) \subseteq T = k[x, y, z]$ in which $\deg(x) = 1$, $\deg(y) = 2$, and $\deg(z) = 3$. Then Q is a homogeneous prime ideal in T. (To see that Q is prime, check that $Q = \operatorname{Ker}(T \twoheadrightarrow k[t])$, where $x \mapsto t$, $y \mapsto t^2$, and $z \mapsto t^3$.)*

We could choose to set all the variables to have degree 1 and homogenize the generators with respect to a new variable of degree 1, say w. In this case, Q transforms into a new ideal $Q_{ghom} = (x^2 - yw, x^3 - zw^2)$ in a standard graded polynomial ring $k[x, y, z, w]$. However,

$$w(xy - zw) = x(x^2 - yw) - (x^3 - zw^2) \in Q_{ghom},$$

but neither w nor $xy - zw$ are elements of Q_{ghom}. Thus Q_{ghom} is no longer prime.

We could instead homogenize all the elements of Q. (It is sufficient to homogenize those generators in a homogeneous Gröbner basis of Q.) In this case we get a new ideal $Q_{hom} = (y^2 - xz, xy - wz, x^2 - wy)$ in $k[x, y, z, w]$. In this case, Q_{hom} is prime, however we have changed both the number and degrees of the minimal generators. One can also check that $\operatorname{reg}_T(T/Q) = 3$, while $\operatorname{reg}_S(S/Q_{hom}) = 1$.

As we saw in the example, the standard homogenization techniques either fail to preserve primeness or perhaps alter the regularity. To compensate for this, McCullough-Peeva introduced a technique called **step-by-step homogenization**. The basic idea is to start with a prime ideal Q in a positively graded ring $T = k[y_1, \ldots, y_r]$ such that $y_i \notin Q$ for all $1 \leq i \leq r$. Set $d_i = \deg(y_i)$. Set

$$S = k\left[x_{1,1}, \ldots, x_{1,d_1}, x_{2,1}, \ldots, x_{2,d_2}, \ldots, x_{r,1}, \ldots, x_{r,d_r}\right],$$

and define $\nu : T \to S$ by setting $\nu(y_i) = \prod_{j=1}^{d_i} x_{i,j}$. Then ν is a graded k-algebra homomorphism. We define the step-by-step homogenization of Q to be $Q_{shom} = \nu(Q) \subseteq S$. With this setup, we can now prove the key fact about step-by-step homogenizaiton

Theorem 5.13 (McCullough-Peeva) *With the notation above, Q_{shom} is a homogeneous prime ideal in S. The graded Betti numbers of S/Q_{shom} over S are the same as those of T/Q over T. Moreover, if Q was nondegenerate, so is Q_{shom}.*

Because polynomial extension are flat maps, the graded Betti numbers are unchanged. Regarding the multiplicity, the crux of the proof can be seen in the following exercise in Eisenbud's book:

Exercise 5.14 *(See Exercise 10.4 in Eisenbud's "Commutative Algebra with a View Toward Algebraic Geometry.") Let a, b be a regular sequence in a domain R, and let $P = R[X]$ be the polynomial ring in one variable over R. Show that $aX - b$ is a prime ideal in P.*

Repeatedly applying the exercise with $a = x_{i,1} \cdots x_{i,d_i-1}$, $X = x_d$, and $b = y_i$ shows that Q_{shom} is prime.

Example 5.15 *Continuing the example above, if we start with $Q = (x^2 - y, x^3 - z)$ in $T = k[x, y, z]$ with $\deg(x) = 1$, $\deg(y) = 2$, and $\deg(z) = 3$, we compute the step-by-step homogenization (with slightly different variable name) by replacing y by $v_1 v_2$ and z by $w_1 w_2 w_3$. Thus our new standard graded polynomial ring is $S = k[x, u_1, u_2, w_1, w_2, w_3]$. The step-by-step homogenization of Q is $Q_{shom} = (x^2 - v_1 v_2, x^3 - w_1 w_2 w_3)$. By the previous theorem, Q_{shom} is a nondegenerate prime ideal with the same graded Betti numbers as T/Q.*

Now we turn our attention to the effect of step-by-step homogenization on the multiplicity of an ideal. Since step-by-step homogenization preserves graded Betti numbers, it also preserves the Euler polynomial $h(t)$ of the free resolution of T/Q. The value of $h(1)$ agrees with the multiplicity in the standard graded case, but not in general! In fact, if Q is an ideal in a positively graded ring that is not standard graded, it's Hilbert function is eventually a quasi-polynomial, i.e. a polynomial whose coefficients are cyclic functions rather than constants.

Example 5.16 *Returning to the Rees-like algebra $S[It, t^2]$ with $I = (x_1 x_2, x_1 x_3, x_2 x_3)$, we found that the defining prime $Q \subseteq T = S[y_1, y_2, y_3, z]$*

had 8 minimal generators. It's Betti table is as below:

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	-	-	-	-
2:	-	-	-	-	-
3:	-	2	-	-	-
4:	-	-	-	-	-
5:	-	6	6	-	-
6:	-	-	-	-	-
7:	-	-	8	6	-
8:	-	-	-	-	-
9:	-	-	-	3	2

We get that the Hilbert Series of T/Q then is:

$$\begin{aligned}\mathrm{Hilb}_{T/Q}(t) &= \frac{(1-2t^4-6t^6+6t^7+8t^9-6t^{10}-3t^{12}+2t^{13})}{(1-t^3)^3(1-t^2)(1-t)^3} \\ &= \frac{(1-t+2t^2)(1+t-t^2)(1+t+t^2)^3}{(1-t^3)^3(1-t^2)}.\end{aligned}$$

So we get that the Euler polynomial of T/Q is

$$h(t) = (1-t+2t^2)(1+t-t^2)(1+t+t^2)^3,$$

and the Euler multiplicity is

$$e_{Euler}(T/Q) = h(1) = 2\cdot 1\cdot 3^3 = 54.$$

After taking the step-by-step homogenization Q' of Q in a larger standard graded polynomial ring T', the resolution and Euler polynomial remain unchanged. (Although the denominator does change.) Thus the multiplicity of T'/Q' is $e(T'/Q') = 54$.

We can compute the Hilbert quasi-polynomial of T/Q from the Hilbert Series. A short calculation yields

$$\dim_K((T/Q)_i) = \begin{cases} \frac{1}{6}i^3+i^2+\frac{1}{3}i+1 & \textit{if } i \textit{ is odd} \\ \frac{1}{6}i^3+i^2+\frac{1}{3}i+\frac{3}{2} & \textit{if } i \textit{ is even and } i\geq 2. \end{cases}$$

Thus we get $e_{Hilb}(T/Q) = 3!\frac{1}{6} = 1$.

It's not hard to see that for any homogeneous ideal I, $S[It,t^2] \to S[t]$ is an integral extension. It follows that the Hilbert multiplicity of both rings is the same, in the sense of a normalized leading coefficient of a Hilbert (quasi)-polynomial. (One must check that the leading coefficient is indeed constant.) Clearly $e_{Hilb}(S[t]) = 1$; thus $e_{Hilb}(S[It,t^2]) \leq 1$ as well. It remains to track what happens to the multiplicity when we take a step-by-step homogenization of Q. The following general result answers this question

Theorem 5.17 *Let Q be prime ideal in a positively graded ring $T = k[y_1, \ldots, y_n]$, where $\deg(y_i) = d_i$. Suppose $y_i \notin Q$ for all i. Let Q' denote the step-by-step homogenization of Q in $T' = k[x_{1,1}, \ldots, x_{1,d_1}, \ldots, x_{n,1}, \ldots, x_{n,d_n}]$. Then*

$$e_{Euler}(T/Q) = e_{Euler}(T'/Q') = e_{Hilb}(T'/Q') = e_{Hilb}(T/Q) \prod_{i=1}^{n} d_i$$

Proof (sketch) Track what happens to the Hilbert Series when replace one degree d variable by d degree 1 variables. The effect is to leave the Euler multiplicity unchanged and to multiply the Hilbert multiplicity by d. ∎

Putting it all together yields the following result.

Theorem 5.18 (McCullough-Peeva) *Let $I = (f_1, \ldots, f_m)$ be a homogeneous ideal in $S = k[x_1, \ldots, x_n]$. Let Q denote the defining prime ideal of the Rees-like algebra of I in $T = S[y_1, \ldots, y_m, z]$. Finally let Q' denote the step-by-step homogenization of Q in a standard graded polynomial ring T'. Then*

$$\begin{aligned} \operatorname{reg}(Q') &\geq \max\{2(\operatorname{maxdeg}(I)+1), \operatorname{maxdeg}(\operatorname{Syz}_1(I)+1)\} \\ e(T'/Q') &= 2\prod_{i=1}^{m}(\deg(f_i)+1) \\ \operatorname{ht} Q' &= m. \end{aligned}$$

Proof Because $\dim(S[It, t^2]) = n+1$, $\operatorname{ht}(Q) = \dim(T) - (n+1) = n+m+1-(n+1) = m$. Since the graded Betti numbers of Q and Q' are the same, so are their heights.

That $e(T'/Q') = 2\prod_{i=1}^{m}(\deg(f_i)+1)$ follows from the free resolutions. We have so far proved $e(T'/Q') \leq 2\prod_{i=1}^{m}(\deg(f_i)+1)$, which is all we need to give counterexamples.

By considering the degree of the generators in $\mathcal{A}$ and $\mathcal{B}$, we get the lower bound on the regularity. ∎

Counterexamples

What we have constructed so far in this chapter can be thought of as a machine that takes in homogeneous ideals in standard graded polynomial rings and outputs nondegenerate prime ideals in standard graded polynomial rings with similar properties.

For $r \geq 1$, Koh constructed in [7] an ideal I_r generated by $22r-3$ quadrics and one linear form in a polynomial ring with $22r-1$ variables, and such that $\operatorname{maxdeg}(\operatorname{Syz}_1(I_r)) \geq 2^{2^{r-1}}$. His ideals are based on the Mayr-Meyer construction in [8]. This leads to a homogeneous

[7] Jee Koh. Ideals generated by quadrics exhibiting double exponential degrees. *J. Algebra*, 200(1):225–245, 1998

[8] Ernst W. Mayr and Albert R. Meyer. The complexity of the word problems for commutative semigroups and polynomial ideals. *Adv. in Math.*, 46(3): 305–329, 1982

prime ideal P_r (in a standard graded polynomial ring R_r) whose multiplicity and maxdeg are:

$$\begin{aligned}\deg(R_r/P_r) &\leq 4 \cdot 3^{22r-3} < 4^{22r-2} < 2^{50r} \\ \operatorname{maxdeg}(P_r) &\geq 2^{2^{r-1}} + 1 > 2^{2^{r-1}}.\end{aligned}$$

Thus for $r \geq 9$, we have counterexamples to not only the Eisenbud-Goto Conjecture, but also the stronger conjecture that the multiplicity of a nondegenerate prime ideal was an upper bound for the degrees of any minimal generator. We even get the following stronger statement.

Theorem 5.19 (McCullough-Peeva) *There is no polynomial bound on the regularity (or projective dimension, or maximal degree of a minimal generator) of a nondegenerate prime ideal purely in terms of its multiplicity.*

One can then ask if there is any such bound. Applying recent work of Ananyan-Hochster (see Chapter 6), we see that there is such a bound. What exactly is the growth rate of this function? That remains a mystery.

6. Stillman's Question

In the final two chapters we consider some effective bounds on the projective dimension and regularity of ideals in polynomial rings over a field. Hilbert's Syzygy Theorem 4.33 shows that the projective dimension of any S-module is at most the number of variables of S. A natural question is to ask if there is a bound on the projective dimension of an ideal in S in terms of the number of generators or their degrees. We begin with the case of three-generated ideals.

Three-generated Ideals

First we consider where $\mathrm{pd}(S/I)$ can be bounded by the minimal number of generators of I, independent of the number of variables. If I is principal, then $\mathrm{pd}(S/I) = 1$. If $I = (f, g)$ has two minimal generators, then either f, g for a regular sequence and hence $\mathrm{pd}(S/I) = 2$, or f, g have a greatest common factor c and we can write $f = cf'$ and $g = cg'$, where f', g' is a regular sequence. In this case, the minimal free resolution of S/I has the form:

$$0 \to S \xrightarrow{\begin{pmatrix} g' \\ -f' \end{pmatrix}} S^2 \xrightarrow{\begin{pmatrix} cf' & cg' \end{pmatrix}} S$$

Once again $\mathrm{pd}(S/I) = 2$.

The case of three-generated ideals is much different.

Example 6.1 (McCullough) *Fix $r \in \mathbb{N}$ and let $S = k[x, y, z_1, \ldots, z_r]$. Set*

$$I = \left(x^r, y^r, \sum_{i=1}^{r} x^{i-1} y^{n-i} z_i \right).$$

Now consider the element $s = x^{r-1}y^{r-1}$. Clearly $s \notin I$ since no term in any of the generators of I divides s. It is easy to check that $xs, ys \in I$. It is only *slightly more difficult to check that $z_j s \in I$ for all $i = j, \ldots, r$. Indeed*

$$\begin{aligned}
z_j s &= x^{r-1} y^{r-1} z_j \\
&= x^{r-j} y^{j-1} \left(x^{j-1} y^{r-j} z_j \right) \\
&= -x^{r-j} y^{j-1} \sum_{i=1}^{r} x^{i-1} y^{r-i} z_i \qquad \text{mod } I \\
&= -x^{r-j} y^{j-1} \sum_{i=1}^{j-1} x^{i-1} y^{r-i} z_i - x^{r-j} y^{j-1} \sum_{i=j+1}^{r} x^{i-1} y^{r-i} z_i \qquad \text{mod } I \\
&= -y^{r} \sum_{i=1}^{j-1} x^{r-j+i-1} y^{j-i-1} z_i - x^{n} \sum_{i=j+1}^{r} x^{i-j-1} y^{r+j-i-1} z_i \qquad \text{mod } I \\
&= 0 \qquad \text{mod } I.
\end{aligned}$$

Now since s represents a nonzero element in S/I annihilated by every variable, there are no elements of S that are nonzerodivisors on S/I. Therefore $\operatorname{depth}(S/I) = 0$. *By the Auslander-Buchsbaum Theorem,* $\operatorname{pd}(S/I) = r + 2$.

It follows from the example that there is no upper bound on $\operatorname{pd}(S/I)$ purely in terms of the number of minimal generators of I.

Example 6.2 *In the specific case $n = 2$ of the previous example, we have $S = k[x, y, z_1, z_2]$ and*

$$I = \left(x^2, y^2, xz_1 + yz_2 \right).$$

We know that $\operatorname{pd}(S/I) = 4$. *The Betti table of S/I is*

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	3	-	-	-
2:	-	-	5	4	1

In fact, three-generated ideals in some sense capture all the pathology of free resolutions. Here we state the graded version of a theorem due to Bruns.

Theorem 6.3 (Bruns) *Let M be a finitely generated graded S-module and let $(\mathbf{F}_\bullet, d_\bullet)$ be its minimal graded free resolution. Then there exists a three generated ideal I whose resolution as the form*

$$\cdots \to F_5 \xrightarrow{d_5} F_4 \xrightarrow{d_4} F_3 \to S^r \to S^3 \to S.$$

Thus by altering the last three free modules, we can convert any free resolution into that of a three-generated ideals. However, if we apply this process to an ideal, the degrees of the generators necessarily grow.

Example 6.4 *Let $S = k[x_1, \ldots, x_5]$ and $m = (x_1, \ldots, x_5)$. Clearly S/m is resolved by a Koszul complex and has Betti table*

	0	1	2	3	4	5
0:	1	5	10	10	5	1

By Bruns' Theorem, there exists a three generated ideal I derived from the resolution of S/m with $\operatorname{pd}(S/I) = 5$. One such ideal is

$$I = (x_4x_5^2, x_2x_4^2 + x_2x_4x_5 + x_1x_5^2, x_2x_3x_4 + x_1x_2x_5 + x_2x_4x_5)$$

and its Betti table is

	0	1	2	3	4	5
0:	1	-	-	-	-	-
1:	-	-	-	-	-	-
2:	-	3	-	-	-	-
3:	-	-	-	-	-	-
4:	-	-	8	10	5	1

The ideal I can be computed in Macaulay2 with:

```
S = QQ[x_1..x_5]
m = ideal vars S
loadPackage "Bruns"
I = brunsIdeal m
```

Note that we started with an ideal generated by linear forms. To find an three-generated ideal with the same projective dimension we needed cubic generators.

Stillman's Question and Equivalent Formulations

The situation with three-generated ideals and computational considerations motivated Stillman to pose the following question

Question 6.5 (Stillman's Question) *Fix positive integers $a_1, \ldots, a_m$. Is there an upper bound, depending only on $a_1, \ldots, a_m$, on $\operatorname{pd}(S/I)$, where $I = (f_1, \ldots, f_m)$ and $\deg(f_i) = a_i$ for $i = 1, \ldots, m$.*

Note that the minimal number of generators $m = \sum_{i=1}^{m} a_i^0$ is part of the data we may reference in such a bound but the number of variables of S is not. Hence HIlbert's Syzygy Theorem 4.33 is not helpful here. One instance of Stillman's Question then asks for an upper bound on the projective dimension of ideals generated by 3 cubics in an unknown number of variables. Example 6.4 shows that the bound must be at least 5, if it exists.

There is a parallel version of Stillman's Question in which projective dimension is replaced by regularity.

Question 6.6 *Fix positive integers $a_1, \ldots, a_m$. Is there an upper bound, depending only on $a_1, \ldots, a_m$, on $\operatorname{reg}(S/I)$, where $I = (f_1, \ldots, f_m)$ and $\deg(f_i) = a_i$ for $i = 1, \ldots, m$?*

Unlike the case for projective dimension, there is no upper bound on the regularity of ideals in a fixed number of variables. Since

$\mathrm{reg}(I) \geq \mathrm{maxdeg}(I)$, to make $\mathrm{reg}(I)$ (equivalently, $\mathrm{reg}(S/I) + 1$) arbitrarily large, we can simply take ideals with arbitrarily large degree generators. There is a doubly-exponential bound ??? in terms of the number of variables and the maximal degree of a minimal generator of I which we discuss in the next chapter.

Caviglia showed that the two versions of Stillman's Question are in fact equivalent, though whatever the hypothetical bounds might be could be quite different.

Theorem 6.7 (Caviglia) *Question 6.5 has an affirmative answer if and only if Question 6.6 has an affirmative answer.*

For a proof, see Theorem 5 in

Jason McCullough and Alexandra Seceleanu. Bounding projective dimension. In *Commutative algebra*, pages 551–576. Springer, New York, 2013

Reductions and Easy Cases

We now consider the situations in which Stillman's Question has a clear positive answer. First note that we can always assume that the coefficient field k is algebraically closed since tensoring a resolution by $\overline{k}$ preserves exactness. If I is principal (the case $m = 1$), then we already know that $\mathrm{pd}(S/I) \leq 1$. Similarly if I has two minimal generators (the case $m = 2$), then $\mathrm{pd}(S/I) = 2$. If I is minimally generated by linear forms (the case $d_1 = d_2 = \cdots = d_m = 1$), then S/I is resolved by a Koszul complex and $\mathrm{pd}(S/I) \leq m$. If I is generated by m monomials of any degree, then $\mathrm{pd}(S/I) \leq m$. We can see this either by noting that Taylor's resolution is a possibly nonminimal free resolution of S/I or by using an inductive argument. In fact, anytime we have a bound on the number of terms or variables among the generators of I, then there is a clear upper bound on $\mathrm{pd}(S/I)$. More generally

For the definition of Taylor's resolution, see Construction 26.5 in

Irena Peeva. *Graded syzygies*, volume 14 of *Algebra and Applications*. Springer-Verlag London, Ltd., London, 2011

Theorem 6.8 *Let $S = k[x_1, \ldots, x_n]$ and let $I = (f_1, \ldots, f_m)$ be a homogeneous ideal of S. If there exists a regular sequence of forms $g_1, \ldots, g_p \in S$ such that $f_1, \ldots, f_m \in k[g_1, \ldots, g_p]$, then $\mathrm{pd}(S/I) \leq p$.*

Proof Since $g_1, \ldots, g_p$ is a regular sequence, $R = k[g_1, \ldots, g_p]$ is a polynomial ring. Let $\overline{I} = I \cap R$. By Hilbert's Syzygy Theorem 4.33, $\mathrm{pd}(R/\overline{I}) \leq p$. Since S is flat over R and since $S/I = R/\overline{I} \otimes_R S$, $\mathrm{pd}_S(S/I) = \mathrm{pd}_R(R/\overline{I}) \leq p$. ■

For instance, if I is generated by m binomials of degree at most d, then $\mathrm{pd}(S/I) \leq 2md$. The difficult cases occur when we do not know how many nonzero terms occur among the minimal generators of I.

Note that toric ideals are all generated by binomials.

Projective Dimension of Three Quadrics

The first nontrivial case of Stillman's Question then is the case when I is generated by 3 quadric forms. By example 6.2 we know the projective dimension of S/I can be at least 4. In this section we show that 4 is the optimal upper bound, therefore giving a positive answer to Stillman's Question in this case.

Theorem 6.9 (Eisenbud-Huneke) *Let $I = (f, g, h)$ where f, g and h are homogeneous minimal generators of degree 2 in a polynomial ring S over a field k. Then* $\operatorname{pd}(S/I) \leq 4$.

We will need several results to prove this theorem. Since $\operatorname{pd}(R/I)$ does not change after tensoring with an extension of the field of coefficients, we may assume that k is infinite.

First we need some terminology. An ideal $I \subseteq S$ is called **unmixed** if $\operatorname{ht}(\mathfrak{p}) = \operatorname{ht}(I)$ for all $\mathfrak{p} \in \operatorname{Ass}(S/I)$.

Exercise 6.10 *Let S/I is Cohen-Macaulay, then I is unmixed.*

Example 6.11 *Not every unmixed ideal is Cohen-Macaulay. The simplest example is $I = (w, x) \cap (y, z) = (wy, wz, xy, xz) \subseteq k[w, x, y, z]$. Since I is the intersection of two height 2 prime ideals, it is unmixed. Since $w + y \notin (w, x) \cup (y, z)$, this is a regular element on S/I. However, $S/(I + (w + y)) \cong k[x, y, z]/(xy, xz, y^2, yz)$. It is easy to check that $(xy, xz, y^2, yz) : y = (x, y, z)$. It follows that* $\operatorname{depth}(S/(I + (w + y)) = 0$ *and hence* $\operatorname{depth}(S/I) = 1$, *while* $\dim(S/I) = 2$. *In particular, S/I is not Cohen-Macaulay.*

Exercise 6.12 *Let $f_1, \ldots, f_c \in S$ is a homogeneous regular sequence with* $\deg(f_i) = d_i$ *for $1 \leq i \leq c$. Then $e(S/(f_1, \ldots, f_c)) = \prod_{i=1}^{c} d_i$.*

Exercise 6.13 *Suppose $I \subseteq J$ are unmixed ideals of the same height and multiplicity. Show that $I = J$.*

Proposition 6.14 *Let $I = (f, g, h)$, where $f, g, h \in S_2$ and* $\operatorname{ht}(I) = 2$. *Then $e(S/I) \leq 3$.*

Proof We may assume that f, g form a regular sequence of quadratic forms. Thus $e(R/(f, g)) = 4$. We have the series of containments $(f, g) \subseteq I \subseteq I^{un}$. Note that (f, g) and I^{un} are unmixed ideals of height two. If $e(R/(f, g)) = e(R/I^{un})$, then $(f, g) = I^{un}$ by the previous exercise. But this would force $(f, g) = (f, g, h)$, contradicting that h is a minimal generator of I. Thus $4 = e(R/(f, g)) > e(R/I^{un}) = e(R/I)$. ■

We also need the following structure theorem for ideals of height two and multiplicity two ideals.

Proposition 6.15 (Engheta) *Let J be a height two unmixed ideal of multiplicity two in a polynomial ring $S = K[x_1, \ldots, x_n$ with $K = \overline{K}$. Then* $\operatorname{pd}(R/J) \leq 3$ *and J is one of the following:*

1. $(x, y) \cap (w, z) = (xw, xz, yw, yz)$ *with independent linear forms* w, x, y, z.

2. (x, yz) *with independent linear forms* x, y, z.

3. *A prime ideal generated by a linear form and an irreducible quadratic.*

4. (x, y^2) *with independent linear forms* x, y.

5. $(x, y)^2 + (ax + by)$ *with independent linear forms* x, y *and* $a, b \in \mathfrak{m}$ *such that* x, y, a, b *form a regular sequence.*

Proof By the associativity formula, the ideal J is of one of the following types: $\langle 2; 1 \rangle$, $\langle 1; 2 \rangle$, or $\langle 1, 1; 1, 1 \rangle$. An ideal of type $\langle 2; 1 \rangle$ is a prime ideal $\mathfrak{p}$ of mutliplicity 2 and height 2. Necessarily these are degenerate, so they must be generated by a linear form and an irreducible quadratic. In particular, $\operatorname{pd}(S/\mathfrak{p}) = 2$.

An ideal of type $\langle 1; 1 \rangle$ is a prime ideal of multiplicity 1, which is necessarily generated by two linear forms. Thus ideals of type $\langle 1, 1; 1, 1 \rangle$ are the intersection of two distinct such ideals, say $J = (x, y) \cap (w, z)$. There are two cases to consider: either $\dim_K(x, y, z, w) =$, putting us in case 1, or $\dim_K(x, y, z, w) = 3$, putting us in case 2. One checks in either case that $\operatorname{pd}(S/J) \leq 3$.

Finally an ideal J of type $\langle 1; 2 \rangle$ is primary to a linear prime $\mathfrak{p} = (x, y)$ and satisfies $e(S/J) = 2$. Note that $J : x = (x, y)$. Therefore $(x, y)^2 \subseteq J$. If J contains a linear form, then after a linear change of variables, we have $(x, y^2) \subseteq J$. Since (x, y^2) is unmixed and multiplicity 2, we have $J = (x, y^2)$.. If J does not contain a linear form, then since $\lambda(S_{\mathfrak{p}}/J_{\mathfrak{p}}) = 2$, J must contain a linear form locally at $\mathfrak{p}$. In other words, there exists an element of the form $ax + by \in J$, with $\operatorname{ht}(x, y, a, b) \geq 3$. Moreover, we can pick such an element of minimal degree. If $\operatorname{ht}(x, y, a, b) = 4$, one check that this ideal is itself unmixed and therefore $\mathfrak{p}$-primary. (We check once more that $\operatorname{pd}(S/J) = 3$ in this case.) If not, we can write $ax + by = c(a'x + b'y)$, where $c \notin \mathfrak{p}$. Since J is $\mathfrak{p}$-primary and $c \notin \mathfrak{p}$, $a'x + b'y \in J$, contradicting the minimality of the degree of $ax + by$. This concludes the proof. ■

We are finally ready to prove the bound.

Proof of Theorem 6.9 By Krull's (generalized) Principal Ideal Theorem, $\operatorname{ht}(I) \leq 3$. If $\operatorname{ht}(I) = 1$, then there are linear forms c, f', g', h' with $f = cf'$, $g = cg'$ and $h = ch'$. Hence $I \cong (f', g', h')$, and so $\operatorname{pd}(R/I) = 3$.

If $\operatorname{ht}(I) = 3$, then f, g, h form a regular sequence and the Koszul complex on f, g, h forms a minimal free resolution of R/I. Again $\operatorname{pd}(R/I) = 3$. Hence we may assume that $\operatorname{ht}(I) = 2$. Moreover, we may assume that f, g form a regular sequence.

Now by Proposition 6.15, $e(R/I) = 1, 2$ or 3.

If $e(R/I) = 3$, consider the short exact sequence

$$0 \to R/((f,g):I) \xrightarrow{h} R/(f,g) \to R/I \to 0.$$

Since f, g form a regular sequence of quadratic forms, we have $e(R/(f,g)) = 4$. Since multiplicity is additive in short exact sequences, $e(R/((f,g):I)) = 1$. As $(f,g):I$ is unmixed, we have $(f,g):I = (x,y)$ for independent linear forms x and y. Therefore, $\mathrm{pd}(R/((f,g):I)) = 2$. Since $\mathrm{pd}(R/(f,g)) = 2$, it follows that $\mathrm{pd}(R/I) \leq 3$.

If $e(R/I) = 2$, we use the same exact sequence above. In this case $(f,g):I$ is an unmixed, height two ideal of multiplicity two. By Proposition 6.15, $\mathrm{pd}(R/((f,g):I))) \leq 3$. It follows that $\mathrm{pd}(R/I) \leq 4$. This completes the proof.

If $e(R/I) = 1$, then by the associativity formula, I^{un} is primary to a height two prime ideal $\mathfrak{p}$ of multiplicity one. Such a prime ideal is generated by two linear forms, say $\mathfrak{p} = (x,y)$. Since $\lambda(R_{\mathfrak{p}}/I_{\mathfrak{p}}) = 1$ and I is $\mathfrak{p}$-primary, $I^{un} = \mathfrak{p}$. Note that

$$(f,g):h = (f,g):I = (f,g):I^{un} = (f,g):(x,y).$$

We leave the following chain of isomorphisms as an exercise

$$\begin{aligned}
\frac{(f,g):(x,y)}{(f,g)} &\simeq \mathrm{Hom}_S(S/(x,y), S/(f,g)) \\
&\simeq \mathrm{Ext}^1_S(S/(x,y), S/(f)) \\
&\simeq \mathrm{Ext}^2_S(S/(x,y), S) \\
&\simeq S/(x,y)
\end{aligned}$$

Now consider the short exact sequence

$$0 \to \frac{(f,g):(x,y)}{(f,g)} \to \frac{S}{(f,g)} \to \frac{S}{(f,g):(x,y)} \to 0.$$

Clearly the middle term has projective dimension two. By the chain of isomorphisms above, so does the first term. Thus

$$\mathrm{pd}\left(\frac{S}{(f,g):(x,y)}\right) = \mathrm{pd}\left(\frac{S}{(f,g):h}\right) \leq 3.$$

Again by the first short exact sequence we have $\mathrm{pd}(S/I) \leq 4$, concluding the proof. ■

Proof of the general case in characteristic 0

Very recently, Ananyan and Hochster [9] gave a proof of Stillman's

[9] Tigran Ananyan and Melvin Hochster. Small subalgebras of polynomial rings and stillman's conjecture. preprint: arXiv:1610.09268, 2016

Question in full generality. In this section we outline the argument in the characteristic 0 case, which is slightly simpler than the general case because we can rely on Euler's formula. For any form $f \in S = k[x_1, \ldots, x_n]$ of degree d we can write

$$f = \frac{1}{d} \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}.$$

In particular, f is in the ideal generated by the partial derivatives $\left(\frac{\partial f}{\partial x_i}\right)$. If d divides the characteristic of k, then clearly this doesn't work. There is a workaround in this case but we'll stay in the easier characteristic 0 situation.

Serre's conditions and prime sequences

We recall Serre's conditions (R_i) and (S_i).

Definition Let R be a Noetherian ring and let i be a nonnegative integer. We say R satisfies **Serre's condition** $(\mathbf{R_i})$ if for all prime ideals $\mathfrak{p}$ of R of height at most i, $R_\mathfrak{p}$ is a regular local ring. We say R satisfies **Serre's condition** $(\mathbf{S_i})$ if for all prime ideals $\mathfrak{p}$ of R, $\operatorname{depth}(R_\mathfrak{p}) \geq \min\{i, \operatorname{ht}(\mathfrak{p})\}$.

Note that every Cohen-Macaulay ring satisfies (S_i) for all i. For any $i \geq j$, Serre's condition (S_i) (resp. (R_i)) implies (S_j) (resp. (S_j)).

The following facts about Serre's conditions are crucial for the proof.

Theorem 6.16 *A Noetherian ring is reduced if and only if it satisfies Serre's conditions (R_0) and (S_1).*

Recall that a Noetherian domain is called **normal** if it is integrally closed in its field of fractions. An arbitrary ring R is normal if $R_\mathfrak{p}$ is a normal domain for all primes p.

Theorem 6.17 *A Noetherian ring is normal if and only if it satisfies Serre's conditions (R_1) and (S_2).*

Theorem 6.18 *Let R be a ring. Assume R is reduced and has finitely many minimal primes. Then the following are equivalent:*

1. *R is a normal ring,*
2. *R is integrally closed in its total ring of fractions, and*
3. *R is a finite product of normal domains.*

Theorem 6.19 *A regular ring is a UFD.*

Theorem 6.20 *Let $\underline{f} = f_1, \ldots, f_m$ be a regular sequence in S and suppose that $S/(\underline{f})_\mathfrak{p}$ is a UFD for every $\mathfrak{p} \in \operatorname{Spec}(S/(\underline{f}))$ with $\operatorname{ht}(\mathfrak{p}) \leq 3$. Then $S/(\underline{f})$ is a UFD.*

A major new concept in the Ananyan-Hochster proof are the following special types of regular sequences.

Definition A sequence of elements $f_1, \dots, f_m \in S$ is a **prime sequence** (respectively $\mathbf{R_i}$**-sequence**, where $i \in \mathbb{N}$), if $f_j \notin (f_1, \dots, f_{j-1})$ and $S/(f_1, \dots, f_j)$ is a domain (respectively, satisfies (R_i)) for $j = 1, \dots, m$.

A very useful observation is the following:

Proposition 6.21 *Let $\underline{f} = f_1, \dots, f_m \in S$ be homogeneous elements of positive degree.*

1. *If $\underline{f}$ is a prime sequence, then $\underline{f}$ is a regular sequence.*
2. *If $\underline{f}$ is an R_i-sequence for some $i \geq 1$, then $S/(\underline{f})$ is a normal domain. In particular, $\underline{f}$ is a prime sequence.*

An R_0-sequence is not necessarily a prime or even regular sequence. For example $f_1 = xy$ and $f_2 = xz$ in $S = k[x,y,z]$. Then $(f_1) = (x) \cap (y)$ and $(f_1, f_2) = (x) \cap (y, z)$ are both intersections of primes. Thus $S/(f_1)$ and $S/(f_1, f_2)$ are reduced and hence satisfy (R_0) but $f_2 = xz$ is a zerodivisor on $S/(xy)$.

Proof 1. Suppose $\underline{f}$ is a prime sequence. Fix $1 \leq j \leq m$. Then $S/(f_1, \dots, f_{j-1})$ is a domain. Since $f_j \notin (f_1, \dots, f_{j-1})$, f_j is a nonzerodivisor on $S/(f_1, \dots, f_{j-1})$. Hence $\underline{f}$ is a regular sequence.

2. Suppose $\underline{f}$ is an R_i-sequence for some $i \geq 1$. Since (R_i) for $i \geq 1$ implies (R_1), we may assume that $\underline{f}$ is an R_1-sequence. We proceed by induction on j by assuming $f_1, \dots, f_{j-1}$ is a prime sequence. By part 1, $f_1, \dots, f_{j-1}$ is a regular sequence. Since $S/(f_1, \dots, f_{j-1})$ is a domain and $f_j \notin (f_1, \dots, f_{j-1})$, f_j is a nonzero divisor on $S/(f_1, \dots, f_{j-1})$. Hence $f_1, \dots, f_j$ is a regular sequence. Therefore $\overline{S} = S/(f_1, \dots, f_j)$ is a complete intersection - hence Cohen-Macaulay - hence (S_2). By assumption $\overline{S}$ satisfies (R_1). By Theorem 6.17, $\overline{S}$ is normal. By Theorem 6.18, $\overline{S}$ is a direct sum of normal domains, say $\overline{S} = D_1 \oplus \cdots \oplus D_p$. Note that every element $(d_1, \dots, d_p) \in D_1 \oplus \cdots \oplus D_p$ where $d_i = 0$ or 1 for all i is an idempotent element of $\overline{S}$. Since $\overline{S}$ is graded, all of these elements have degree 0, and since $\overline{S}_0 \cong k$, they can be identified with elements of k. But if $p \geq 2$, then we have at least 4 roots in k of the polynomial $x^2 - x$, a contradiction. Therefore $p = 1$ and $\overline{S}$ is a normal domain. In particular, $f_1, \dots, f_p$ is a prime sequence.

∎

Strength and k-collapse

Definition Let $F \in K[x_1, \dots, x_n]$ with $\deg(F) > 0$. We say that F has a ***k*-collapse** for $k \in \mathbb{N}$, if F is in an ideal generated by k elements of strictly smaller positive degree. We say that F has **strength** k if it has a $(k+1)$-collapse but no k-collapse.

Exercise 6.22 *If F is a nonzero linear form, then F has strength $+\infty$.*

Exercise 6.23 *If F is a nonzero form of positive degree then F has strength at least 1 if and only if F is irreducible.*

Exercise 6.24 *Let F be a nonzero quadratic form. If F has a 2-collapse then $S/(F)$ is not a UFD.*

Singular locus

One of the key observations of Ananyan-Hochster was that F has a small collapse if and only if the singular locus of f is large (i.e. has small codimension). One direction is easy. Suppose f has a k-collapse. Write $f = \sum_{i=1}^{k} g_i h_i$, where $\deg(h_i) < \deg(f)$ and $\deg(g_i) < \deg(f)$ for all $1 \leq i \leq k$. Then all partial derivatives of f are contained in the ideal $(g_1, \ldots, g_k, h_1, \ldots, h_k)$. We remark that the following conditions are equivalent:

Recall that the singular locus of a ring R is the set $\{\mathfrak{p} \mid R_{\mathfrak{p}} \text{ is not regular}\}$. There are subtleties relating smoothness and regularity of rings that do not concern us here and we omit the details for simplicity. For details, see Matsumura's Commutative Rings textbook.

1. The ring $S/(f)$ satisfies Serre's (R_i) condition.
2. The codimension of the singular locus of $S/(f)$ is at least $i+1$ in $S/(f)$.
3. The height of $(f) + \left(\frac{\partial f}{\partial x_i}\right)$ is at least $i+2$ in S.

Thus if f has a k-collapse, the height of $(f) + \left(\frac{\partial f}{\partial x_i}\right)$ is at most $2k+1$. By above this means the codimension of the singular locus of $S/(f)$ is at most $2k$. Ananyan-Hochster prove a very surprising converse to this statement, see Theorem 6.25 below.

We give one final definition first. An integer-valued function on $\mathbb{N}^h$ is **ascending** if it is nondecreasing in each input when the others are held fixed. By taking maximums over preceding values, all bounding functions considered below can be made ascending. A d-tuple of integer-valued functions on $\mathbb{N}^h$ is ascending if all its entries are ascending functions.

Theorem 6.25 *There exists an integer ${}^{\eta}A(d) \geq d-1 \geq 0$, ascending as a function of $\eta, d \in \mathbb{Z}_+$, such that if $S = K[x_1 \ldots, x_N]$ is a polynomial ring in N variables over an algebraically closed field K and $f \in S$ is a form of degree $d \geq 1$ of strength at least ${}^{\eta}A(d)$, then the codimension of the singular locus in $S/(f)$ is at least $\eta + 1$. (i.e. $S/(f)$ satisfies Serre's (R_η) condition.)*

Theorem 6.26 *There is an ascending function ${}^{\eta}\mathcal{A}(n,d)$, independent of K and N, such that for all polynomial rings $S = K[x_1, \ldots, x_N]$ over an algebraically closed field K and all ideals I generated by a graded vector space V whose nonzero homogeneous elements have positive degree of at most d, if no homogeneous element of $V - \{0\}$ is in an ideal generated by ${}^{\eta}\mathcal{A}(n,d)$ forms of strictly lower degree, then S/I satisfies Serre's (R_η) condition.*

Let V be a finite dimensional graded vector subspace of S spanned by forms of positive degree. If d is an upper bound for the degree of any element of V, we may write $V = V_1 \oplus \cdot \oplus V_d$, where V_i denotes the ith graded piece, we shall say V has **dimension sequence** $(\delta_1, \ldots, \delta_d)$ where $\delta_i = \dim_K(V_i)$.

Theorem 6.27 *There is an ascending function ${}^{\eta}\mathcal{B}(n,d)$, independent of K and N, such that for all polynomial rings $S = K[x_1, \ldots, x_N]$ over an algebraically closed field K and all graded vector subspaces V of S of dimension at most n whose homogeneous elements have positive degree at most d, the elements of V are contained in a subring $K[g_1, \ldots, g_B]$, where $B \leq {}^{\eta}\mathcal{B}(n,d)$ and $g_1, \ldots, g_B$ is an R_η-sequence of forms of degree at most d.*

Theorem 6.28 *There is an ascending function $C : \mathbb{Z}_+ \times \mathbb{Z}_+ \times \mathbb{N} \to \mathbb{Z}_+$ with the following property. If S is a polynomial ring over any field K and M is a graded module that is the cokernel of an $m \times n$ matrix whose entries have degree at most d, then the following quantities are bounded by $C(m,n,d)$:*

1. $\mathrm{pd}(M)$
2. $\mathrm{reg}(M)$
3. $\beta_{ij}(M)$ *for all i and j*
4. *The number of primary components of M*
5. *The number and degrees of generators of each primary component in some primary decomposition of M.*
6. *The minimum number of generators of every associated prime ideal.*

The outline of the proof works as follows:

Theorem 6.25 in degree at most d
$\Longrightarrow$ Theorem 6.26 in degree at most d
$\Longrightarrow$ Theorem 6.27 in degree at most d
$\Longrightarrow$ Theorem 6.28 in degree at most d
$\Longrightarrow$ Theorem 6.25 in degree at most $d+1$.

Thus we are finished after a massive inductive argument.

We close this chapter with one recent application. Recall that in Theorem 5.19 we showed that there is no polynomial bound on the regularity nondegenerate prime ideals purely in terms of their multiplicity. We now have the tools to prove there is such a (non-polynomial) bound. First we need the following theorem:

Theorem 6.29 (Eisenbud-Huneke-Vacsoncelos) *If K is a perfect field and $I \subseteq S = K[x_1, \ldots, x_n]$ is a homogeneous equidimensional radical ideal then I is generated up to radical by forms of degree $\leq e(S/I$.[10]*

[10] David Eisenbud, Craig Huneke, and Wolmer Vasconcelos. Direct methods for primary decomposition. *Invent. Math.*, 110(2):207–235, 1992

An immediate application of this theorem combined with the Ananyan-Hochster results is the following:

Theorem 6.30 (Cavgilia-McCullough-Peeva-Varbaro) *There is an ascending function $P(e)$ with the following property. Let $\mathfrak{p}$ be a nondegenerate prime ideal in $S = K[x_1, \ldots, x_n]$, where K is algebraically closed and suppose that $e(S/\mathfrak{p}) = e$. Then* $\mathrm{pd}(S/\mathfrak{p})$, $\mathrm{reg}(S\mathfrak{p})$ *and all graded Betti numbers of $S/\mathfrak{p}$ are bounded by $P(e)$.*[11]

[11] Giulio Caviglia, Jason McCullough, Irena Peeva, and Matteo Varbaro. Regularity of prime ideals. preprint, 2017

Proof Since $K = \overline{K}$, we have $h := \mathrm{ht}(\mathfrak{p}) < e(S/\mathfrak{p})$. By the previous theorem, $\mathfrak{p}$ is generated up to radical by forms of degree $\leq e(S/\mathfrak{p})$. Therefore we can find a regular sequence $g_1, \ldots, g_h$ of forms of degree at most e in $\mathfrak{p}$. Moreover $\mathfrak{p}$ is a minimal prime of $(g_1, \ldots, g_h)$. By Theorem 6.28, the minimal number of generators of $\mathfrak{p}$ is at most $C(h, 1, e)$. By another application of Theorem 6.28 we see that $\mathrm{reg}(S/\mathfrak{p})$, $\mathrm{pd}(S/\mathfrak{p})$, and all $\beta_{ij}(S/\mathfrak{p})$ are bounded by a formula depending only on $e(S/\mathfrak{p})$. ■

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