Nordfjordeid Commutative Algebra Workshop - Problem Set 1 (McCullough)

(1) Let $0 \to L \to M \to N \to 0$ be a short exact sequence of finitely generated graded S-modules. Show that
 (a) $\mathrm{pd}_S(L) \leq \max\{\mathrm{pd}_S(M), \mathrm{pd}_S(N) - 1\}$
 (b) $\mathrm{pd}_S(M) \leq \max\{\mathrm{pd}_S(L), \mathrm{pd}_S(N)\}$
 (c) $\mathrm{pd}_S(N) \leq \max\{\mathrm{pd}_S(L) + 1, \mathrm{pd}_S(M)\}$.
Similarly, show that
 (a) $\mathrm{reg}_S(L) \leq \max\{\mathrm{reg}_S(M), \mathrm{reg}_S(N) + 1\}$
 (b) $\mathrm{reg}_S(M) \leq \max\{\mathrm{reg}_S(L), \mathrm{reg}_S(N)\}$
 (c) $\mathrm{reg}_S(N) \leq \max\{\mathrm{reg}_S(L) - 1, \mathrm{reg}_S(M)\}$.

(2) Let $I = (m_1, m_2, \ldots, m_t) \subseteq S = k[x_1, \ldots, x_n]$ be a monomial ideal. Set $d_i = \deg(m_i)$.
 (a) Show that for $1 \leq i \leq t$:
$$(m_1, \ldots, m_{i-1}) : m_i = \left(\frac{m_1}{\gcd(m_1, m_i)}, \ldots, \frac{m_{i-1}}{\gcd(m_{i-1}, m_i)}\right).$$
 Note that for ideals I, J, the ideal $I : J$ is
$$I : J = \{s \in S \mid sj \in I \text{ for all } j \in J\}.$$
 (b) Show that
$$0 \to \frac{S}{(m_1, \ldots, m_{i-1}) : m_i}(-d_i) \xrightarrow{m_i} \frac{S}{(m_1, \ldots, m_{i-1})} \to \frac{S}{(m_1, \ldots, m_i)} \to 0$$
 is a graded short exact sequence of S-modules for $1 \leq i \leq t$.
 (c) Show that
$$\mathrm{pd}_S(S/I) \leq t \qquad \text{and} \qquad \mathrm{reg}_S(S/I) \leq \sum_{i=1}^{t} (d_i - 1).$$

(3) Let $I = (x^2, xy, y^2) \subseteq S = k[x, y]$. Compute the minimal graded free resolutions of S/I. Is S/I Cohen-Macaulay? (You may want to ask Macaulay2 for help.)

(4) What about $I = (wy, wz, xy, xz) \subseteq k[w, x, y, z]$? Is S/I Cohen-Macaulay?

(5) Let M be a finitely generated S-module. Show that M is Cohen-Macaulay if and only if $\mathrm{codim}(M) = \mathrm{pd}_S(M)$. (Recall $\mathrm{codim}(M) = \dim(S) - \dim(M)$. If $M = S/I$, then $\mathrm{codim}(M) = \mathrm{ht}(I)$.)

(6) Recall that $\mathrm{ht}(I) = \min\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}(S/I)\}$. Let $\mathrm{bight}(I) = \max\{\mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Ass}(S/I)\}$. Show that
$$\mathrm{pd}_S(S/I) \geq \mathrm{bight}(I).$$
Bonus: Find an example where the inequality is strict.

Nordfjordeid Commutative Algebra Workshop - Problem Set 2 (McCullough)

(1) Let $I = (x^2, y^2, ax + by) \subseteq k[x, y, a, b]$. The graded Betti table of S/I is

	0	1	2	3	4
0:	1	-	-	-	-
1:	-	3	-	-	-
2:	-	-	5	4	1

Find the Hilbert Series, Hilbert polynomial, multiplicity of S/I, dimension and depth of S/I. Is S/I Cohen-Macaulay?

(2) Let $S = k[x_1, \dots, x_n]$. Show that

$$\mathrm{Hilb}_S(t) = \frac{1}{(1-t)^n}.$$

(3) Let $f_1, \dots, f_m \in S$ be a regular sequence of forms of degrees $d_1, \dots, d_m$. Show that

$$\mathrm{Hilb}_{S/(f_1,\dots,f_m)}(t) = \frac{\prod_{i=1}^m (1 - t^{d_i})}{(1-t)^n}.$$

(4) Let $f_1, \dots, f_m \in S$ be a regular sequence of forms of degrees $d_1, \dots, d_m$. Show that

$$e(S/(f_1, \dots, f_m)) = \prod_{i=1}^m d_i.$$

(5) Suppose $I \subseteq J \subseteq S$ are homogeneous ideals of the same height. Show that $e(J) \leq e(I)$.

(6) (Harder) If $f_1, \dots, f_m \in S = k[x_1, \dots, x_n]$ is a homogeneous regular sequence then $k[f_1, \dots, f_m]$ is isomorphic to a polynomial ring $k[y_1, \dots, y_m]$.

Nordfjordeid Commutative Algebra Workshop - Problem Set 3 (McCullough)

Recall that an ideal $I \subseteq S$ is a primary ideal if $xy \in I$ implies that $x \in I$ or $y^n \in I$ for some $n > 0$. In this case $\sqrt{I} = \mathfrak{p}$ is a prime ideal and we say that I is $\mathfrak{p}$-primary.

(1) Show that $(x, y^2) \subseteq k[x, y]$ is (x, y)-primary and $e(S/(x, y^2)) = 2$.

(2) Let x, y be independent linear forms in a polynomial ring S. Show that $I = (x, y)^2 + (ax + by)$ is (x, y)-primary with $e(S/I) = 2$ if and only if $\mathrm{ht}(x, y, a, b) = 4$.

(3) Show that if I is $\mathfrak{p}$-primary and $x \in \mathfrak{p}$, then $I : x = S$ or $I : x$ is $\mathfrak{p}$-primary and $I : x \supsetneq I$.

(4) The associativity formula says that

$$e(S/I) = \sum_{\mathfrak{p} \in \mathrm{Ass}(S/I) \text{ with } \mathrm{ht}(\mathfrak{p}) = \mathrm{ht}(I)} e(S/\mathfrak{p}) \lambda(S_{\mathfrak{p}}/I_{\mathfrak{p}}).$$

Use the associativity formula to show that if $I \subseteq J$, $\mathrm{ht}(I) = \mathrm{ht}(J)$, $e(S/I) = e(S/J)$ and both I and J are unmixed (all associated primes have height equal to height of I), then $I = J$.

(5) Show that if I is (x, y)-primary and $e(S/I) = 2$, then after perhaps a linear change of variables, $I = (x, y^2)$ or $I = (x, y)^2 + (ax + by)$, where $\mathrm{ht}(x, y, a, b) = 4$.

(6) (Hard) What about if I is (x, y)-primary and $e(S/I) = 3$?

(7) Buchsbaum-Eisenbud acyclicity theorem: Let $F_\bullet$ be a finite free complex of finitely generated S-modules

$$0 \to F_s \xrightarrow{\phi_s} F_{s-1} \xrightarrow{\phi_{s-1}} \cdots \xrightarrow{\phi_1} F_0.$$

For $i = 1, \ldots, s$, set $r_i = \sum_{j=i}^{s} (-1)^{j-i} \operatorname{rank}(F_j)$. Then $F_\bullet$ is acyclic if and only if for each $1 \le i \le s$, $\mathrm{ht}(I_{r_i}(\phi_i)) \ge i$. Here if $\phi : G \to H$ is a map of finitely generated free modules, $I_r(\phi)$ denotes the ideal generated by the $r \times r$-minors of any matrix representing ϕ.
Use the Buchsbaum-Eisenbud theorem to show that the following is a resolution of $k[x, y, a, b]/(x^2, xy, y^2, ax + by)$.

$$0 \to S \xrightarrow{\phi_3} S^4 \xrightarrow{\phi_2} S^4 \xrightarrow{\phi_1} S,$$

where

$$\phi_1 = \begin{pmatrix} x^2 & xy & y^2 & xa + yb \end{pmatrix} \qquad \phi_2 = \begin{pmatrix} -y & 0 & -a & 0 \\ x & -y & -b & -a \\ 0 & x & 0 & -b \\ 0 & 0 & x & y \end{pmatrix} \qquad \phi_3 = \begin{pmatrix} a \\ b \\ -y \\ x \end{pmatrix}$$