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Dep. of Mathematics, Univ. of Tr.heim

Uncountable Intersections of Random Sets

By Gunnar Taraldsen
University of Trondheim

We prove compact-randomness of the intersection of a measurable family of compact-random sets. This generalizes results on the intersection of countable families of random sets. The main tools are given by the theory of Souslin spaces and the idea of countable separation.

An example of a random set is given by the energy spectrum of an electron in a disordered solid, or more generally by the spectrum of a closed linear random operator. If the solid is ergodic the spectrum equals a fixed set. This self-averaging result fails if the electron in addition is influenced by an electric field. Our idea is to single out the part of the spectrum which depends on the bulk properties of the solid. The intersection result makes it possible to prove self-averaging for this part of the spectrum.

This work indicates the importance of the link between the theory of random sets and the theory of linear random operators.

1. Introduction A classical reference for the theory of random sets is [7]. Updated supplements are [11], [8], [1], and the references found therein. A random set F in a set X is described by assignment of a consistent probability to the hit event $F \bigcap C \neq \emptyset$ for every fixed set C in a suitable family of sets. If F and G are random sets it is natural to ask if the union $F \bigcup G$ and intersection $F \bigcap G$ also are random sets. Clearly the union $F \bigcup G$ hits C iff F hits C or G hits C, so the union is a random set. Similarly a countable union of random sets is again a random set, since a hit event is given by the union of individual hit events: The family $\mathcal{E}$ of events contains the individual hit events and is closed under countable unions (and complements). If the intersection $F \bigcap G$ hits C then F and G hits C, but the converse is false so it is not clear whether the intersection is a random set. In fact [6, p.166,ex.2] gives an example which shows that the intersection of two closed open-random sets in a Polish space may fail to be an open-random set. In relation to our results it is then important to note that closed- implies open- implies compact-randomness of closed random sets in a metric space. In a sigma-compact space compact- implies closed-randomness.

We use the following concepts from descriptive set theory [2]: A Polish space is a separable, complete metric space. An example is the set of irrational numbers which is a Polish space in the relative topology from the real line with the usual topology, but with an unusual metric. A Souslin space is a metric space which is the continuous image of a Polish space. The family $\mathcal{B}$ of Borel sets in a topological space is the sigma-algebra generated by the family $\mathcal{F}$ of closed sets. A

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closed random set F is a measurable mapping $F : \Omega \to \mathcal{F}$, where Ω is a probability space. F is compact-random if $F^{-1}(K) := \{\omega \mid F(\omega) \cap K \neq \emptyset\}$ is an event for every compact set K. With this choice we can assign a consistent probability for hitting compact sets. A probability space is complete if all subsets of null events are events: $P(E) = 0$ and $A \subset E$ implies $A \in \mathcal{E}$. Our main theorem can now be stated.

Theorem 1.1 (Intersection of Closed Random Sets) *Let Ω be a complete probability space and let B be a Borel set in a Souslin space S. If $F_s(\omega)$ are closed sets in a metric space such that $\{(s, \omega) \in S \times \Omega \mid F_s(\omega) \cap K = \emptyset\}$ is product-measurable for any compact set K, then $\cap_{s \in B} F_s$ is a compact-random set. The completeness assumption on the probability space is unnecessary if B is countable.*

The proof of this theorem is found in the next section, where we use the term measurable family of compact-random sets to denote the measurability condition stated above. The identity $(\cup_s F_s)^{-1}(C) = \cup_s F_s^{-1}(C)$ gives $\mathcal{C}$-randomness of the union of a measurable family of $\mathcal{C}$-random sets when the probability space is complete and the index set is a Souslin space. $\mathcal{C}$ is the family of sets which defines the hit events. We rely on the Souslin measurable projection theorem. An alternative approach is given by the theory of separable and measurable processes [5], but this seems less straightforward. In the remainder of this section we present the model which motivated our efforts. The main message is the importance of the link between the theory of random operators and sets.

A quantum mechanical model for the electron in a solid is given by the Schrödinger operator $H = -\Delta + V(\vec{Q})$. The domain of this differential operator is a subspace of the Hilbert space of functions $f : \sigma(\vec{Q}) \to \mathbf{C}$ with $\|f\| := \sqrt{\int |f|^2} < \infty$. The spectrum $\sigma(\vec{Q})$ of the position observable represents the possible electron positions. A typical choice is $\sigma(\vec{Q}) = \mathbf{R}^3$. $-\Delta$ is given by the Laplace differential operator, and represents the kinetic energy of the electron. $V(\vec{Q})$ is a multiplication operator given by the potential energy function $V : \sigma(\vec{Q}) \to \mathbf{R}$, and depends on the solid. Under suitable, often rather technical [4], conditions on the potential the Schrödinger operator is realized as a self-adjoint operator and the spectrum $\sigma(H)$ is a closed set of real numbers representing the possible energies of the electron. In many cases it is reasonable to consider a random solid, either because of actual uncertainty or because of an untractable deterministic model. Anyway the result is a model where the potential V is a stochastic process, and $V(\vec{Q})$ and H are examples of self-adjoint random operators [9], and their spectra are closed random sets [13]. A most interesting case, both from a theoretical and applied point of view, follows from ergodicity of the potential V with respect to spatial shifts. As a result the event $\sigma(H) \cap U$ has probability zero or one for any open set U, and the energy spectrum equals a fixed set with probability one. In the case of self-adjoint operators this self-averaging result is due to Pastur [10].

We will now sketch how our main result on uncountable intersection of random sets generalizes Pastur's result to the case of a potential which is not stationary, nor ergodic. The motivation is the Stark potential $V(\vec{r}) = V_s(\vec{r}) + Ex$. V_s represents the solid and $E > 0$ is the result of an external electric field in the positive x-direction. We will assume the randomness of the solid to be entirely due to a sequence $A_1, A_2, \ldots$ of independent random elements in a Souslin space. Typically A_i specifies properties of the i'th atom, such as its position and electric charge. The result is an energy spectrum $\Sigma(A) := \sigma(H)$. Our idea is now to single out the part of the spectrum which only depends on the tail of A. This tail-part Σ^t of the spectrum is characterized by the property of being the largest closed set contained in Σ such that $\Sigma^t(A') = \Sigma^t(A)$ whenever A' and A have equal tails. At the end of the next section we prove randomness of the tail-spectrum as an application of our main result. From Kolmogorovs zero-one law [5, p.102-] it follows now that the event $\Sigma^t \cap U \neq \emptyset$ has probability zero or one. The final conclusion is the self-averaging of the tail-spectrum. In some applications the tail-spectrum equals the essential spectrum.

2. Proof of the Main Result

First we prove the result in the case of a countable intersection. The basic idea is the concept of a countable separant for compacts.

Lemma 2.1 *In a second countable Hausdorff space there exists a countable basis which Hausdorff separates the compacts.*

Proof. The family $\mathcal{B}$ of finite unions of sets from a countable basis is a countable separant for the compacts: Let M and N be disjoint compact sets. The Hausdorff property gives disjoint open sets U' and V' with $M \subset U'$ and $N \subset V'$. U' is a union of sets from the basis. This gives a covering of M and the compactness of M gives a finite subcovering. The union U of the sets in the finite subcovering belongs to $\mathcal{B}$. A set V in $\mathcal{B}$ with the property $N \subset V \subset V'$ is found likewise. The conclusion is the existence of two disjoint open sets U and V in the basis $\mathcal{B}$ with $M \subset U$ and $N \subset V$. □

A compact Hausdorff space with a countable Hausdorff separant for compacts is second countable, since the separant also is a basis. In the remainder of this section all the proofs are valid for closed compact-random sets in a space with second countable compacts. In particular the case of a second countable Hausdorff space and the case of a metric space. In the following lemma we do not require the probability space to be complete and this is essential in the proof of our main theorem.

Lemma 2.2 *The intersection of two closed compact-random sets in a metric space is a compact-random set.*

Proof. Let K be compact, let $\mathcal{B}$ be a countable separant in K, and let F and G be closed compact-random sets. With $(U, V) \in \mathcal{B}' \Leftrightarrow [U, V \in \mathcal{B}, U \cap V = \emptyset]$ the claim follows from:

$$\{\omega \mid F(\omega) \cap G(\omega) \cap K = \emptyset\} = \bigcup_{(U,V)\in\mathcal{B}'} \{\omega \mid F(\omega) \cap K \cap U^c = \emptyset \text{ and } G(\omega) \cap K \cap V^c = \emptyset\}.$$

□

The previous proof is a modification of the separation ideas found in [12] where open-randomness of the intersection of two compact open-random sets in a separable metric space is proven. A slight modification of our proof gives closed-randomness of the intersection of two compact closed-random sets in a second countable Hausdorff space.

Theorem 2.2 *A countable intersection of closed compact-random sets in a metric space is a compact-random set.*

Proof. It is sufficient to prove the claim for a sequence $F_1 \supset F_2 \supset \cdots$ of closed compact-random sets, because compact-randomness is conserved under finite intersections. The finite intersection property of a compact set K gives

$$[\bigcap_i F_i]^{-1}(K) = \bigcap_i F_i^{-1}(K)$$

and the claimed measurability follows. □

Reference [3, p.87, Thm. III.40] proves open-randomness of a countable intersection of open-random closed sets in a Polish space in the case of a complete probability space. The following depends on the Souslin measurable projection theorem [3, p.75, Thm. III.23]: Let Ω be a complete probability space and let S be a Souslin space. The projection into Ω of a product measurable set in $\Omega \times S$ is measurable.

Theorem 2.3 *Let the probability space be complete, and consider a measurable family of closed compact-random sets in a metric space. The intersection of this family of sets is a compact-random set if the index set is a Souslin space.*

Proof. Let $\{F_s\}_{s\in S}$ be the family of compact-random sets. The finite intersection property of a compact set K gives:

$$\{\omega \mid [\cap_{s\in S} F_s(\omega)] \cap K = \emptyset\} = \bigcup_{n=1}^{\infty} \bigcup_{\vec{s}\in S^n} \{\omega \mid [\cap_{i=1}^n F_{s_i}(\omega)] \cap K = \emptyset\}.$$

S^n is also a Souslin space. By a measurable family we mean compact-measurability of $(\omega, s) \mapsto F_s(\omega)$ so $(\omega, \vec{s}) \mapsto F_{s_i}(\omega)$ is compact-measurable with respect to the product sigma-algebra in $\Omega \times S^n$. Compact-randomness of the intersection $\cap_{s\in S} F_s$ is now a consequence of the Souslin projection theorem and the compact-measurability of $(\omega, \vec{s}) \mapsto [\cap_{i=1}^n F_{s_i}(\omega)]$. □

Corollary 2.1 *Let S be a Souslin space and Σ a compact-measurable closed-valued multifunction from $S^{\mathbf{N}}$ into a metric space. Let $A_1, A_2, \ldots$ be S valued random variables. The tail-part $\Sigma^t(A_1, A_2, \ldots)$ is a compact-random set if the probability space is complete.*

Proof. $\Sigma_n(a_1, \ldots) := \bigcap_{\vec{s} \in S^n} \Sigma(s_1, \ldots, s_n, a_{n+1}, \ldots)$ is compact-measurable with respect to the completion of the Borel-field in $S^{\mathbf{N}}$ with respect to the image measure from $A_1, A_2, \ldots$. The claim follows from $\Sigma^t := \bigcap_n \Sigma_n$. □

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Department of Mathematics and Statistics - AVH
University of Trondheim
N-7055 Dragvoll
NORWAY

gunnar@matstat.unit.no