A BRATTELI DIAGRAM FOR COMMUTING HOMEOMORPHISMS OF THE CANTOR SET

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Abstract. This paper presents a construction of a sequence of Kakutani-Rohlin Towers and a corresponding simple Bratteli Diagram, $\mathcal{B}$, for a general minimal aperiodic $\mathbb{Z}^d$ action on a Cantor Set, X, with $d \geq 2$. This is applied to the description of the orbit structure and to the calculation of the ordered group of coinvariants. For each minimal continuous $\mathbb{Z}^d$ action on X there is a minimal continuous $\mathbb{Z}$ action on X which has the same orbits on the set $X \setminus Y$, where Y is of the form $\cup_{v\in\mathbb{Z}^d} T^v Y^\sharp$, where $Y^\sharp$ is closed and nowhere dense, and Y is null with respect to every probability measure invariant under the action. Also the ordered top cohomology of $(X, \mathbb{Z}^d)$, H^d, is a simple dimension group. There is a positive order embedding surjection $K_0(\mathcal{B}) \longrightarrow H^d$ which becomes an order isomorphism upon quotienting by the infinitessimals. So there is no ordered cohomological obstruction to continuous orbit equivalence between minimal $\mathbb{Z}^d$ actions and $\mathbb{Z}$ actions in general.

Note This is a rewrite of an October 1997 script of the same title. The content is the same, but the presentation has been changed completely, and I hope for the better. This new presentation reflects ideas which have grown only in the intervening 15 months-or-so as a result of seminars and informal discussion. In this connection, I am particularly grateful to the members of Christian Skau's Operator Algebras Seminar at NTNU Lade, Trondheim.

§1 **Introduction** The construction of Kakutani-Rohlin Tower partitions for $\mathbb{Z}$ actions on topological or measurable spaces is a well-established and effective tool in the study and construction of dynamical systems (see [**Kak**] [**R**] or any modern elementary text in Ergodic Theory). In the category of continuous actions on the Cantor set (which this paper considers exclusively) the construction of Kakutani-Rohlin towers is the starting point for the description of a complete invariant for topological orbit-equivalence for minimal Cantor $\mathbb{Z}$ actions, a deep investigation completed by Giordano, Herman, Puntam and Skau [**HPS**] [**GPS**]. These same tower constructions make this invariant effectively computable in many interesting cases, [**Pu**] [**Fo**] [**DHS**]. These techniques also show that the phenomenon of topological orbit equivalence is at least as rich as the theory of countably generated torsion-free abelian ordered groups with simple dimension group order and unit.

The connection between Kakutani-Rohlin towers and orbit equivalence is particularly well developed in the measurable dynamical category. A complete classification of the orbit structure of ergodic amenable group actions has been found in various degrees of generality by Connes, Dye, Feldman, Katznelson, Kreiger and Weiss and others ([**D**] [**KW**][**CFW**][**K**]). Here the phenomenon of orbit equivalence (among ergodic systems) is equivalent to the theory of von Neuman factors of types II_1, II_∞, and $III_\lambda : 0 \leq \lambda \leq 1$ - a classification which, among topological dynamical systems with invariant measures, is cruder than that of topological orbit equivalence.

On the other hand, in the Borel measurable category, the action of a countable group on a polish space in general has an even cruder classification; all such ergodic systems are orbit-equivalent by pseudo-homeomorphism, a result established by Sullivan, Weiss and Wright ([**DJK**] [**SWW**]) without the use of tower-like constructions.

But all these investigations leave unresolved the orbit classification of general group actions in the topological dynamical category and it is this important problem that this paper starts to address by examining minimal $\mathbb{Z}^d$ actions on the Cantor set.

The best reason why the usual construction of towers for minimal $\mathbb{Z}$ actions cannot be generalized quickly to $\mathbb{Z}^d$ actions on the Cantor set, X, is the unsuitability of the "first return" argument [**Pu**]; there is no natural linear order on $\mathbb{Z}^d$, $d \geq 2$ for the purpose. Furthermore, attempts to generalize the arguments suitable for the Measure Preserving Category founder on the fact that, in the continuous topological category, we cannot neglect any points.

In this paper rather we cultivate a different picture of the Kakutani-Rohlin tower construction: it is, equivalently, a continuous family of partitions of $\mathbb{Z}^d$, indexed by elements of X, with certain further properties. It turns out that this picture suggests more easily a suitable construction of Kakutani-Rohlin partitions for $\mathbb{Z}^d$ actions. This new picture and its correspondence with the classical Vershik construction of iterated Kakutani-Rohlin towers and the associated Bratteli diagram is explained fully in sections 2 to 4 of this paper.

With this equivalence between partition-valued maps and the Vershik construction in mind, section 5 starts to build the geometric aparatus needed to construct the relevant partitions in $\mathbb{Z}^d$. In section 6 we introduce the dynamics to this construction.

Definition 1.1 Suppose that that $(X, \mathbb{Z}^d)$ is a Cantor system; that is X is a Cantor set with $\mathbb{Z}^d$ action by homeomorphisms. We write $T^v x$ for the action of $v \in \mathbb{Z}^d$ on $x \in X$.

It is *free* if, for any given $x \in X$, the equation $T^v x = x$ implies $v = 0$.

It is *minimal* if every orbit is dense.

We refer to [**HPS**] or to section 4 for the definitions used in connection with Bratteli diagrams. But note that all the Bratteli diagrams in this paper have a unique vertex at level 0, and that every other vertex in the diagram is the range of some edge.

In section 6, we prove the following result.

Theorem 1.2 *Suppose that $(X, \mathbb{Z}^d)$ is a minimal free Cantor system, then there is a simple unordered Bratteli diagram, $\mathcal{B}$, and a homeomorphism, ϕ, from X to $P(\mathcal{B})$, the path space of $\mathcal{B}$, such that*

i/ if p and q are cofinal in $P(\mathcal{B})$, then $\phi^{-1}(p)$ and $\phi^{-1}(q)$ are coorbital in X

ii/ the points in X for which the converse is false, Y, is a set of zero measure with respect to every invariant probability measure on X, and has the form $\cup_{v \in \mathbb{Z}^d} T^v Y^\sharp$, where $Y^\sharp$ is closed and nowhere dense.

Section 7 gives a second application of the construction: the calculation of ordered top cohomology of the dynamical system modulo its infinitessimals, a known invariant for continuous orbit equivalence.

Definition 1.3 Define

$$H^d(\mathbb{Z}^d; C(X; \mathbb{Z})) = C(X; \mathbb{Z}) / \langle f - Tf : f \in C(X; \mathbb{Z}), T \in \mathbb{Z}^d \rangle$$

where $C(X; \mathbb{Z})$ is the group of continuous integer valued functions on X. When the context is clear, this is sometimes abbreviated H^d. H^d is considered as an ordered group with unit (see Defn. 7.1): The order on H^d is induced from the usual non-strict order on $C(X; \mathbb{Z})$ ($[f] \geq 0$ iff there is a $g \in [f]$ such that $g(x) \geq 0$ for all $x \in X$) and the order unit is the class which contains the constant 1 function.

Given an ordered group with unit, G, we write $\tilde{G} = G/InfG$, where $InfG$ is the subgroup of infinitessimals (see Defn 7.1). We sometimes call this group, the coinfinitessimals of G. We note the alternative description

$$\tilde{H}^d = C(X; \mathbb{Z}) / \{ f \in C(X; \mathbb{Z}) : \mu(f) = 0 \quad \forall\ \mu \text{ a } \mathbb{Z}^d \text{ invariant probability measure on } X \ \}$$

The main result of Section 7 is:

Theorem 1.4 *Suppose that $(X, \mathbb{Z}^d)$ is a free minimal Cantor system. Then, with its canonical ordering and unit, $H^d(\mathbb{Z}^d, C(X; \mathbb{Z}))$ and $\tilde{H}^d(\mathbb{Z}^d, C(X; \mathbb{Z}))$ are each simple dimension groups.*

Furthermore, ϕ in Theorem 1.2 induces an affine homeomoephism between the $\mathbb{Z}^d$ invariant probability measures with the normalized states on $K_0(\mathcal{B})$, where $\mathcal{B}$ is the Bratteli diagram of Theorem 1.2. So $\tilde{H}^d(\mathbb{Z}^d, C(X; \mathbb{Z})) \equiv \tilde{K}_0(\mathcal{B})$, where $\tilde{K}_0(\mathcal{B})$ is its dimension group mod its infintessimals (see Defn 7.1).

This is a partial generalization to $\mathbb{Z}^d$ actions of a result for $\mathbb{Z}$ actions found in [**HPS**].

From these two results we deduce a result about the orbit structure of such $\mathbb{Z}^d$ actions, comparing it with the orbit structure of $\mathbb{Z}$ actions.

Theorem 1.5 *Suppose that $(X, \mathbb{Z}^d)$ is a free minimal Cantor system, then there is a minimal transformation, S, on X and a set Y, which is $\mathbb{Z}^d$-invariant, and S-invariant, nowhere dense and of zero measure with respect to every $\mathbb{Z}^d$ and S-invariant measure, so that on $X \setminus Y$, the $\mathbb{Z}^d$ orbits and S-orbits coincide.*

Moreover a probability measure on X is $\mathbb{Z}^d$-invariant if and only if it is S-invariant. Also $\tilde{H}^d(\mathbb{Z}^d, C(X;\mathbb{Z})) \equiv \tilde{H}^1(\langle S\rangle, C(X;\mathbb{Z}))$.

Proof of Theorem 1.5 The first part follows quickly from Theorem 1.2 and the work of [**HPS**]. Construct on $\mathcal{B}$ a Vershik order which has a single maximal path and a single minimal path; we may ensure that the max path and min path are both members of the set $\phi^{-1}(Y)$. By [**HPS**] we know that the Vershik map, which we write S, can be extended to the whole of the path space of $\mathcal{B}$ and it has orbits equal to the cofinal classes with the exception of the single orbit which contains both the maximal path and the minimal path. In particular the orbits of S equal the $\mathbb{Z}^d$ orbits on the set $\phi^{-1}(X \setminus Y)$ as required.

The second part follows from Theorem 1.4 since, [**HPS**], the states on $K_0(\mathcal{B})$ are precisely the S-invariant probability measures on X, and $\tilde{K}_0(\mathcal{B}) = \tilde{H}^1(\langle S\rangle, C(X;\mathbb{Z}))$. •

Theorem 1.5 should be compared with the main result (Theorem 1.8) of [**SWW**] which produces a pseudo-homeomorphism of X as an orbit equivalence between the $\mathbb{Z}^d$ action and the action of ρ. Upon removal of a meagre set, this pseudo-homeomorphism becomes continuous, but there is no guarantee that it could be extended continuously to the whole of X. In this paper, rather, we obtain a homeomorphism of X which is an orbit equivalence except on a meagre, zero-measure set. There seems to be no necessary implication between the two results. Indeed the construction of this paper, which retrieves the coinfinitessimal group of the dynamics, distinguishes between $\mathbb{Z}^d$ actions which are confused by the psuedo-homeomorphic orbit-equivalences of [**SWW**].

We observe without proof that if two Cantor minimal systems, $(X, \mathbb{Z}^d)$ and $(Y, \mathbb{Z}^k)$, are orbit equivalent, then $\tilde{H}^d(\mathbb{Z}^d, C(X;\mathbb{Z})) = \tilde{H}^k(\mathbb{Z}^k, C(Y;\mathbb{Z}))$ as ordered groups with unit. Since we know, [**HPS**], that all simple infinitessimal-free dimension groups, not equal to $\mathbb{Z}$, are realized as $\tilde{H}^1$ of some minimal $\mathbb{Z}$ action on a Cantor set, this paper shows that there is no ordered cohomological obstruction to the orbit equivalence of each minimal $\mathbb{Z}^d$ action with some $\mathbb{Z}$ action.

Although we define all we need for dynamics, ordered groups and geometry, for reasons of space this paper assumes a working knowledge of (unordered) Bratteli diagrams and their combinatorics. See [**HPS**] and [**Sk**] for further details in this context.

Finally, we note the recent PhD thesis of Priebe **[Pr]** to which the author's attention was drawn only after the completion of the work of this paper. There, Voronoi domains such as $K(w, R)$ of Definition 5.1 below are used to describe the structure of two dimensional substitution tiling systems and to prove interesting results about their combinatorial structure.

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§**2 The picture for $\mathbb{Z}$ actions** To introduce the fundamental construction and to draw the analogy between $\mathbb{Z}$ actions and $\mathbb{Z}^d$ actions effectively, we first present the description of Kakutani-Rohlin partitions for $\mathbb{Z}$ actions in a way that is most suitable for generalization. The reader can then take the opportunity to compare the familiar "skyscraper" picture with our construction: a continuous family of partitions of $\mathbb{Z}$ indexed by X.

Consider a homeomorphism, T, from the Cantor Set, X, onto itself. A *tower* is a disjoint union of clopen sets of the form $T^n U$ where n runs from 0 to some $N \geq 0$. The set U is termed the *base* of the tower and the sets $T^n U$ are the *levels* of the tower. A *traverse* of such a tower is a set of the form $\{T^n x : 0 \leq n \leq N\}$ where $x \in U$.

An elementary argument shows that every minimal Cantor system can be covered by a finite disjoint union of towers, the union of whose bases (called the *base of the partition*) can be pre-defined arbitrarily (see **[Pu]**). The subsequent partition of X into levels is called a *Kakutani-Rohlin (KR) Partition* for the system (X, T). The union of the tower bases is called the *base of the partition.* It is clear that if we are given a KR partition and its base, then we can reconstruct the towers and the traverses.

Given a KR partition of X and $x \in X$, we may partition $\mathbb{Z}$ into intervals $\{a_j, ..., a_{j+1} - 1\} : j \in \mathbb{Z}$, where a_j is a biinfinite strictly monotone sequence of integers, such that, for each $j \in \mathbb{Z}$, $\{T^k x : a_j \leq k < a_{j+1}\}$ is a traverse of a tower in the partition. Given the KR partition, such a partition of $\mathbb{Z}$ is uniquely defined by x and is written $\mathcal{T}(x)$.

Note the following three properties.

P1 The partition $\mathcal{T}(x)$ is the same as the partition $\mathcal{T}(Tx)$ shifted positively by one unit: $\mathcal{T}(x) = \mathcal{T}(Tx) + 1$. This is the *shift invariance* of the construction.

P2 The atoms of the partitions $\mathcal{T}(x) : x \in X$ are each an interval of uniformly bounded length, bounded uniformly in x also.

P3 The map $x \mapsto \mathcal{T}(x)$ is uniformly continuous in the sense that for all integers $R > 0$, there is an $\epsilon > 0$ such that if $d(x, y) < \epsilon$, then the partitions $\mathcal{T}(x)$ and $\mathcal{T}(y)$ are the same when restricted to the finite subset $\{-R, -R+1, ..., R-1, R\}$. This is the *continuity* of the construction.

Note also the reversibility of this procedure. A map from X to the set of partitions of $\mathbb{Z}$ with the three properties above can be constructed as above from a KR partition of X.

An important development due to Vershik [**V**] is the idea of refinement of a KR Partition to a subsequently defined finer KR Partition. We describe this now.

Definition 2.1 Consider a sequence of clopen partitions of X, $\mathcal{Q}_n$, *refining* in the sense that each is refined by its successor. If, for any $x \neq y \in X$ there is an n such that x and y are in different atoms of $\mathcal{Q}_n$, then we say that $\mathcal{Q}_n$ *refine to points.*

Theorem 2.2 [**Pu**] [**V**] *Given a minimal Cantor system, there is a sequence of refining KR Partitions, $\mathcal{Q}_n$, which refine to points and whose bases form a nested sequence of clopen sets which tend to a single point, x_0. A traverse of a tower of the nth partition is a union of traverses from the $n-1$st partition.* •

From this the work of Herman, Putnam and Skau, [**HPS**], building on the that of Vershik [**V**] and of [**Pu**], produces an ordered Bratteli diagram, $\mathcal{B}$, by methods that are now standard in the subject.

Theorem 2.2 can also be expressed in terms of partition-valued functions. Let $x \mapsto \mathcal{T}_n(x)$ be the map from X to partitions of $\mathbb{Z}$, constructed for each of the KR partitions $\mathcal{Q}_n$. Let $A_n(x)$ be the atom of $\mathcal{T}_n(x)$ which contains 0. Then the following properties are implied by the conclusions of Theorem 1.2:

P4 For $x \in X$, $\mathcal{T}_n(x)$ refines $\mathcal{T}_{n+1}(x)$.

P5 There is a point $x_0 \in X$ such that the following two statements are equivalent for all $x \in X$:
a/ $\cup_n A_n(x) \neq \mathbb{Z}$
b/ $x = T^k x_0$ for some $k \in \mathbb{Z}$.

With some care about the refinement of the partitions to points, the properties P1 to P5 are not only necessary but sufficient to define a refining sequence of KR partitions (see section 4 ahead) which obey the conclusions of Theorem 2.2.

Thus, in the case of $\mathbb{Z}$ actions, we pass from partition-valued functions on X (with properties P1 to P3) to KR decompositions, and from sequences of partition-valued functions on X (with additional properties P4, P5) to Bratteli diagrams, all quite naturally and freely. Now we make this sketch more general and definite.

§**3 From Vershik models to Bratteli diagrams** This section describes a process, well-known in the case of $\mathbb{Z}$ actions and easily generalized, whose formalization we find convenient for the further development to other group actions.

Definition 3.1 Suppose that A is a finite subset of $\mathbb{Z}^d$. We write e_j for the standard unit basis of $\mathbb{R}^d$, $1 \le j \le d$, and we say that $a \in A$ is *on the boundary of* A, if at least one of the $2d$ points, $a \pm e_j : 1 \le j \le d$, is not in A. The collection of points on the boundary of A is written ∂A.

Suppose that $(X, \mathbb{Z}^d)$ is a Cantor system (1.1). A *tower* (in $(X, \mathbb{Z}^d)$) is a collection of disjoint clopen sets of the form $T^v U$ where $v \in A$ and A is a finite subset of $\mathbb{Z}^d$ containing 0. We can sometimes equate a tower with the union of its elements without confusion. We make no assumptions yet about A.

The *levels* of the tower are the sets $T^v U : v \in A$ and the *boundary* of the tower is the union $\cup\{T^v U : v \in \partial A\}$. Note that the boundary of a tower is independent of the choice of U and A to represent it.

A *traverse* of the tower $\{T^v U : v \in A\}$ is a set of the form $\{T^v x : v \in A\}$ for $x \in U$.

A *KR decomposition*, $\mathcal{Q}$, for $(X, \mathbb{Z}^d)$ is a collection of towers covering X and whose levels are all mutually disjoint. Thus we may think of a KR decomposition as a partition of X into clopen sets (the levels); and this set of levels is itself partitioned into towers. We write $|\mathcal{Q}|$ for the underlying partition of X into levels.

The *boundary of the decomposition* is the union of the boundaries of its towers, a subset of X.

Now we develop the important idea of refinement.

Definition 3.2 We say that a KR decomposition, $\mathcal{Q}$, *refines* another KR decomposition $\mathcal{Q}'$, if $|\mathcal{Q}|$ refines $|\mathcal{Q}'|$ as a partition in the usual sense and if each traverse of a tower in $\mathcal{Q}$ is a union of traverses of towers in $\mathcal{Q}'$.

A *Vershik model* for $(X, \mathbb{Z}^d)$ is a sequence of refining KR decompositions, $(\mathcal{Q}_n)$. If the $|\mathcal{Q}_n|$ refine to points then we say that it is *faithful*.

In a Vershik model, $(\mathcal{Q}_n)$, write Y_n for the boundary of the KR decomposition, $\mathcal{Q}_n$. Owing to the fact that $\mathcal{Q}_n$ refines $\mathcal{Q}_{n-1}$, the Y_n form a decreasing sequence of clopen sets whose (necessarily non-empty closed) intersection is written $Y^\sharp$. We call $Y^\sharp$ the *boundary of the model* $(\mathcal{Q}_n)$.

Write $Y = \cup_{v \in \mathbb{Z}^d} T^v Y^\sharp$.

And finally some notation for Bratteli diagrams.

Definition 3.3 Suppose that $\mathcal{B} = (V_n, E_n)$ is a Bratteli diagram with a single vertex at level 0. Let $P(\mathcal{B})$ be the space of infinite paths in $\mathcal{B}$ starting from V_0, a space which is a Cantor set with the following basic clopen sets, $U(p)$:

Suppose that p is a finite path from V_0 to $v \in V_n$. Write $U(p)$ for the set of paths in $P(\mathcal{B})$ whose initial segment is p.

Thus we are equipped for the main construction of the section.

Theorem 3.4 *Suppose that $(X, \mathbb{Z}^d)$ is a Cantor system with a faithful Vershik model, $(\mathcal{Q}_n)$. Then there is a canonical Bratteli diagram, $\mathcal{B}$, and a canonical homeomorphism $\phi : P(\mathcal{B}) \longrightarrow X$ for which*

a/ If $p, q \in P(\mathcal{B})$ are cofinal, then $\phi(p)$ and $\phi(q)$ are coorbital, and

b/ $x \in X \setminus Y$ implies $\phi^{-1}(x)$ is cofinal with $\phi^{-1}(T^v x)$ for all $v \in \mathbb{Z}^d$.

c/ If $(X, \mathbb{Z}^d)$ is free, then the converse to b/ is true: If $\phi^{-1}(x)$ is cofinal with $\phi^{-1}(T^v x)$ for all $v \in \mathbb{Z}^d$, then $x \in X \setminus Y$.

Moreover, for each $n \geq 1$, the partition $|\mathcal{Q}_n|$ corresponds, by ϕ, precisely to the partition of $\mathcal{P}(\mathcal{B})$ into the sets $U(p)$ as p runs over finite paths to level n.

Proof The construction proceeds exactly as for the case of a $\mathbb{Z}$ action [**HPS**] but the details are important for the further properties a/ and b/, so we give them here.

Suppose that $\mathcal{Q}_1, \mathcal{Q}_2, ...$ is a Vershik model for $(X, \mathbb{Z}^d)$. Define $\mathcal{Q}_0$ as the trivial decomposition (one tower of one level, X) which is clearly the crudest possible.

For each $n \geq 0$, let V_n be the collection of towers in $\mathcal{Q}_n$; these are considered abstractly as the vertices of our Bratteli diagram at level n.

Suppose that $\kappa \in V_n$ and that $L = \{T^v x : v \in A\}$ is a traverse of κ. Since $\mathcal{Q}_n$ refines $\mathcal{Q}_{n-1}$, we know that L is a union of traverses of towers of $\mathcal{Q}_{n-1}$: $L = \cup_j L_j$ where each $L_j = \{T^v x_j : v \in A_j\}$ is a traverse of tower κ'_j in $\mathcal{Q}_{n-1}$ and j is some (finite) indexing. The L_j are necessarily disjoint and the κ'_j are uniquely determined by L_j. Suppose that $\kappa' \in V_{n-1}$. Then we connect vertex κ to vertex κ' multiply by as many edges as there are indices j such that $\kappa'_j = \kappa'$. If e is such an edge then, in common with [**HPS**], we say that κ is the *range* of e and κ', the *source*.

We do this for all $n \geq 1$ and for each $\kappa \in V_n$ and so define a Bratteli diagram, $\mathcal{B}$.

Note that, given $\kappa = \{T^v U : v \in A\}$, this construction is independent of the choice of traverse. For if $L = \{T^v x : v \in A\}$ and $L' = \{T^v x' : v \in A\}$ are two traverses of the same tower, then $T^v x$ and $T^v x'$ are in the same $|\mathcal{Q}|$ atom whenever $v \in A$, hence in the same $|\mathcal{Q}_{n-1}|$ atom (by refinement) and hence in the same $\mathcal{Q}_{n-1}$ tower.

In fact each edge comes with further useful information. With the notation above, an edge, e, connecting $\kappa \in V_n$ with $\kappa' \in V_{n-1}$ say, corresponds to a traverse of κ', L_j, which is a subset of our chosen traverse of κ, L. Thus to e we can associate a collection, $\Lambda(e)$, of $|\mathcal{Q}_n|$ atoms: $\Lambda(e) = \{W \in |\mathcal{Q}_n| : L_j \cap W \neq \emptyset\}$. By the argument of the last paragraph, $\Lambda(e)$ is independent of the choice of L, and so is a well-defined labelling of edges.

We note three technical points which come directly from the details of this construction.

- **A** The tower κ can be expressed as a disjoint union: $\cup\Lambda(e) = \kappa$, where the union is taken over all edges e whose range is κ; this follows from the KR refinement property. Thus two distinct edges between V_{n-1} and V_n cannot share the same label.
- **B** if W, W' are distinct elements of $\Lambda(e_n)$, then they are contained in distinct atoms of $|\mathcal{Q}_{n-1}|$; this is because each element of a traverse of a tower is contained in a distinct atom of the tower.
- **C** if W' is a level of the tower $\kappa' \in V_{n-1}$ then, for each edge, e, with source κ', there is an atom, $W \in \Lambda(e)$ such that $W \cap W' \neq \emptyset$; this is because a traverse of a tower intersects each of the tower's levels.

Now we complete the construction of ϕ.

Given $x \in X$ and $n \geq 1$, there is a unique atom, $W_n \in |\mathcal{Q}_n|$ such that $x \in W_n$, and so (by **A** above) there is a unique edge e_n such that $W_n \in \Lambda(e_n)$. This edge connects κ'_n with κ_n if and only if x is in some atom of tower κ'_n and is in some atom of κ'_n. Therefore, we see that $\kappa'_n = \kappa_{n-1}$ for all $n \geq 1$ (where κ_0 is the unique vertex at level 0). In other words, this collection of edges, $e_1, e_2, \ldots$ forms an infinite path in $\mathcal{B}$, and we have defined a map $X \to P(\mathcal{B})$. Since all the atoms are clopen sets, the map (which is in fact ϕ^{-1}) is continuous by construction.

To reverse this map, take an infinite path $e_1, e_2, \ldots$, together with the labels $\Lambda(e_n)$. Let $\tilde{\Lambda}(e_n) = \cup\{W : W \in \Lambda(e_n)\}$ a finite union of clopen atoms of $|\mathcal{Q}_n|$. Consider the following claim which we shall prove by induction on N:

$$\text{Claim:} \qquad \bigcap_{1\leq n\leq N} \tilde{\Lambda}(e_n) \text{ is a single atom of } |\mathcal{Q}_N|$$

From this, compactness and the refinement to points assumed in the conditions of the Theorem, we find a unique $x : \{x\} = \cap\tilde{\Lambda}(e_n)$. It is then straightforward that this map from (e_n) to x, which we will write ϕ, inverts the map above.

We prove our claim. First, suppose that $e = e_1$ is an edge which connects the single element $\kappa' \in V_0$ with some $\kappa \in V_1$. Since $\mathcal{Q}_0$ is the trivial decomposition, each traverse of the tower in $\mathcal{Q}_0$ is a single point and so, by construction, $\Lambda(e_1)$ is a set containing a single atom of $\mathcal{Q}_1$, and induction starts.

To continue the induction, suppose that $\cap_{1\leq n\leq N-1}\tilde{\Lambda}(e_n)$ is a single atom of $|Q_{N-1}|$ and that e_N connects $\kappa' \in V_{N-1}$ with $\kappa \in V_N$. Suppose that W, W' are distinct elements of

$\Lambda(e_N)$ which, by **B** above, are thus in distinct atoms of $\mathcal{Q}_{N-1}$. In particular, at most one element of $\Lambda(e_N)$ intersects $\cap_{1\le n\le N-1}\tilde{\Lambda}(e_n)$. However, $\cap_{1\le n\le N-1}\tilde{\Lambda}(e_n)$ is a level in the tower κ', the source of e_N, since the elements of $\Lambda(e_{N-1})$ are each a level of κ'. Thus by **C** above, at least one element of $\Lambda(e_N)$ intersects $\cap_{1\le n\le N-1}\tilde{\Lambda}(e_n)$. So induction continues.

Thus we have a homeomorphism, $\phi : P(\mathcal{B}) \longrightarrow X$. Also, we note that the partition of $P(\mathcal{B})$ which distinguishes paths according to their edges up to level n (i.e. $e_1, ..., e_n$) maps by ϕ precisely to the partition $|\mathcal{Q}_n|$, thus we have the last part of the Theorem.

To proceed to parts a/ and b/, here is a convenient construction which redescribes Λ. Suppose that $x \in X$ and that $\{T^v x : v \in A_n\}$ is a traverse of the $\mathcal{Q}_n$ tower, κ_n, in which x sits. This defines A_n uniquely, and we have $A_1 \subset A_2 \subset ....$ Write $\phi^{-1}(x) = (e_1(x), e_2(x), ...)$. Then $\Lambda(e_n(x))$ is precisely the collection of $|\mathcal{Q}_n|$ atoms which contain some element of $\{T^v x : v \in A_{n-1}\}$.

As a consequence, if $v \in A_k$, then $\Lambda(e_n(x)) = \Lambda(e_n(T^v x))$ for all $n > k$, whence we deduce, by **A** above, that $e_n(x) = e_n(T^v x)$ under the same conditions.

Now, suppose that two paths, $p = (e_n(x))$ and $q = (e_n(y))$ are cofinal, so that there is an N for which $e_n(x) = e_n(y)$ for all $n > N$. Since the source of $e_N(x)$ and of $e_N(y)$ agree, x and y are in the same tower $\kappa \in V_N$. Let U be the atom of $|\mathcal{Q}_N|$ which contains x so we can write $\kappa = \{T^v U : v \in A_N\}$.

We know that $y \in T^w U$ for some $w \in A_N$, so let $x' = T^w x$. Therefore, as above, $e_n(x') = e_n(x)$ for all $n > N$ and so $e_n(x') = e_n(y)$ for $n > N$. On the other hand, x' and y are in the same $|\mathcal{Q}_N|$ atom (i.e. $T^w U$) and hence $e_n(x') = e_n(y)$ for all $n \le N$ as well. Therefore we have $x' = y$ and conclude part a/ of the theorem.

For parts b/ and c/, we observe that the argument for part a/ above shows that if $x \in X$, $v \in \cup_k A_k$ (with the A_k constructed for x as in part a/) and $y = T^v x$, then $\phi^{-1}(x)$ and $\phi^{-1}(y)$ are cofinal. Conversely, if $y = T^v x$ and $\phi^{-1}(x)$ and $\phi^{-1}(y)$ are cofinal, then $v \in \cup_k A_k$ provided that the action is free.

So consider $\cup_k A_k$: $\cup_k A_k \ne \mathbb{Z}^d$ if and only if there is a v such that $v \in \cap_k \partial A_k$. This, in turn is equivalent to $T^v x \in Y_k$ for all k, which is equivalent to $x \in Y$.

Putting these two arguments together shows that if $x \notin Y$, then $\phi^{-1}(T^v x)$ is cofinal with $\phi^{-1}(x)$ for all $v \in \mathbb{Z}^d$, i.e. part b/, and in the free case, that if $x \in Y$, then there is a $w \notin \cup_k A_k$ and $\phi^{-1}(T^w x)$ is not cofinal with $\phi^{-1}(x)$, i.e. part c/. •

Remark 3.5 Thus by making $Y^\sharp$ smaller, we make better the connection between the orbits of the dynamical system and the cofinal classes of the Bratteli diagram.

In the case of a minimal $\mathbb{Z}$ action on the Cantor set, the set $Y^\sharp$ can be made equal to two points in the same orbit: $\{T^{-1}x_0, x_0\}$. The construction in this paper for $d > 1$ has $Y^\sharp$ uncountable in general.

We conclude this section be showing that faithfulness of a Vershik model can always be ensured artificially by a well-known process of symbol splitting [**HPS**] which we formalize in the following Lemma.

Lemma 3.6 *Suppose that $\mathcal{Q}$ is a KR decomposition and that $\mathcal{P}$ is a partition of X into clopen sets. Then there is a unique crudest KR decomposition, written $\mathcal{Q}\vee\mathcal{P}$, which refines $\mathcal{Q}$ and whose underlying partition of X, $|\mathcal{Q}\vee\mathcal{P}|$, refines $\mathcal{P}$ (and automatically $|\mathcal{Q}|$).*

We have the further properties:

a/ Suppose that $\mathcal{Q}$ is a KR decomposition and that $\mathcal{P}$ is a partition of X into clopen sets. Then the boundary of $\mathcal{Q}\vee\mathcal{P}$ equals the boundary of $\mathcal{Q}$.

b/ If in addition, $\mathcal{Q}$ refines $\mathcal{Q}'$ as a KR decomposition, and $\mathcal{P}$ refines $\mathcal{P}'$ as a clopen partition, then $\mathcal{Q}\vee\mathcal{P}$ refines $\mathcal{Q}'\vee\mathcal{P}'$ as a KR decomposition.

Proof We give the construction from which the properties claimed are easily checked.

Suppose that the towers of $\mathcal{Q}$ are $\kappa_1, \kappa_2, ..., \kappa_k$ which are represented as $\kappa_j = \{T^v U_j : v \in A_j\}$. For each j consider the partition $\mathcal{P}_j = \vee_{v\in A_j} T^{-v}\mathcal{P}$ restricted to U_j, i.e. the partition of U_j whose atoms (if non-empty) are of the form $U_j \cap \cap_{v\in A_j} T^{-v} W_v$ where W_v is some selection of atoms of $\mathcal{P}$. Extend this partition to each of the levels $T^v U_j$ by translation. Combine these partitions in the disjoint towers into a partition, $\mathcal{P}'$, of X.

The atoms of $\mathcal{P}'$ are organized canonically into towers: Suppose that $U' \subset U_j$ is an atom of $\mathcal{P}'$, then in fact $U' \in \mathcal{P}_j$ by construction. Therefore, $\kappa' = \{T^v U' : v \in A_j\}$ is a tower each of whose levels is an atom of $\mathcal{P}'$, and each a subset of a distinct level of κ_j. The collection of towers which can be formed this way make up a KR decomposition, $\mathcal{Q}\vee\mathcal{P}$, whose underlying partition, $|\mathcal{Q}\vee\mathcal{P}| = \mathcal{P}'$, refines both $\mathcal{P}$ and $|\mathcal{Q}|$. •

Proposition 3.7 *Suppose that $\mathcal{Q}_n$ is a Vershik model for a Cantor system $(X, \mathbb{Z}^d)$, then there is a faithful Vershik model, $\mathcal{Q}'_n$, such that $\mathcal{Q}'_n$ refines $\mathcal{Q}_n$ for each n. These two models have the same boundary.*

Consequently the Bratteli diagram of Theorem 3.4 can be formed in the presence of a general Vershik model, and the same conclusions can be drawn.

Proof Let $\mathcal{P}_n$ be a refining sequence of clopen partitions of X which refine to points. Let $\mathcal{Q}'_n = \mathcal{Q}_n \vee \mathcal{P}_n$. Lemma 3.6 completes the first part and the second part is simply an observation. •

§**4 From partition-valued maps to Vershik models** Suppose that $(X, \mathbb{Z}^d)$ is a Cantor system and let $\mathcal{T}$ be a function from X to the set of partitions of $\mathbb{Z}^d$. We redraft for this case the properties discussed in section 2.

P1 If $x \in X$ and $v \in \mathbb{Z}^d$, then $\mathcal{T}(T^v x)$ is the partition $\mathcal{T}(x)$ shifted by $-v$.

P2 The atoms of the partitions $\mathcal{T}(x) : x \in X$ are each a finite set of uniformly bounded diameter, bounded uniformly in x also.

P3 The map $x \mapsto \mathcal{T}(x)$ is uniformly continuous in the sense that for all $r > 0$, there is an $\epsilon > 0$ such that if $d(x, y) < \epsilon$, then the partitions $\mathcal{T}(x)$ and $\mathcal{T}(y)$ are the same when restricted to the ball of radius r: $\mathbb{Z}^d \cap B(0, r)$.

We write $A(x)$ for the atom of $\mathcal{T}(x)$ which contains 0.

Lemma 4.1 *Suppose that $(X, \mathbb{Z}^d)$ is a Cantor system. A map $\mathcal{T} : X \longrightarrow$ partitions of $\mathbb{Z}^d$, with properties P1, P2 and P3, defines canonically a KR decomposition. The boundary of this decomposition can be described as the set $\{x \in X : 0 \in \partial A(x)\ \}$.*

Proof The partition which underlies the decomposition has atoms $U(A) = \{x \in X : A(x) = A\}$, which are clopen (by P2 and P3), disjoint, covering, and non-empty for finitely many choices of A.

We put two atoms $U(A)$ and $U(A')$ in the same tower if $A = A' - v$ for some $v \in \mathbb{Z}^d$. Property P1 shows then that $T^v U(A) = U(A')$, and so the tower can be described as $\{T^v U(A) : v \in A\}$.

The boundary of such a tower is $\cup_{v \in \partial A} T^v U(A) = \cup_{0 \in \partial A'} U(A')$ which fits the description in the Lemma. •

Likewise we introduce the idea of refinement. Suppose that $\mathcal{T}_n : n \geq 1$ is a sequence of partition-valued maps from X.

P4 For each $x \in X$ and $n \geq 1$, $\mathcal{T}_n(x)$ refines $\mathcal{T}_{n+1}(x)$.

Write $A_n(x)$ for the atom of $\mathcal{T}_n(x)$ which contains 0.

Proposition 4.2 *Suppose that $(X, \mathbb{Z}^d)$ is a Cantor system. Suppose that $\mathcal{T}_n : n \geq 1$ is a sequence of functions $X \longrightarrow$ partitions of $\mathbb{Z}^d$ which obeys P4. Suppose also that each of the $\mathcal{T}_n$ obey P1, P2 and P3, so that we may form canonical KR decompositions, $\mathcal{Q}_n$. Then $(\mathcal{Q}_n)$ is a Vershik model for $(X, \mathbb{Z}^d)$.*

In this case $Y = \{x \in X : \cup_n A_n(x) \neq \mathbb{Z}^d\}$ (recall that $Y = \cup_{v \in \mathbb{Z}^d} T^v Y^\sharp$ where $Y^\sharp$ is the boundary of the Vershik model).

Proof The first part is immediate by definition and construction.

For the second part we use Lemma 4.1 to write

$$Y^\sharp = \{x \in X : 0 \in \partial A_n(x) \ \ \forall\ n \geq 1\ \}$$

Thus $x \in Y^\sharp$ if and only if there is $v \in \mathbb{Z}^d$ (within one unit of 0) not in $\cup_n A_n(x)$. By P1, the description of Y follows. •

So we produce Vershik models from partition-valued functions, and hence Bratteli diagrams which recover orbital information outside of the set Y. We now examine sufficient conditions on the partitions to restrict the size of Y. First a topological condition.

Definition 4.3 Write $||.||$ for the Euclidean norm on $\mathbb{R}^d$ and consider $\mathbb{Z}^d$ placed canonically in $\mathbb{R}^d$.

For $w \in \mathbb{Z}^d$ and $r > 0$, write $B(w, r) = \{v \in \mathbb{R}^d : ||v - w|| \leq r\}$.

Suppose that A is a finite subset of $\mathbb{Z}^d$. The *inradius* of A is the supremum of $r > 0$ for which we can find $v \in \mathbb{Z}^d$ so that $B(v, r) \cap \mathbb{Z}^d \subset A$.

Given a partition, $\mathcal{T}$, of $\mathbb{Z}^d$ into finite sets, we define the *upper inradius* of $\mathcal{T}$ as the supremum of the inradii of the atoms. We write this $\overline{r}(\mathcal{T})$.

The *lower inradius*, $\underline{r}(\mathcal{T})$, is the infemum (minimum) of the inradii of the atoms of $\mathcal{T}$.

Lemma 4.4 *Suppose that $(X, \mathbb{Z}^d)$ is a Cantor system, and that $\mathcal{T}_n$ is a sequence of partition-valued functions obeying properties P1 to P4 above. Suppose that $\overline{r}(\mathcal{T}_n) \to \infty$. Then $Y^\sharp$, the boundary of the corresponding Vershik model, is nowhere dense.*

Proof The conditions on the inradius shows that there is a sequence of points, $x_n \in X$ and an unbounded sequence of real numbers, r_n, such that $B(0, r_n) \cap \mathbb{Z}^d \subset A_n(x_n)$. A simple subsequence argument and compactness show that we can find $x \in X$ and a subsequence $r_{n_k} \to \infty$ such that $B(0, r_{n_k}) \cap \mathbb{Z}^d \subset A_{n_k}(x)$ for all k.

Suppose that U_k is the atom of $|\mathcal{Q}_{n_k}|$ which contains x and define

$$W_k = \bigcup \{T^v U_k : v \in B(0, r_{n_k} - d) \cap \mathbb{Z}^d\}$$

which is an open subset of X disjoint from Y_{n_k}, the boundary of $\mathcal{Q}_{n_k}$. Then $\cup_k W_k$ is an open subset of X which contains the orbit of x, is hence also dense in X. But $Y^\sharp$ is disjoint from this set and is hence nowhere dense as required. Recall that $Y^\sharp$ is necessarily closed. •

Now a measure-theoretical version of the same Lemma.

Definition 4.5 Write $|A|$ for the cardinality of a finite set A.

A finite subset $A \subset \mathbb{Z}^d$ is said to be ϵ-Følner if $|A \Delta (A + v)| \leq \epsilon |A| \, ||v||$ for each $v \in \mathbb{Z}^d$.

Given a partition, $\mathcal{T}$, of $\mathbb{Z}^d$ into finite sets, write $\lambda(\mathcal{T})$ for the supremum of the best Følner constants for A, as A runs over atoms of $\mathcal{T}$. I.e.

$$\lambda(\mathcal{T}) = \sup\{ \ \inf\{\epsilon > 0 : A \text{ is } \epsilon\text{-Følner } \} : \ A \in \mathcal{T} \ \}$$

Lemma 4.6 *Suppose that $(X, \mathbb{Z}^d)$ is a Cantor system, and that $\mathcal{T}_n$ is a sequence of partition-valued functions obeying properties P1 to P4 above. Suppose $\lambda(\mathcal{T}_n) \to 0$. Then $Y^\sharp$, the boundary of the corresponding Vershik model, is of zero measure with respect to every $\mathbb{Z}^d$ invariant probability measure on X.*

Proof We give the argument for $d = 2$, but it proceeds for other d with perhaps a change of numerical constants.

Suppose that μ is an invariant probability measure on X which, without loss of generality, we may assume to be ergodic. By the ergodic theorem we can find $y \in X$ such that for any clopen set, W, the averages,

$$\frac{1}{k^2}|\{v \in F_k : T^v y \in W\}| \to \mu(W)$$

converge, where $F_k = \{0, 1, 2, ..., k-1\}^2 \subset \mathbb{Z}^2$ (see [**F**]).

For given n, consider the set $\Delta_k = \{v \in F_k : T^v y \in Y_n\}$ and consider k very large compared with the maximal diameter, D_n, of the atoms of $\mathcal{T}_n(y)$ (P2). Consider the collection of atoms of $\mathcal{T}_n(y)$ contained entirely in F_k. The cardinality of the union of these is at least $(k-2D_n)^2$ since this union contains the set $\{D_n, D_n+1, ..., k-D_n-2, k-D_n-1\}^2$. The cardinality of the union of their boundaries is at most $5\lambda(\mathcal{T}_n)(k-2D_n)^2$ therefore, accounting for all the elements of Δ_k which are contained in $\{D_n, D_n+1, ..., k-D_n-2, k-D_n-1\}^2$. Adding in the unaccounted $4kD_n - 4D_n^2$ extra elements gives the bound

$$|\Delta_k| \leq 5\lambda(\mathcal{T}_n)(k-2D_n)^2 + 4kD_n - 4D_n^2$$

Whence, letting k tend to infinity, we deduce that $\mu(Y_n) \leq 5\lambda(\mathcal{T}_n)$, implying that $\mu(Y^\sharp) = 0$ as required. •

And finally a condition to ensure simplicity (see [**HPS**]) of the Bratteli diagram.

Proposition 4.7 *Suppose that $(X, \mathbb{Z}^d)$ is a minimal Cantor system and that $\mathcal{T}_n$ is a sequence of partition-valued functions obeying properties P1 to P4 above. Suppose that $\underline{r}(\mathcal{T}_n) \to \infty$. Then the Bratteli diagram, $\mathcal{B}$, produced from the Vershik model defined in 4.2 is simple.*

Proof Suppose that $\kappa' \in V_n$ is a tower. We seek an $m > n$ such that each tower $\kappa \in V_m$ has some level intersecting some level of κ'. By the construction of $\mathcal{B}$, this implies that there is a path from κ' to κ. Since n and κ' are arbitrary, this implies simplicity in $\mathcal{B}$.

Suppose that $\delta > 0$ is chosen so that every level of κ' contains a ball of radius at least δ. By minimality there is a number, r, such that for any choice of $x \in X$ and any ball, U, of radius δ in X, there is some $v \in \mathbb{Z}^d$, $||v|| \leq r$, such that $T^v x \in U$.

Find an m such that $\underline{r}(\mathcal{T}_m) \geq r+1$, and choose any $\kappa \in V_m$. Therefore, by construction, there is a level U in κ such that $T^v U$ is also a level of κ for each $||v|| \leq r$. Pick $x \in U$, so that $T^v x : ||v|| \leq r$ is a collection of points which is both contained in the levels of κ, and hits some (indeed every) level of κ'. •

With the constructions of sections 3 and 4 and these three lemmas therefore, we have reduced the proof of Theorem 1.2 in the introduction to the construction of a sequence of partition-valued functions on X with certain properties, P1 to P4, and further conditions on the inradii and Følner properties of the atoms involved. In the next two sections, we construct these partitions.

§**5 Regions in** $\mathbb{Z}^d$ In this section we develop a geometric construction of partitions of $\mathbb{Z}^d$. For simplicity we make the argument for the case $d=2$. All the constructions of this section proceed identically for $d>2$, perhaps with some change of constants.

Definition 5.1 Suppose that $R \subset \mathbb{Z}^2$ is a non-empty set, which we consider also as a subset of $\mathbb{R}^2$, $\mathbb{Z}^2$ having been put into $\mathbb{R}^2$ in usual position. Define the regions:

$$K(w,R) = \{v \in \mathbb{R}^2 : ||v-w|| \leq ||v-w'|| \ \forall w' \in R\}$$

for each $w \in R$. These are often referred to as *Voronoi domains* or *Dirichlet cells* in the literature.

Elementary geometric considerations give us

Lemma 5.2 *a/ For each $w \in R$, $K(w,R)$ is a closed convex polygonal subset of $\mathbb{R}^2$, whose interior contains w.*

b/ $\cup_{w\in R} K(w,R) = \mathbb{R}^2$.

c/ $IntK(w,R) \cap IntK(w',R) \neq \emptyset$ implies that $w=w'$.

d/ If $K(w,R) \cap K(w',R) = A \neq \emptyset$, then A is a connected subinterval of the perpendicular bisector of the straight line from w to w' (A could be a single point in some circumstances).

Proof All of these facts follow from the equation, easily established,

$$K(w,R) = \bigcap \{H_{w,v} : v \in R, v \neq w\}$$

where $H_{w,v}$ is the closed half plane containing w and bounded by the perpendicular bisector of the straight line joining v and w. •

Here are a few geometric parameters which will be needed to control the later construction:

Definitions 5.3 A subset, F, of $\mathbb{Z}^2$ is *syndetic*, if there is a number, N, so that for all $w \in \mathbb{Z}^2$, there is a $v \in F$ with $||v - w|| \leq N$. N is a *(syndetic) bound* for F and we say that F is N*-syndetic* (see **[F]**).

A subset, F, of $\mathbb{Z}^2$ is M*-spread* if $v, w \in F$ and $v \neq w$ implies that $||v - w|| \geq M$.

Lemma 5.4 *Suppose that $R \subset \mathbb{Z}^2$ is N-syndetic and M-spread, then for each $w \in R$,*

a/ $B(w, M/2) \subset K(w, R) \subset B(w, 2N)$,

b/ $K(w, R)$ is determined by the set of points: $B(w, 4N) \cap R$, and

c/ The ratio of the length of boundary of $K(w, R)$ to its area is at most $2/M$.

Proof The description of $K(w, R)$ in the proof of 5.2, together with the definitions of syndetic and spread, give part a/ immediately.

b/ Simple geometric considerations show that only close neighbours of w influence the intersection in the proof of 5.2: $K(w, R) = \bigcap\{H_{w,v} : v \in R, 0 < ||v - w|| \leq 4N\}$. A better constant than 4 is, of course, possible.

c/ By Lemma 5.2 a/, the area of $K(w, R)$ can be divided into triangles, each with w as a vertex whose opposite base is a face of the polygon $K(w, R)$. By Lemma 5.2 d/, the altitude dropped from w is at least $M/2$ in length and so we deduce that the length of the base stands in ratio $2/M$ to the area of the triangle. Combining all the triangles shows that the length of the boundary of $K(w, R)$ is at most ratio $2/M$ of the area of $K(w, R)$. The result follows quickly from this fact and the convexity of $K(w, R)$. •

Definition 5.5 Suppose that K is a subset of $\mathbb{R}^2$. Write K^c for the complement of K in $\mathbb{R}^2$.

When C is positive, write $B(K, C) = \{v \in \mathbb{R}^2 : \exists w \in K : ||v - w|| \leq C\}$ and $B(K, -C) = B(K^c, C)^c$. This is without confusion with the notation $B(w, C)$ of Definition 5.3.

Suppose that A is a subset of $\mathbb{Z}^2$. Then say that A is C*-close* to K if $B(K, -C) \cap \mathbb{Z}^2 \subset A \subset B(K, C)$. Note that if $C' > C$ then C-closeness implies C'-closeness.

Lemma 5.6 *Suppose that $A \subset \mathbb{Z}^2$ is C-close to K a compact convex subset of $\mathbb{R}^2$ with area μ and perimeter length λ. Then $\mu(K) - (C + \sqrt{2})\lambda \leq |A| \leq \mu(K) + (C + \sqrt{2})\lambda + \pi(C + \sqrt{2})^2$*

Proof Set $J = A + [0, 1]^2$, the union of closed unit squares with bottom left corner in A. Then it is clear that $\mu(J) = |A|$ and that $B(K, -(C + \sqrt{2})) \subset J \subset B(K, C + \sqrt{2})$.

The convex set $B(K, -(C + \sqrt{2}))$ is formed from K by cutting away rectangles (possibly overlapping or redundant) of width $C + \sqrt{2}$ along the inside of each edge of K. Adding up for each edge we deduce that $\mu(J) \geq \mu(B(K, -(C + \sqrt{2}))) \geq \mu(K) - (C + \sqrt{2})\lambda$ therefore.

On the other hand, $B(K, C+\sqrt{2})$ is a union of the set K, of rectangles of width $C+\sqrt{2}$ placed outside along each of the edges of K, and of sectors of discs of radius $C+\sqrt{2}$ centred at each of the corners of K. Since K is convex, the angle of arc included in a sector at one corner does not overlap the angle of arc included in another corner's sector. Thus the total area accounted by the sectors is at most the area of a disc of radius $C+\sqrt{2}$. This contributes to the following upper estimate of the area of J therefore: $\mu(J) \leq \mu(B(K, C+\sqrt{2})) \leq \mu(K) + (C+\sqrt{2})\lambda + \pi(C+\sqrt{2})^2$.

Thus we arrive at the bounds claimed in the statement of the Lemma. •

Lemma 5.7 *Suppose that $R \subset \mathbb{Z}^2$ is N-syndetic and M-spread and that $w \in R$. Suppose that $A \subset \mathbb{Z}^2$ is C-close to $K(w, R)$. Then, provided $8(C+\sqrt{2}) \leq M$,*

a/ $B(w, M/2 - C) \cap \mathbb{Z}^2 \subset A \subset B(w, 2N + C)$, and

b/ A is $20(C+\sqrt{2})/M$-Følner.

Proof Part a/ is immediate from Lemma 5.4 a/.

Part b/: We wish to estimate $|A\Delta(A+v)|$ from above and to do this we use the formula $|A\Delta(A+v)| = 2|A| - 2|A \cap (A+v)|$ and estimate $|A|$ from above and $|A \cap (A+v)|$ from below.

Write $K = K(w, R)$ and λ for the length of perimeter of K. μ is Lebesgue measure in $\mathbb{R}^2$,

Let $J = A + [0,1]^2$ as in Lemma 5.6 and note that $|A \cap (A+v)| = \mu(J \cap (J+v))$. As in Lemma 5.6 we have $J \cap (J+v) \supset B(K, -(C+\sqrt{2}+||v||))$ and the same argument gives $|A \cap (A+v)| \geq \mu(K) - \lambda(C+\sqrt{2}+||v||)$.

Using the upper estimate for $|A|$ in Lemma 5.6, we have

$$\begin{aligned} |A\Delta(A+v)| &\leq 2(\mu(K) + (C+\sqrt{2})\lambda + \pi(C+\sqrt{2})^2) - 2(\mu(K) - \lambda(C+\sqrt{2}+||v||)) \\ &= 4\lambda(C+\sqrt{2}) + 2\pi(C+\sqrt{2})^2 + 2\lambda||v|| \end{aligned}$$

However, Lemma 5.4c/ and Lemma A show that $\lambda \leq 2\mu(K)/M \leq 2(|A| + \lambda(C+\sqrt{2}))/M$ which implies that $\lambda(1 - 2(C+\sqrt{2})/M) \leq 2|A|/M$ whence, by the provision on C and M, $\lambda \leq 8|A|/3M$.

Also part a/ of this Lemma shows that $J \supset B(w, M/2 - C - \sqrt{2})$ and so, assuming the conditions provided on C and M as before, $|A| \geq \pi(M/4)^2$. So $\pi(C+\sqrt{2})^2 \leq |A|(4(C+\sqrt{2})/M)^2 \leq 2|A|(C+\sqrt{2})/M$.

Therefore we have

$$|A\Delta(A+v)| \leq 32|A|(C+\sqrt{2})/3M + 4|A|(C+\sqrt{2})/M + 16|A|\ ||v||/3M$$

We know what happens when $||v|| = 0$ and otherwise we have $||v|| \geq 1$, so we get the estimate:

$$|A\Delta(A+v)| \leq (32(C+\sqrt{2})/3M + 4(C+\sqrt{2})/M + 16/3M)|A|\ ||v|| \leq 20|A|\ ||v||(C+\sqrt{2})/M$$

as required. •

Remark 5.8 A more precise argument can give better bounds but this estimate is both sufficient for subsequent applications and, more importantly, generalizable, with adaption only of the numerical constants, to higher dimensions.

Definition 5.9 Suppose that $R \subset \mathbb{Z}^2$ and that $C > 0$. Then a partition, $\mathcal{T}$, of $\mathbb{Z}^2$ is a (R, C)-*regionalisation* if, for each atom, $A \in \mathcal{T}$, there is an $w \in R$ such that A is C-close to $K(w, R)$.

Lemma 5.10 *Suppose that R is N-syndetic and M-spread, that $8(C + \sqrt{2}) \leq M$ and that $\mathcal{T}$ is an (R, C)-regionalisation. Then there is a bijection between $\mathcal{T}$ and R; an atom, A, of $\mathcal{T}$ is in correspondence with $w \in R$ if and only if A is C-close to $K(w, R)$ if and only if $w \in A$.*

Proof Since, under the conditions above and by Lemma 5.7 a/, $B(w, 3C) \subset A$, A can be contained in $B(K(w', R), C)$ only if $w' = w$. Conversely, by the same estimate, the atom of $\mathcal{T}$ which contains w is necessarily C-close to $K(w, R)$. •

Definition 5.11 Define on $\mathbb{Z}^2$ the lexicographical order: $(a, b) <_{lex} (c, d)$ if $a < c$, or if $a = c$ and $b < d$. This is shift invariant.

Proposition 5.12 *Suppose that $R \subset \mathbb{Z}^2$ is N-syndetic and M-spread, that $8(C+\sqrt{2}) \leq M$ and that $\mathcal{T}$ is a (R, C)-regionalisation. Suppose further that $R' \subset R$ is N'-syndetic and M'-spread with $M' \geq 2N+C$, then there is a (R', C')-regionalisation, $\mathcal{T}'$, which encrudens $\mathcal{T}$, where $C' = 2N + C$.*

Proof Given the data of (5.12) and $w \in R$, let $A(w)$ be the atom of $\mathcal{T}$ which is C-close to $K(w, R)$. By Lemma 5.10 this is well defined and, by Lemma 5.7 a/, $A(w) \subset B(w, 2N+C)$.

Given $w \in R'$, let

$$D(w) = \{u \in R \cap K(w, R') : ||u - w'|| = ||u - w|| \text{ and } w' \in R' \text{ implies } w' \geq_{lex} w\}$$

Thus $D(w)$ is essentially $R \cap K(w, R')$ except for a resolution of ambiguities exactly on the boundary using the lexicographical order.

Construct $K(w, R')$ as above for each $w \in R'$ and let $A'(w) = \cup\{A(v) : v \in D(w)\}$. Then it is clear that $A'(w) \subset B(K(w, R'), 2N + C)$ and also that $A'(w)$ contains every lattice point at least distance $2N + C$ from $K(w, R')^c$ as is required to show that $A'(w)$ is $(2N + C)$-close to $K(w, R')$.

The construction ensures that the $A'(w) : w \in R'$ are the atoms in a partition of $\mathbb{Z}^2$.

•

Remark 5.13 Note the independence of the parameter C' from N' and M', assuming that the latter is chosen large enough.

Lemma 5.14 *Given the assumptions, construction and notation of 5.12.*

a/ The construction is shift-invariant in the sense that if $v \in \mathbb{Z}^2$, the partition $\mathcal{T}''$, produced from the sets $R+v$, $R'+v$ and the partition $\mathcal{T}+v$, is the partition, $\mathcal{T}'$, produced from R, R' and $\mathcal{T}$, but shifted by v: $\mathcal{T}'' = \mathcal{T}' + v$. And $\mathcal{T}''$ is a $(R'+v, C')$-regionalisation with the same parameter C'.

b/ The construction is locally defined in the sense that if $\mathcal{T}'$ is the partition of $\mathbb{Z}^2$ produced from $\mathcal{T}$, R and R', then, for any $r > 0$, the restriction of $\mathcal{T}'$ to the ball $B(0, r)$ is determined by the restrictions of $\mathcal{T}$, R and R' to the ball $B(0, r + 6N' + 2C')$.

Proof a/ All the details of the construction are shift-invariant.

b/ Suppose that $v \in \mathbb{Z}^d$, $||v|| \leq r$. By Lemma 5.7a/ we know that v is contained in some $A'(w)$ where $w \in R'$ and $||w - v|| \leq 2N' + C'$. Thus if we can determine $A'(w)$ for all such w, we find the atom of $\mathcal{T}'$ which contains v (Lemma 5.10) and we are done.

So suppose that $w \in R'$ and that $||w - v|| \leq 2N' + C'$, so that $||w|| \leq 2N' + C' + r$. The information given in the ball $B(0, r + 6N' + 2C')$ is sufficient therefore to define R, R' and $\mathcal{T}$ within the ball $B(w, 4N' + C')$.

As in Lemma 5.4b/, the information in the ball $B(w, 4N')$ is enough to determine $D(w)$ (in the proof of 5.12), and the additional C' is sufficient (by Lemma 5.7a/) to determine $A(u)$ (knowing $\mathcal{T}$ by assumption) for all $u \in D(w)$. The formula in the proof of 5.12 finds $A'(w)$, therefore. •

§**6 Incorporating the dynamics** Here we continue to consider the case $d = 2$ noting again that the higher dimensional cases proceed with only a change of constants. Using the dynamics, we specify the subsets R etc. to which the previous section can be applied.

Suppose that$(X, \mathbb{Z}^2)$ is Cantor system. It will be assumed throughout this section that $(X, \mathbb{Z}^2)$ is minimal and free (see Definition 1.1).

Definition 6.1 For $x \in X$ and a clopen subset, U, of X, write $R(x, U) = \{v \in \mathbb{Z}^2 : T^v x \in U\}$.

Lemma 6.2: *Suppose that $U \subset X$ is clopen non-empty.*

i/ For any $x \in X$, $R(x, U)$ is syndetic.

ii/ Given $M > 0$, there is a $\delta > 0$ such that, for every clopen $W \subset U$ with $diam W < \delta$, and every $x \in X$, $R(x, W)$ is M-spread.

Proof i/ See [**F**].

ii/ Suppose not, and so there is sequence of $x_n \in U$, $n \geq 1$ and a sequence of clopen sets $W_n = \{y \in U : d(x_n, y) \leq 2^{-n}\}$ so that $v_n, v_n + w \in R(x'_n, W_n)$ for some $w \in \mathbb{Z}^2 \setminus \{0\}$ fixed and $v_n \in \mathbb{Z}^2$ and $x'_n \in X$. We may suppose, without loss of generality, that the $x_n \to x_0 \in U$ and so we have $d(T^{v_n} x'_n, x_0) \leq 2^{-n} + d(x_n, x_0) \to 0$. Similarly, we have $T^{v_n+w} x'_n \to x_0$, from which we deduce that $T^w x_0 = x_0$, contradicting freeness of the action. •

Construction 6.3 We will define inductively a sequence of clopen sets $...U_n \subset U_{n-1} \subset ... \subset U_0 = X$, strictly ascending sequences of real numbers N_n, M_n, another sequence of real numbers C_n, and, for each $x \in X$, a sequence of partitions of $\mathbb{Z}^2$, $\mathcal{T}_n(x)$, for which the following hypothesis holds, for $n \geq 1$:

H(n): i/ $(C_n + \sqrt{2})/M_n \leq 4^{-n-4}$, and $C_n \geq 2N_{n-1} + C_{n-1}$.
For each $x \in X$,
ii/ $R(x, U_n)$ is N_n-syndetic and M_n-spread,
iii/ $\mathcal{T}_{n-1}(x)$ refines $\mathcal{T}_n(x)$,
iv/ $\mathcal{T}_n(x)$ is a $(R(x, U_n), C_n)$-regionalisation, and
v/ $\mathcal{T}_n(T^v x) = \mathcal{T}_n(x) - v$ for all $v \in \mathbb{Z}^2$.
vi/ The map $x \mapsto \mathcal{T}_n(x)$ is uniformly continuous.

We start the induction at $n = 1$ and to make sense of the statement, we set $U_0 = X$, $N_0 = M_0 = 1$ and $\mathcal{T}_0(x) = \mathcal{T}_0$, the partition of $\mathbb{Z}^2$ into single points, for all $x \in X$, and we find $C_0 = 1/2$.

Set $C_1 = 3 > 2N_0 + C_0$ and set $M_1 = 4^5(3 + \sqrt{2})$. Find, by Lemma 6.2ii/, a $\delta > 0$ so that for any clopen set U with $diam(U) < \delta$ we know that $R(x, U)$ is M_1-spread for any choice of x. Choose such an atom, U_1 say. Let N_1 be the uniform syndetic constant for $R(x, U_1)$. It is clear that we have $H(1)$ parts i/ and ii/ already.

Since the case $R = \mathbb{Z}^2$, $\mathcal{T}_0(x) = \mathcal{T}_0$, $N_0 = M_0 = 1$, $C_0 = 1/2$ and $R' = R(x, U_1)$ is so simple, we make a special case of it; Although the conditions on M_0 and C_0 for Lemma 5.10 do not hold strictly, never-the-less the conclusion of 5.10 does hold trivially. Similarly, the construction of 5.12 and the conclusions of 5.14 also hold in this case.

Therefore, for each $x \in X$, we pursue the construction of Proposition 5.12 to form $\mathcal{T}'$ which we will write $\mathcal{T}_1(x)$. From 5.12, properties H(1) iii/ and iv/ follow directly and property H(1) v/ follows by Lemma 5.14 a/.

H(1) vi/ holds by Lemma 5.14b/, since if x and y are sufficiently close that $R(x,U_1)\cap B(0,r+6N_1+5)=R(y,U_1)\cap B(0,r+6N_1+5)$, then $\mathcal{T}_1(x)$ and $\mathcal{T}_1(y)$ agree on the set $B(0,r)\cap\mathbb{Z}^2$.

A similar argument shows that induction continues:

First define $M_{n+1}=4^{n+4}(2N_n+C_n+\sqrt{2})$ and, by Lemma 6.2 ii/, pick a $\delta>0$ so that for any clopen set $U\subset U_n$ with $diam(U)<\delta$ we know that $R(x,U)$ is M_{n+1}-spread for any choice of x. Let U_{n+1} be such a clopen set. Let N_{n+1} be the syndetic bound on R' and let $C_{n+1}=2N_n+C_n$. Thus we confirm $H(n+1)$ parts i/ and ii/ directly.

Now, fix x and write $R=R(x,U_n)$, $\mathcal{T}=\mathcal{T}_n(x)$ and $R'=R(x,U_{n+1})$ and pursue the construction of Proposition 5.12, for which the data are appropriate, to form $\mathcal{T}'=\mathcal{T}_{n+1}(x)$. The properties iii/ and iv/ required by $H(n+1)$ are found from the conclusion of Proposition 5.12 directly. Property v/ comes from 5.14a/.

Property vi/ follows from 5.14b/, as before: If x and y are sufficiently close that $R(x,U_{n-1})$, $R(x,U_n)$ and $\mathcal{T}_{n-1}(x)$ agree with $R(y,U_{n-1})$, $R(y,U_n)$ and $\mathcal{T}_{n-1}(y)$ respectively when restricted to the ball $B(0,r+6N_n+2C_n)$ (as can be ensured uniformly by hypothesis) then $\mathcal{T}_n(x)$ and $\mathcal{T}_n(y)$ agree when restricted to the ball $B(0,r)$.

Proposition 6.4 *Given the construction above, the sequence of functions $x\mapsto\mathcal{T}_n(x)$ produced obey the properties P1 to P4 in section 4. Moreover, independently of the choice of $x\in X$,*

a/ Each atom of $\mathcal{T}_n(x)$ is 4^{-n}-Følner.

b/ Each atom of $\mathcal{T}_n(x)$ is of inradius at least $M_n/4\geq 4^n$.

Proof The properties P1, P3 and P4 can each be read directly from v/, vi/ and iii/ of the induction hypothesis. Property P2 follows from ii/ and iv/ and Lemma 5.7a/ showing that the diameter of an atom of $\mathcal{T}_n(x)$ is at most $2N_n+C_n$.

The Følner estimate follows from i/, ii/, iv/ and Lemma 5.7 b/.

The inradius estimate follows from i/, ii/, iv/ and Lemma 5.7 a/. •

We repeat that there is no restriction on extending the arguments above to the case $d\geq 3$. Therefore, by combining Proposition 6.4 with Lemmas 4.4, 4.6 and Prop 4.7 we have proved Theorem 1.2.

§**7 Ordered Cohomology** Here we recall some basic definitions about ordered groups. These are extracted from [**HPS**], [**Bl**] and [**E**].

Definitions 7.1 An *ordered group* is a countable abelian group, G, together with a positive cone, G^+ with the properties: $G^+\cap(-G^+)=\{0\}$, $G^++G^+\subset G^+$ and $G^+-G^+=G$. If

$a - b \in G^+$, we write $a \geq b$, and, if in addition, $a \neq b$, we write $a > b$. Often, we suppress mention of G^+.

An *order unit* is an element $a \in G^+$ such that for all $b \in G$, there is an $n \geq 1$ such that $na \geq b$. All the ordered groups we consider here will have an order unit.

An ordered group, G, with order unit 1, is *simple* if $a > 0$ implies there is an $n \geq 1$ such that $na \geq 1$.

An ordered group is *unperforated* if $na > 0$ and $n > 0$ implies that $a > 0$.

The *infinitessimals* of G are defined $Inf(G) = \{a \in G : na \leq 1 \; \forall \; n \in \mathbb{Z}\}$. Write $\tilde{G}$ for the ordered group with unit $G/Inf(G)$.

A homomorphism $h : G \longrightarrow H$ of ordered groups is *positive* if $a \geq 0$ implies $h(a) \geq 0$. It is *strictly positive* if the same holds with strict inequality.

A map $h : G \longrightarrow H$ is an *order embedding* if $a \in G$ and $h(a) > 0$ implies that there is an $a' \in G$ such that $a' > 0$ and $h(a) = h(a')$.

If G and H have order units, 1_G and 1_H respectively, then h is *unital* if $h(1_G) = 1_H$.

If G has an order unit, then a *normalised state* on G is a positive unital homomorphism from G to $\mathbb{R}$ (with usual order and unit 1). The set of such, $\Sigma(G)$, is a closed Choquet simplex in the space of all homomorphisms $G \longrightarrow \mathbb{R}$ with weak* topology.

A positive unital homomorphism $h : G \longrightarrow H$ induces a continuous affine map $h^* : \Sigma(H) \longrightarrow \Sigma(G)$: $h^*(\mu) = \mu \circ h$.

Lemma 7.2 *Suppose that G is a simple dimension group with unit and that $h : G \longrightarrow H$ is a unital positive homomorphism onto H, a torsion-free simple ordered group with unit, not equal to 0 nor to $\mathbb{Z}$. If $h^* : \Sigma(H) \longrightarrow \Sigma(G)$ is onto, then H is a dimension group and h factors through to an ordered group isomorphism between $\tilde{G}$ and $\tilde{H}$.*

Proof Recall the result of [**EHS**] in which it is shown that an unperforated ordered torsion-free group with unit, not isomorphic to $\mathbb{Z}$, is a dimension group iff it has the strict Reisz interpolation property: $a, b < c, d$ implies there is an f: $a, b < f < c, d$.

It is well-known (see [**HPS**] or [**Bl**] for example) that $a \in G$ is strictly positive if and only if $\sigma(a) > 0$ for all $\sigma \in \Sigma(G)$. Furthermore, $a \in Inf(G)$ if and only if $\sigma(a) = 0$ for all $\sigma \in \Sigma(G)$.

Thus if $a \in \ker h$ and $\sigma \in \Sigma(G)$, we have, by assumption, $\tau \in \Sigma(H)$ such that $\sigma = \tau \circ h$. Thus $\sigma(a) = \tau(h(a)) = 0$ and so $a \in Inf(G)$.

Also simplicity of H shows that if $b > 0$, $b \in H$, then $\tau(b) > 0$ for all $\tau \in \Sigma(H)$. In particular, if $h(a) = b$ then $h^*\tau(a) = \tau(h(a)) > 0$ for all $\tau \in \Sigma(H)$. The fact that h^* is onto then shows that $\sigma(a) > 0$ for all $\sigma \in \Sigma(G)$ and the previous paragraph implies that $a > 0$. Thus h is an order embedding.

From this we deduce that H is unperforated since $nb > 0$, $n > 0$ and $b = h(a)$ implies that $na > 0$. Since G is unperforated, $a > 0$ which implies that $b > 0$ ($b \geq 0$ by positivity of h, but $b \neq 0$ since $nb \neq 0$).

If $a > 0$, then the simplicity of G shows that $a \notin Inf(G)$ and so $h(a) \neq 0$. However h is positive, so that $h(a) > 0$ therefore; h is strictly positive.

So if $h(a), h(b) < h(c), h(d)$ are four related elements of H, then, for all $\tau \in \Sigma(H)$, the same inequalities hold between $\tau(h(a)), \tau(h(b)), \tau(h(c)), \tau(h(d))$ for all choices of τ. By assumption, $\tau \circ h$ covers all choices of $\sigma \in \Sigma(G)$ and so we deduce that $a, b < c, d$. The strict Reisz interpolation property applies to G and so we find f so that $a, b < f < c, d$ and therefore, by strict positivity of h, the inequalities: $h(a), h(b) < h(f) < h(c), h(d)$ as required.

The isomorphism of $\tilde{G}$ and $\tilde{H}$ follows from the fact that $\ker h \subset Inf(G)$ and that h^* is a bijection. •

Lemma 7.3 *Suppose that $(X, \mathbb{Z}^d)$ is a Cantor system, then*

a/ $H^d(\mathbb{Z}^d; C(X;\mathbb{Z}))$ is torsion-free.

b/ If $(X, \mathbb{Z}^d)$ is minimal, then $H^d(\mathbb{Z}^d; C(X;\mathbb{Z}))$ is simple.

c/ The normalized states on $H^d(\mathbb{Z}^d; C(X;\mathbb{Z}))$ are in canonical affine correspondence with the $\mathbb{Z}^d$ invariant probability measures on X.

Proof a/ Is the main result of section 5 in **[FH]**.

b/ Suppose that $[g] > 0$, $[g] \in H^d(\mathbb{Z}^d; C(X;\mathbb{Z}))$ and we may assume that g itself is a positive function not equal to 0. Let U be the clopen set which supports g and note that by minimality (see **[F]** for example), we have $\cup_{v \in F} T^{-v} U = X$ for some finite set $F \subset \mathbb{Z}^d$. From this we deduce $\sum_{v \in F} T^v g \geq 1$ and hence $n[g] \geq [1]$ where $n = |F|$, as required.

c/ Straightforward using the weak density of the complex algebra $C(X;\mathbb{Z}) \otimes \mathbb{C}$ in $C(X)$. •

Now return to the Bratteli diagram, $\mathcal{B}$, produced in section 4 from a faithful Vershik model. Recall the equation between $|\mathcal{Q}_n|$ and the finite path partition (Theorem 3.4).

Definition 7.4 Let p be a finite path from level 0 to level n. Write g_p for the indicator function of $U(p)$ (Defn. 3.3).

Lemma 7.5 *If p and q are two finite paths with the same terminus, then g_p and g_q are H^d cohomologous.*

Proof From Theorem 3.4, $U(p)$ and $U(q)$ are two atoms of $|\mathcal{Q}_n|$ in the same tower, κ say. Then by definition, there is a $v \in \mathbb{Z}^d$ such that $T^v U(p) = U(q)$, as required. •

Definition 7.6 Canonically, $\mathcal{B}$ defines a direct limit dimension group (see **[El]** **[EHS]** **[HPS]** etc) $\mathbb{Z} = \mathbb{Z}^{V_0} \xrightarrow{\beta_0} \mathbb{Z}^{V_1} \xrightarrow{\beta_1} \ldots \xrightarrow{\beta_{n-1}} \mathbb{Z}^{V_n} \xrightarrow{\beta_n} \ldots$ which has a natural order unit

($1 \in \mathbb{Z}^{V_0} \equiv \mathbb{Z}$) and which is simple if the diagram is simple. This ordered group with unit will be written $K_0(\mathcal{B})$ for various reasons.

We abbreviate $H^d(\mathbb{Z}^d; C(X;\mathbb{Z}))$ as H^d.

Construction 7.7 For each $\kappa \in V_n$, choose $p(\kappa)$, a finite path from level 0 to κ, and let g_κ be the indicator function of $U(p(\kappa))$.

Suppose that p is a finite path from V_0 to $\kappa \in V_n$. The clopen set $U(p)$ can be expressed as a disjoint union, $U(p) = \cup^* U(p')$ where star expresses the fact that the union is over the set of paths p' from V_0 to V_{n+1} which agree with p to level n. Note that if $\kappa' \in V_{n+1}$, then the number of such paths which terminate at κ' is equal to, $(\beta_n)_{\kappa,\kappa'}$ the number of edges from κ to κ'.

This is the key to

Lemma 7.8 *With the notation above,*

$$[g_\kappa] = \sum_{\kappa' \in V_{n+1}} (\beta_n)_{\kappa,\kappa'} [g_{\kappa'}]$$

the square brackets indicating the H^d equivalence class among elements of $C(X;\mathbb{Z})$.

Proof The union $U(p) = \cup^* U(p')$ noted above, can be applied to the case $p = p_\kappa$, implying that $g_\kappa = \sum^* g_{p'}$ where star indicates the same index set as the union. However, the H^d equivalence class of $g_{p'}$ depends only on its terminus (Lemma 7.5), so $[g_{p'}] = [g_{\kappa'}]$ where κ' is the terminus of p', and the sum above is implied by the path count above. •

Construction 7.7 ctd. Consider the correspondence: $\alpha_n : \mathbb{Z}^{V_n} \longrightarrow H^d$ defined $\alpha_n(a) = \sum_{\kappa \in V_n} a_\kappa [g_\kappa]$, where $a = (a_\kappa : \kappa \in V_n) \in \mathbb{Z}^{V_n}$.

From Lemma 7.8 we deduce $\alpha_{n+1}(\beta_n(a)) = \alpha_n(a)$ immediately, and so produce a map $K_0(\mathcal{B}) \xrightarrow{\alpha} H^d$ therefore.

Proposition 7.9 *Suppose that $(X, \mathbb{Z}^d)$ is a free Cantor system and that $\mathcal{T}_n$ is a sequence of $\mathbb{Z}^d$-partition-valued maps with properties P1-P4 (section 4). Suppose that, with added refinement if necessary (3.7), $\mathcal{Q}_n$ is a faithful Vershik model produced from this (4.2), giving canonically a Bratteli diagram, $\mathcal{B}$ (3.4). Then the map $\alpha : K_0(\mathcal{B}) \longrightarrow H^d$ constructed above is a positive unital ordered group homomorphism onto and $\alpha^* : \Sigma(H^d) \longrightarrow \Sigma(K_0(\mathcal{B}))$ is 1-1.*

Suppose further that $\lambda(\mathcal{T}_n) \to 0$ (Defn 4.5), then α^* is onto.

Proof α is positive and unital by construction.

To show it is onto, consider a map $C(X;\mathbb{Z}) \longrightarrow H^d(\mathcal{B})$ defined as follows: Suppose that $f \in C(X;\mathbb{Z})$ is measurable with respect to $|\mathcal{Q}_n|$, then by Lemma 7.5, we may write

$[f] = \sum_{\kappa\in V_n} a_\kappa[g_\kappa]$ for some integer values a_κ. Hence we produce an element of $\mathbb{Z}^{V_n}$. We see that this definition is independent of the choice of n large enough by the same argument as in 7.7.

It is clear that this map is the diagonal in a commutative triangle

$$\begin{tikzcd} C(X;\mathbb{Z}) \arrow[r] \arrow[dr] & H^d \\ & K_0(\mathcal{B}) \arrow[u, "\alpha"'] \end{tikzcd}$$

where the horizontal map is the canonical quotient. This shows that α is onto.

Concerning the states, it is immediate from the surjectivity of α that α^* is 1-1.

To show that α^* is onto under the conditions assumed, suppose that σ is a normalized state on $K_0(\mathcal{B})$ and pick a level n. Thus σ_n, defined as the composition $\mathbb{Z}^{V_n} \longrightarrow H^d(\mathcal{B}) \xrightarrow{\sigma} \mathbb{R}$ is represented as a row vector of positive real numbers: $(s_\kappa : \kappa \in V_n)$.

For each $\kappa \in V_n$, find a traverse $L(\kappa)$. Then the definition of $\lambda(\mathcal{T}_n)$ etc. shows that $|L(\kappa)\Delta T^v L(\kappa)| \le 2\lambda(\mathcal{T}_n)\ ||v||\ |L(\kappa)|$ for all $v \in \mathbb{Z}^d$.

For $f \in C(X)$, let $\mu_n(f) = \sum_{\kappa\in V_n} s_\kappa \sum_{x\in L(\kappa)} f(x)$, clearly a positive measure on X.

The Følner property above implies that $|\mu_n(f - T^v f)| \le \sum_\kappa s_\kappa 2\lambda(\mathcal{T}_n)||v||\ ||f||_\infty |L(\kappa)|$. However, since $\sigma(1) = 1$, and since 1 is represented as the element $(|L(\kappa)| : \kappa \in V_n)$ in $\mathbb{Z}^{V_n}$, we see that $\sum_\kappa s_\kappa |L(\kappa)| = 1$. Hence we have: $\mu_n(1) = 1$ and $|\mu_n(f - T^v f)| \le 2\lambda(\mathcal{T}_n)||v||\ ||f||_\infty$. Therefore, since we assume that $\lambda(\mathcal{T}_n) \to 0$, any weak* limit point of the measures μ_n is a probability measure on X, invariant under the action of $\mathbb{Z}^d$.

We note that if $f \in C(X)$ is $\mathcal{Q}_n$ measurable, then $\mu_m(f) = \mu_n(f)$ for all $m \ge n$, since the σ_n factor through σ via the connecting maps, β_n.

Therefore, fix a weak* limit point, μ', of μ_n, so that μ' is a state on H^d (Lemma 7.3). The observation of the last paragraph shows that $\sigma_n = \mu' \circ \alpha_n$ and so $\sigma = \mu' \circ \alpha$ on H^d, as required. •

Proof of Theorem 1.4 With the construction of Section 6, extended without complication to $d > 2$, we have the conditions necessary to be able to apply Proposition 7.9. Then Lemma 7.2, using 7.3, completes the proof. •

§**8 Concluding remarks** We note that many of the arguments above are valid in greater generality than required by the immediate problem. We can hope to generalize the results of this paper in several ways: to more general group actions: to non-minimal systems: and with more demanding restrictions on the nature of the atoms of the partitions, $\mathcal{T}_n(x)$.

The definitions, constructions and arguments of sections 3 and 4 all apply equally to general discrete group actions (with a suitable discrete metric on the group). However, we need a result such as Lemma 5.7 to restrict the size of the boundary at each stage. Lemma 5.7b/ implies the amenability of $\mathbb{Z}^d$ by the classical Følner condition. We have been unable to resolve whether the conclusion of 5.7b/ (or some equally useful formulation) is equivalent to amenability, but it seems likely to be the case. Such an equivalence would generalize Theorem 1.2 to free minimal amenable group actions, and generalize the first part of Theorem 1.5 accordingly. Without amenability we could still hope for such a generalization but only of the topological restriction of Y.

Likewise, the results of section 3 and 4 have little need for freedom and no need for minimality. Only in section 6 are both these properties used, and then with full force. By using an idea of "hitting set", due to Handelman **[B]**, we may avoid minimality but still find syndetic return times; and we can hope in specific examples to achieve spreading as well.

With more care over the constructions of section 5, whose main properties are shift-invariance and locality (Lemma 5.14), it is not hard to modify the constructions to produce partitions of $\mathbb{Z}^d$ whose atoms have nice combinatorial properties. We could require, for example, that if A is an atom of such a partition, then, as a subset of $\mathbb{R}^d$, $A + [0, 1]^d$ is homeomorphic to the closed unit ball. All such modifications are made keeping properties P1-P4, and the inradius and Følner properties intact, of course. We should not expect always the perfectly rectangular atoms which are found in the measure-preserving case **[KW]**.

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