FRONT TRACKING FOR ONE-DIMENSIONAL NONLINEAR ADVECTION EQUATIONS WITH VARIABLE COEFFICIENTS

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ABSTRACT. A new front tracking method is developed for the variable coefficient equation $u_t + V(x,t)f(u)_x = 0$. The method is a generalization of Dafermos' method for the constant coefficient case and is well-defined also for certain discontinuous velocities V. We give an explicit inequality stating the stability with respect to flux function, velocity, and initial data. The numerical method is unconditionally stable and has linear convergence. It is well suited for numerical calculations, as is demonstrated in three examples.

1. INTRODUCTION

We consider the nonlinear, variable coefficient equation

$$u_t + V(x,t)f(u)_x = 0, \qquad u(x,0) = u_0(x). \tag{1}$$

This equation describes the nonlinear advection of a scalar quantity u in a one-dimensional velocity field $V(x,t)$. A motivation for this study is that (1) occurs as an essential step in various operator splittings; for instance, dimensional splitting for multi-dimensional, variable coefficient equations [11], two-phase flow with capillary forces [8], and convection–diffusion equations [7]. The focus here will be on the following two questions: How is a numerical approximation to a possibly discontinuous solution of (1) computed efficiently? Moreover, how does the solution depend on flux function, velocity, and initial data?

Existence and uniqueness of discontinuous solutions were proved by Kružkov [9] under certain requirements for a more general equation. He proposed an *entropy condition* to identify the unique physical solution.

Dafermos [2] suggested a method to study initial value problems for the nonlinear hyperbolic equation $u_t + f(u)_x = 0$. He observed that given a piecewise linear flux function, the solution of the Riemann problem only contains constant states separated by discontinuities moving along straight lines. Therefore, if the initial data are piecewise constant, a local solution can be found by superposition of simple Riemann problems.

The solution will remain piecewise constant for all time, since all wave interactions lead to new Riemann problems. By solving these Riemann problems, a global solution can be established for *any* piecewise linear flux function. This was proved by Holden, Holden and Høegh–Krohn [5, 6], who verified that given a finite number of constant states in the initial

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data, there is only a finite number of constant states for all time. Hence, the algorithm is well-defined and has a finite number of steps, even in infinite time! In the general case, approximating an arbitrary f by a sequence of piecewise linear functions and u_0 by a sequence of step functions gives a compact sequence of approximate solutions converging to the solution of the Cauchy problem. The algorithm naturally leads to a numerical method called *front tracking*. The method is first order, it has no associated time step, and has proved to be very efficient.

We show how these ideas carry over to the variable coefficient equation (1). The Riemann problem for this equation has the same permissible discontinuities as $u_t + f(u)_x = 0$, but instead of following straight lines, the discontinuities travel along paths satisfying an ordinary differential equation. By carefully choosing an approximation to the velocity, one is able to compute these paths explicitly. The form of the shock paths is not essential to Dafermos' method, and it can therefore be extended to the variable coefficient case. We show that the new method is well-defined under some mild restrictions on the velocity V, and that it can be used to obtain a sequence of approximations converging to the weak entropy solution of the Cauchy problem (1) in the sense of Kružkov [9]. For $V = V(x)$ an obvious alternative would be to transform (1) using $\frac{\partial}{\partial y} = V\frac{\partial}{\partial x}$. Although this approach would work well if V has constant sign, we believe that it gets rather cumbersome when V changes sign in the domain. Moreover, how to define a similar approach for time dependent velocities is a difficult task, if possible, and we will therefore not pursuit it further here.

Using the Kružkov entropy condition, we turn to investigate how solutions are sensitive to changes in flux function, velocity, and initial data. Our analysis is inspired by a theorem proved by Lucier [12] for the constant coefficient case. The dependence result can then be applied to prove stability and linear convergence of the new front tracking scheme.

The outline of the paper is as follows. In §2 a sequence of approximate weak solutions are constructed. We show that the limit is the unique entropy solution in the Kružkov sense, thus giving a partial answer to our first question. In §3, the second question is fully answered in the form of an explicit inequality stating the stability of solutions with respect to f, V, and u_0. The second question is answered in §4, which contains a more thorough discussion of the front tracking construction used as a numerical method. Using the stability result from §3 we verify that the method has linear convergence. In §5 three numerical examples are discussed.

2. Construction of a Weak Solution

Kružkov [9] proved the existence of an entropy solution having a linear modulus (i.e., bounded variation) for the more general equation

$$u_t + \frac{d}{dx}\varphi(x,t,u) + \psi(x,t,u) = 0 \tag{2}$$

under certain assumptions on φ and ψ. He suggested an entropy condition to identify the unique physical solution: u is an entropy solution of (2), if for all constants k and all

$\phi \in C_0^1$, the following inequality is valid

$$
\begin{aligned}
0 \leq \iint_{\mathcal{D}_T} \big\{ &|u-k|\phi_t + \operatorname{sgn}(u-k)[\varphi(x,t,u) - \varphi(x,t,k)]\phi_x \\
&- \operatorname{sgn}(u-k)[\varphi_x(x,t,k) - \psi(x,t,u)]\phi \big\}\, dxdt \\
&+ \int_{\mathbf{R}} |u(x,0)-k|\phi(x,0)\, dx \; - \int_{\mathbf{R}} |u(x,T)-k|\phi(x,T)\, dx.
\end{aligned}
$$

This condition both defines a generalised (or weak) solution, and characterises permissible discontinuities. For continuous velocities, equation (1) may be transformed without changing the generalized solutions by adding and subtracting a term $V_x(x,t)f(u)$ to yield an equation on the general form (2),

$$u_t + \frac{\partial}{\partial x}\big(V(x,t)f(u)\big) - V_x(x,t)f(u) = 0.$$

Recall that for smooth velocities a weak solution of (1) is defined to satisfy

$$\int_{\mathbf{R}}\int_0^\infty \big\{u\phi_t + f(u)\big(V(x,t)\phi\big)_x\big\}\, dtdx + \int_{\mathbf{R}} u(x,0)\phi(x,0)\, dx = 0 \tag{3}$$

for an arbitrary test function ϕ. If $V(x,\cdot)$ is merely Lipschitz continuous, the weak solution can be defined similarly. This follows from a standard argument in which we replace V by a smooth approximation $V^\varepsilon = \omega_\varepsilon * V$ and then subsequently let ε tend to zero. Here ω_ε is a standard mollifier function. In this case, $(V(x,t)\phi)_x$ denotes the generalized derivative (this notation will be assumed throughout the paper). With piecewise constant initial data, the initial value problem (1) is decomposed into a sequence of simple Riemann problems of the form

$$u_0(x) = \begin{cases} u^-, & \text{when } x < 0 \\ u^+, & \text{when } x > 0. \end{cases}$$

Generally, Riemann problems admit both smooth waves (rarefactions) and discontinuities (shocks). However, for a piecewise linear flux function the solution is a step function where rarefactions coincide with sequences of (small) shocks.

A weak solution is not necessarily unique. To get a unique solution, an entropy condition must be introduced. Here we use a simplified version of a condition proposed by Oleinik [14]. Let u be between u^+ and u^-. Then the jump from u^- to u^+ is permissible provided

$$\operatorname{sgn} V(x,t)\frac{f(u)-f(u^+)}{u-u^+} \leq s \operatorname{sgn} V(x,t) \leq \operatorname{sgn} V(x,t)\frac{f(u)-f(u^-)}{u-u^-}, \tag{4}$$

for u between u^- and u^+. The associated shock speed s is given by

$$s = (f(u^+) - f(u^-))/(u^+ - u^-).$$

If V is positive, the entropy condition is the same as for $u_t + f(u)_x = 0$. For a negative velocity, we can replace f by $-f$ and V by $-V$, or in other words, change the role of the upper and the lower convex envelope when determining f_c. Thus, the permissible shocks are similar to those for the constant coefficient case, but the shock curves are no longer straight lines. However, as we shall see later, they are possible to compute explicitly in certain cases.

Next, we outline the exact solution of the Riemann problem, assuming for simplicity that $V(x,t) > 0$. Suppose that $u^- < u^+$ and let f_c denote the lower convex envelope of f relative to the interval $[u^-, u^+]$. The case $u^- > u^+$ is treated symmetrically, with f_c denoting the upper concave envelope. Since f is piecewise linear, then so is f_c. Let $\{u_i\}_{i=1}^N$ denote the breakpoints of f_c, such that $u_1 = u^-$ and $u_N = u^+$. Then the solution of the Riemann problem is

$$u(x,t) = \begin{cases} u_1, & \text{for } x < x_1(t) \\ u_i, & \text{for } x_{i-1}(t) \le x \le x_i(t), \quad i = 2, \dots, N-1, \\ u_N, & \text{for } x > x_{N-1}(t), \end{cases}$$

where each $x_i(t)$ satisfies a Rankine–Hugoniot condition

$$\frac{dx_i}{dt} = V(x_i,t)\Big(\frac{f_c(u_{i+1}) - f_c(u_i)}{u_{i+1} - u_i}\Big) = V(x_i,t)s_i. \tag{5}$$

There exists a unique shock path satisfying the ordinary differential equation (5). If the function $V(x,t)$ has a simple form the shock paths can be computed explicitly. For instance, if $V = V(x) = (ax+b)$ for nonzero a, the path of a shock with speed s starting from x_0 at time t_0 is given by

$$x(t) = \big(x_0 + \frac{b}{a}\big) \exp\big[as(t-t_0)\big] - \frac{b}{a}.$$

Similarly, if $V = V(t) = (ct+d)$, the path is given by

$$x(t) = s\big(\tfrac{1}{2}ct^2 + dt\big) - s\big(\tfrac{1}{2}ct_0^2 + dt_0\big).$$

To avoid solving differential equations for each shock path, we therefore consider velocities on the above form; that is, either piecewise linear in x and piecewise constant in t, or piecewise constant in x and piecewise linear in t. Other approximations are also possible.

If the velocity has constant sign, the construction of a weak solution from Holden *et al.* [6] can be repeated, only modifying the shock paths according to (5): Connect the solutions of the local Riemann problems to compute the (global) solution of the initial value problem up to the first time when two waves from neighbouring Riemann problems interact. This interaction defines a new Riemann problem with left and right states given by the values immediately to the left and to the right of the collision point. The Riemann problem is then solved as outlined above, thereby giving the global solution until the next interaction occurs, and so on.

When V changes sign, special care has to be taken as the entropy condition depends on the sign of V. Assume first that $V = V(x)$. Since waves cannot cross a point or a region of zero velocity, domains with different sign of the velocity are not connected. Moreover, we see that $V(x)$ can also contain discontinuities, as long as the sign is preserved across each jump. This follows if V is replaced by a smooth approximation. Hence the only restriction is that $\operatorname{sgn} V(x^-) = \operatorname{sgn} V(x^+)$ at every point of discontinuity.

Assume now that $V = V(t)$. At points $t = \tau$ where the velocity changes sign we simply 'restart' the algorithm with initial data $u(x,\tau^-)$ and use $\operatorname{sgn} V(\tau^+)$ in the entropy condition to determine which envelope to choose for each initial Riemann problem. Finally, if the velocity is zero in an interval, $V(t) = 0$ for $t \in [t_1, t_2]$, then the solution remains constant in this interval, $u(x,t) = u(x, t_1^-)$. Similarly, the algorithm can be defined for arbitrary $V(x,t)$ satisfying the two requirements above.

Summing up, we have the following result

Lemma 2.1. *Suppose that f is continuous and piecewise linear and that u_0 is a bounded step function of bounded variation. Furthermore, suppose that V is bounded and piecewise linear/constant in x, t and satisfies the two properties:*

(i) $\operatorname{sgn} V(x^-,t) = \operatorname{sgn} V(x^+,t)$ *at every point of discontinuity,*
(ii) *for each finite $0 < T$ there is a finite number of τ's less than T such that* $\operatorname{sgn} V(x,\tau^-)$ *is not equal* $\operatorname{sgn} V(x,\tau^+)$ *for some x.*

Then the solution of equation $u_t + V(x,t)f(u)_x = 0$ can be computed by the front tracking algorithm. The solution $u(x,t)$ is a step function for fixed times t, and $u(x,t) \in \{u_0\} \cup \{u | f' \text{ is discontinuous at } u\}$.

Gimse and Risebro [3, 4] investigated a related problem corresponding to a piecewise constant velocity in the equation $u_t + (V(x)f(u))_x = 0$. They constructed a weak solution, and proposed another condition, called minimum jump, to ensure uniqueness.

Next, we show that the construction above is well-defined. Let $u_1 < u_2 < \cdots < u_M$ be the values u can take. A curve of discontinuity of u is called a *shock front* or simply a front. A shock front where u has the value u_i on one side and u_j on the other side, is said to contain $|i-j|$ shock lines. The following result can be proved by slightly modifying the proof of Theorem 2.1 in Holden *et al.* [6]:

Lemma 2.2. *Let $\{\tau_i\}$ be all the points such that* $\operatorname{sgn} V(x,\tau_i^-)$ *is not equal* $\operatorname{sgn} V(x,\tau_i^+)$*. Let N be the number of intervals where f is linear. Let $L(t)$ be the number of shock lines in $u(\cdot,t)$ and $F(t)$ the number of shock fronts. Then the function*

$$G(t) = L(t)N + F(t)$$

is strictly decreasing for every shock collision in the intervals $t \in (\tau_i, \tau_{i+1})$.

The result means that the total number of shock fronts is bounded between each restart in the algorithm. If u_0 has a finite number of steps, and if f is linear on a finite number of intervals, $u(x,t)$ is constant on a finite number of domains. Suppose that $|V| < M$ and $|f'| < K$. Then the values of u_0 outside the interval $[x - MKt, x + MKt]$ does not influence $u(x,t)$. So, even for an infinite number of steps in u_0, the construction is well-defined provided the number of restarts is finite.

The solution satisfies a maximum principle by construction, $\|u\|_\infty \le \|u_0\|_\infty$. Furthermore, all waves have finite speed of propagation and the L^1-norm is therefore Lipschitz continuous, $\|u(t_2) - u(t_1)\|_1 \le C|t_2 - t_1|$. A refined argument shows that the total number of shock lines $L(t)$ in $u(\cdot,t)$, which is a measure on the total variation, is diminishing. When two or more fronts collide, at least three values must meet. Let $u_1,\dots,u_K$ be the values that meet in the collision. After the collision, there will be at least one new shock front, unless the values u_1 and u_K are equal. When $u_1 = u_K$, all the intermediate values disappear and the total variation decreases in the collision. If the collision produces new shocks, u_1 will be the value to the left of the leftmost shock and u_K the value to the right of the rightmost shock, and the values in between will form a monotone sequence. Accordingly, all of the values $u_2,\dots,u_{K-1}$ that are not between u_1 and u_K disappear and the total variation decreases. At each restart in the algorithm the wave structure may change, but this does not increase the total variation.

The observations can be summarized in the following lemma:

Lemma 2.3. *Suppose that f is continuous and piecewise linear, V is as in Lemma 2.1, and u_0 is a bounded step function of bounded variation. Then the solution of $u_t + V(x,t)f(u)_x = 0$ satisfies*

(i) *a maximum principle,* $\|u\|_\infty \le \|u_0\|_\infty$,
(ii) *Lipschitz continuity of the L^1-norm,* $\|u(t_2) - u(t_1)\|_1 \le C|t_2 - t_1|$,
(iii) *TVD-property,* $\mathrm{TV}(u) \le \mathrm{TV}(u_0)$.

In the general case we can approximate f by a piecewise linear function, u_0 by a piecewise constant function, and V by a piecewise linear or a piecewise constant function in x and t. This will give a family $\{u^\Delta(t)\}$ of approximate solutions. Now by using Lemma 2.3 and a standard application of Helly's theorem (see for instance Ref. [13]), we can show that there exists a convergent subsequence of $\{u^\Delta(t)\}$. Moreover, this sequence converges in $L^1_{\mathrm{loc}}(\mathbf{R}\times[0,T])$ for any $T>0$, and the limit $u(x,t)$ is the solution of (1). The above construction is valid for all velocities that are continuous in x and possibly discontinuous in t, since each shock path then is the solution of a Caratheodory differential equation. The velocity may be discontinuous in x as long as the sign is preserved across each discontinuity.

We can now prove that both the approximate solutions and the limit satisfy a Kružkov entropy condition. By construction, an approximate solution is constant within the subdomain bounded by the shock paths (and the lines $t=0$ and $t=\tau_i$). Choose a domain D such that it contains only one shock curve $x(t)$, see Figure 1. The solution is constant and therefore smooth in D, except along the curve $x(t)$. Assume that it attains the values u^+ and u^- in D^+ and D^-, respectively. Let ϕ be a test function with compact support in D. By construction, the solution then satisfies equation (3). The same is true for any constant k. Choosing $k > \sup u(x,t)$, gives $\mathrm{sgn}(u-k) = -1$ and $|u-k| = k-u$. The (in)equality is then derived by subtracting (3) with u from (3) with k

$$0 = \iint\limits_D \left\{ |u-k|\phi_t + \mathrm{sgn}(u-k)[f(u)-f(k)]\big(V(x,t)\phi\big)_x \right\} dt\,dx.$$

A similar argument is valid for $k < \inf u(x,t)$, and it remains to establish the inequality for $k \in [\inf u(x,t), \sup u(x,t)]$. We substitute $u=k$ in (4), multiply by $|V(x,t)|$, and use

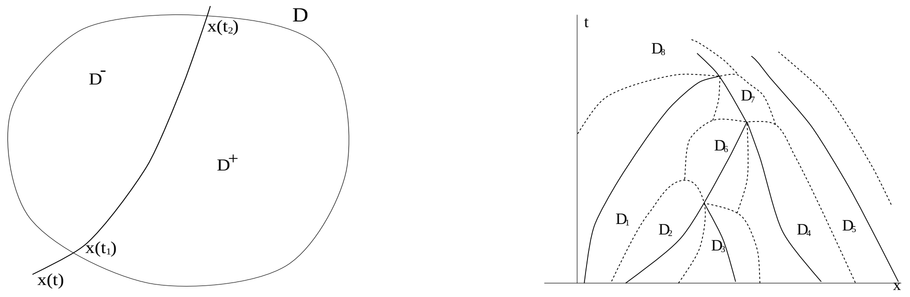

FIGURE 1. (Left) Domain for integration around a shock curve. (Right) The global domain divided into subdomains. The solid lines are shock curves and the dotted lines boundaries between subdomains.

the Rankine-Hugoniot condition (5). The entropy condition now reads

$$V(x,t)\frac{f(u^+)-f(k)}{u^+-k} \le \frac{dx}{dt} \le V(x,t)\frac{f(u^-)-f(k)}{u^--k}$$

for any constant k between u^+ and u^-. This inequality may be further rearranged to

$$\begin{aligned}-|u^+-k|\frac{dx}{dt} &+ \operatorname{sgn}(u^+-k)[f(u^+)-f(k)]V(x,t)\\ &\le -|u^--k|\frac{dx}{dt} + \operatorname{sgn}(u^--k)[f(u^-)-f(k)]V(x,t).\end{aligned}$$

We multiply by a test function, integrate the inequality along $x(t)$ from t_1 to t_2, and use the divergence theorem to derive the desired inequality on the domain D

$$\begin{aligned}0 &\le \int_{\partial D^-} \operatorname{sgn}(u^--k)\phi(x,t)\big\langle u^--k, [f(u^-)-f(k)]V(x(t),t)\big\rangle \cdot \vec{n}_{\partial D^-}\,ds\\ &+ \int_{\partial D^+} \operatorname{sgn}(u^+-k)\phi(x,t)\big\langle u^+-k, [f(u^+)-f(k)]V(x(t),t)\big\rangle \cdot \vec{n}_{\partial D^+}\,ds\\ &= \iint_D \big\{|u-k|\phi_t + \operatorname{sgn}(u-k)[f(u)-f(k)]\big(V(x,t)\phi\big)_x\big\}\,dxdt.\end{aligned}$$

Hence, entropy inequality is established on D.

The next step is to divide the global domain into a countable number of subdomains D_i, as in Figure 1, and use the above inequality for each subdomain. Note that the compact support of ϕ in D_i was superfluous, since contributions along the boundaries of subdomains cancel when added together. The only exceptions are along the lines $t = 0$ and $t = T$.

Hence, we have verified that the constructed solution satisfies the Kružkov condition

$$
0 \le \iint_{\mathcal{D}_T} \left\{ |u-k|\phi_t + \operatorname{sgn}(u-k)[f(u)-f(k)]\big(V(x,t)\phi\big)_x \right\} dt\,dx
+ \int_{\mathbf{R}} |u(x,0)-k|\phi(x,0)\,dx - \int_{\mathbf{R}} |u(x,T)-k|\phi(x,T)\,dx. \tag{6}
$$

By using the bounded convergence theorem for Lebesgue integrals [16], the same result follows in the general case if V is continuous in x. This gives an alternative proof of the existence of a solution of (1) satisfying the entropy condition (6).

3. Continuous Dependence

We now investigate how the variable coefficient equation (1) is sensitive to changes in flux function, velocity, and initial data. In other words, we determine the difference in L^1–norm for solutions of the two equations

$$
u_t + U(x,t)f(u)_x = 0, \qquad u(x,0) = u_0(x), \tag{7}
$$

$$
v_t + V(x,t)g(v)_x = 0, \qquad v(x,0) = v_0(x). \tag{8}
$$

The corresponding entropy conditions are on the form (6). Here v_0, V, and g are considered as perturbations of u_0, U, and f. Assume that U and V are continuous and possess derivatives U_x and V_x. Let $\mathcal{D}_T$ denote the domain $\mathbf{R}\times[0,T]$. Once we have establish continuous dependence, an error bound follows immediately as we will see in §4.

Let us first construct a suitable class of test functions. Assume that $\omega \in C_0^\infty(\mathbf{R})$ is a non-negative, even function such that $\int_{\mathbf{R}}\omega(x)dx = 1$, $\operatorname{sgn}(x)\omega'(x) \le 0$, and $\omega(x)\equiv 0$ for $|x|\ge 1$. Then $\int_{\mathbf{R}}|x|\,|\omega'(x)|dx = 1$. For any $\varepsilon > 0$ define the function $\omega_\varepsilon(x) = \varepsilon^{-1}\omega(x/\varepsilon)$. Then there is a constant M_ω, independent of ε, such that $|\omega_\varepsilon(x)| \le M_\omega/\varepsilon$ and $|\omega_\varepsilon'(x)| \le M_\omega/\varepsilon^2$ for all x. Based on ω_ε, we construct test functions

$$
\phi^\varepsilon = \phi^\varepsilon(x-\xi, t-\tau) = \omega_\varepsilon(x-\xi)\,\omega_\varepsilon(t-\tau).
$$

For later use, introduce the following shorthand notations

$$
F(w,z) = \operatorname{sgn}(w-z)[f(w)-f(z)],
$$

$$
\Gamma_\varepsilon(w,z) = \iint_{\mathbf{R}^2} |w(x)-z(y)|\omega_\varepsilon(x-y)\,dx\,dy.
$$

Choose test function ϕ^ε and constant $v(y,\tau)$ in the entropy condition for (7). After integration with respect to τ and y, it reads

$$
0 \le \iint_{\mathcal{D}_T}\iint_{\mathcal{D}_T} \left\{ |u(x,t)-v(y,\tau)|\phi_t^\varepsilon + F(u(x,t),v(y,\tau))\big(U(x,t)\phi^\varepsilon\big)_x \right\} dt\,dx\,d\tau\,dy
+ \int_0^T \Gamma_\varepsilon(u(\cdot,0),v(\cdot,\tau))\omega_\varepsilon(\tau)\,d\tau - \int_0^T \Gamma_\varepsilon(u(\cdot,T),v(\cdot,\tau))\omega_\varepsilon(T-\tau)\,d\tau.
$$

Similarly, let the entropy condition for (8) be given in the variables (y,τ), choose $k = u(x,t)$, and integrate with respect to x and t to get a similar inequality. Note that $\phi_x^\varepsilon = -\phi_y^\varepsilon$ and $\phi_t^\varepsilon = -\phi_\tau^\varepsilon$. Taking this and the symmetry of ϕ^ε into account, we add the two integrated entropy conditions to obtain

$$\begin{aligned}
&\int_0^T \Gamma_\varepsilon(u(\cdot,T),v(\cdot,\tau))\omega_\varepsilon(T-\tau)\,d\tau + \int_0^T \Gamma_\varepsilon(v(\cdot,T),u(\cdot,t))\omega_\varepsilon(T-t)\,dt \\
&\quad\le \int_0^T \Gamma_\varepsilon(u(\cdot,0),v(\cdot,\tau))\omega_\varepsilon(0-\tau)\,d\tau + \int_0^T \Gamma_\varepsilon(v(\cdot,0),u(\cdot,t))\omega_\varepsilon(0-t)\,dt \\
&\qquad + \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T} \phi_x^\varepsilon \big[F(u(x,t),v(y,\tau))U(x,t) - G(u(x,t),v(y,\tau))V(y,\tau)\big]\,dtdxd\tau dy \\
&\qquad + \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T} \phi^\varepsilon \big[F(u(x,t),v(y,\tau))U_x(x,t) + G(u(x,t),v(y,\tau))V_y(y,\tau)\big]\,dtdxd\tau dy.
\end{aligned}$$

The inequality above may be written as

$$L_1(\varepsilon) + L_2(\varepsilon) \le R_1(\varepsilon) + R_2(\varepsilon) + R_3(\varepsilon) + R_4(\varepsilon). \tag{9}$$

The dependence result will follow by bounding the terms as ε tends to zero.

Starting with the left-hand side, we can show that

$$\lim_{\varepsilon\to 0} L_1(\varepsilon) = \frac{1}{2}\int_{\mathbf{R}} |u(x,T)-v(x,T)|\,dx.$$

First, recall that ω_ε has compact support for $|x|\le\varepsilon$, and integrates to unity. Hence it is obvious that $\int_0^T \omega_\varepsilon(T-t)\,dt = 1/2$. Then observe that

$$\begin{aligned}
&\Big| L_1(\varepsilon) - \frac{1}{2}\int_{\mathbf{R}} |u(x,T)-v(x,T)|\,dx \Big| \\
&\quad \le \frac{M_\omega}{\varepsilon}\int_{\mathbf{R}}\int_{T-\varepsilon}^{T} |v(y,\tau)-v(y,T)|\,dyd\tau + \frac{1}{2}\frac{M_\omega}{\varepsilon}\int_{\mathbf{R}}\int_{x-\varepsilon}^{x+\varepsilon} |v(x,T)-v(y,T)|\,dydx.
\end{aligned}$$

Using a lemma proved by Kružkov [9, Lemma 2] we see that both terms on the right-hand side of the inequality tend to zero with ε. A similar argument may be used to prove that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\big[L_1(\varepsilon)+L_2(\varepsilon)\big] &= \int_{\mathbf{R}} |u(x,T)-v(x,T)|\,dx = \|u(\cdot,T)-v(\cdot,T)\|_1, \\
\lim_{\varepsilon\to 0}\big[R_1(\varepsilon)+R_2(\varepsilon)\big] &= \int_{\mathbf{R}} |u(x,0)-v(x,0)|\,dx = \|u_0-v_0\|_1.
\end{aligned}$$

For $R_4(\varepsilon)$ we observe that

$$\begin{aligned}|R_4(\varepsilon)| \leq& \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T} \phi^\varepsilon |f(u(x,t)) - f(v(y,\tau))||U_x(x,t)|\,dtdxd\tau dy \\ &+ \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T} \phi^\varepsilon |g(u(x,t)) - g(v(y,\tau))||V_y(y,\tau)|\,dtdxd\tau dy \\ \leq& \big(\|f\|_{\mathrm{Lip}}\|U_x\|_\infty + \|g\|_{\mathrm{Lip}}\|V_x\|_\infty\big) \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T} \phi^\varepsilon |u(x,t) - v(y,\tau)|\,dtdxd\tau dy\end{aligned}$$

and repeat the argument used for $L_1(\varepsilon)$ to yield the upper bound

$$\lim_{\varepsilon\to 0} |R_4(\varepsilon)| \leq \big(\|f\|_{\mathrm{Lip}}\|U_x\|_\infty + \|g\|_{\mathrm{Lip}}\|V_x\|_\infty\big) \int_0^T \|u(\cdot,t) - v(\cdot,t)\|_1\,dt.$$

The remaining term $R_3(\varepsilon)$ must be split into two terms $R_{31}(\varepsilon)+R_{32}(\varepsilon)$, one with a difference in velocities and one with a difference in flux functions. There are two possible ways of rearranging the integrand

$$FU - GV = F(U-V) + (F-G)V = G(U-V) + (F-G)U.$$

Here we choose the first form. Because the two velocities U and V in $R_{31}(\varepsilon)$ are evaluated at different variables (x,t) and (y,τ), this term must be further rearranged. Since ω_ε has compact support, it is obvious that $\int_{\mathbf{R}} h(y)\omega_\varepsilon'(x)dx = 0$ for any integrable function h. Thus $R_{31}(\varepsilon)$ may be rewritten as follows

$$\begin{aligned}R_{31}(\varepsilon) &= \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T} F(u(x,t),v(y,\tau))[U(x,t) - V(y,\tau)]\phi^\varepsilon_x(x-y,t-\tau)\,dtdxd\tau dy \\ &= \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T} \big[F(u(x,t),v(y,\tau))[U(x,t) - V(y,\tau)] \\ &\qquad\qquad - F(u(y,t),v(y,\tau))[U(y,t) - V(y,\tau)]\big]\phi^\varepsilon_x(x-y,t-\tau)\,dtdxd\tau dy.\end{aligned}$$

The integrand (except the test function) can be manipulated to yield three terms

$$\begin{aligned}F(x,t,y,\tau)&[U(x,t) - V(y,\tau)] - F(y,t,y,\tau)[U(y,\tau) - V(y,\tau)] \\ &= F(x,t,y,\tau)[U(x,t) - U(y,t)] \\ &\quad + [F(x,t,y,\tau) - F(y,t,y,\tau)][(U(y,t) - U(y,\tau)) + (U(y,\tau) - V(y,\tau))],\end{aligned}$$

and the integral is split in three parts

$$R_{31}(\varepsilon) = R_{311}(\varepsilon) + R_{312}(\varepsilon) + R_{313}(\varepsilon).$$

Using the symmetry, the integrand of $R_{31}(\varepsilon)$ could have been rearranged using V instead of U. The first term $R_{311}(\varepsilon)$ is straightforward to dominate

$$\begin{aligned}|R_{311}(\varepsilon)| &\leq \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T} |F(u(x,t),v(y,\tau))|\,|U(x,t)-U(y,t)|\,|\phi^\varepsilon_x(x-y,t-\tau)|\,dt\,dx\,d\tau\,dy\\ &\leq \|U_x\|_\infty\|f\|_{\mathrm{Lip}}\iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T} |u(x,t)-v(y,\tau)|\,|x-y|\,|\phi^\varepsilon_x(x-y,t-\tau)|\,dt\,dx\,d\tau\,dy.\end{aligned}$$

Recall that $\int_{\mathbf{R}}|x||\omega_\varepsilon'(x)|dx=1$ and that $\phi^\varepsilon_y=-\phi^\varepsilon_x$. Thus

$$\int\limits_{\mathbf{R}}|u(x,t)-v(x,t)|dx=\int\limits_{\mathbf{R}}\iint\limits_{\mathcal{D}_T}|u(x,t)-v(x,t)|\,|x-y|\,|\phi^\varepsilon_x(x-y,t-\tau)|\,dx\,d\tau\,dy.$$

It is obvious that $|x-y|\,|\phi^\varepsilon_x(x-y,t-\tau)|\leq M_\omega/\varepsilon$, since ω_ε is of compact support in the interval $[-\varepsilon,\varepsilon]$, and the derivative is bounded by M_ω/ε^2. This is all we need in order to repeat the argument for $L_1(\varepsilon)$ and obtain

$$\lim_{\varepsilon\to 0}|R_{311}(\varepsilon)|\leq\|U_x\|_\infty\|f\|_{\mathrm{Lip}}\int_0^T\|u(\cdot,t)-v(\cdot,t)\|_1\,dt.$$

If the function $f(u)$ satisfies a Lipschitz condition on the interval $[-M,M]$ with constant L, then the function $F(u,v)=\mathrm{sgn}(u-v)[f(u)-f(v)]$ also satisfies the Lipschitz condition in u and v with the constant L, see [9, Lemma 3]. Hence it follows that

$$\begin{aligned}|R_{312}(\varepsilon)| &\leq \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T}|U(y,t)-U(y,\tau)|\,|F(u(x,t),v(y,\tau))-F(u(y,t),v(y,\tau))|\,|\phi^\varepsilon_x|\,dt\,dx\,d\tau\,dy\\ &\leq \sup_{\tau,t<T,|t-\tau|<\varepsilon}\|U(\cdot,t)-U(\cdot,\tau)\|_\infty\|f\|_{\mathrm{Lip}}\iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T}|u(x,t)-u(y,\tau)|\,|\phi^\varepsilon_x|\,dt\,dx\,d\tau\,dy\end{aligned}$$

since $\omega_\varepsilon(t-\tau)$ is zero when $|t-\tau|\geq\varepsilon$. For the integral $I(\varepsilon)$, we use that $\omega_\varepsilon(x)$ and $|x|\,|\omega_\varepsilon'(x)|$ integrate to unity and find that

$$\begin{aligned}I(\varepsilon) &\leq \int\limits_{\mathbf{R}}\iint\limits_{\mathcal{D}_T}|u(x,t)-u(y,t)|\,|\omega_\varepsilon(x-y)_x|\,dx\,dy\,dt\\ &= \iint\limits_{\mathcal{D}_T}|\omega_\varepsilon'(z)|\,|z|\int\limits_{\mathbf{R}}\frac{|u(x,t)-u(x-z,t)|}{|z|}\,dx\,dz\,dt\ \leq\ T\,\mathrm{TV}(u_0).\end{aligned}$$

Now introduce the shorthand $\nu_T(U)=\sup_{0<t<T}\|\|U(\cdot,t^+)-U(\cdot,t^-)\|_\infty$. Thus we have

$$\lim_{\varepsilon\to 0}|R_{312}(\varepsilon)|\leq\nu_T(U)\|f\|_{\mathrm{Lip}}T\,\mathrm{TV}(u_0).$$

Note that if $U(\cdot,t)$ is continuous, $R_{312}(\varepsilon)\to 0$.

We have now established the dependence on initial data. Suppose that $u(x,t)$ and $v(x,t)$ are two solutions of (1), with initial data u_0 and v_0. Let $\{\tau_i\}$ denote the discontinuities in the velocity, that is, $U(x,\tau_i^-) \neq U(x,\tau_i^+)$ for $i>0$, and let $\tau_0=0$. Then for $t\in[\tau_i,\tau_{i+1}\rangle$

$$\|u(\cdot,t)-v(\cdot,t)\|_1 \leq \|u(\cdot,\tau_i^+)-v(\cdot,\tau_i^+)\|_1 + \gamma\int_{\tau_i^+}^t \|u(\cdot,s)-v(\cdot,s)\|_1\,ds$$

for a finite constant $\gamma = 3\|V_x\|_\infty\|f\|_{\mathrm{Lip}}$. Applying a Gronwall inequality gives

$$\|u(\cdot,t)-v(\cdot,t)\|_1 \leq e^{\gamma(t-\tau_i^+)}\|u(\cdot,\tau_i^+)-v(\cdot,\tau_i^+)\|_1.$$

This inequality can be used to prove uniqueness of the solution. Obviously

$$\|u(\cdot,\tau_i^+)-u(\cdot,\tau_i^-)\|_1 = 0,$$

and by choosing equal initial data $u_0=v_0$, the uniqueness follows by induction on i.

To establish the continuous dependence on flux function and velocity, the two remaining terms $R_{313}(\varepsilon)$ and $R_{32}(\varepsilon)$ need to be bounded. Fortunately, most of the work has already been done

$$\begin{aligned}
|R_{313}(\varepsilon)| \leq& \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T} |U(y,t)-V(y,\tau)|\,|F(u(x,t),v(y,\tau))-F(u(y,t),v(y,\tau))|\,|\phi_x^\varepsilon|\,dt\,dx\,d\tau\,dy\\
\leq& \|U-V\|_\infty\|f\|_{\mathrm{Lip}}\iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T}|u(x,t)-u(y,t)|\,|\omega_\varepsilon(x-y)_x|\,\omega_\varepsilon(t-\tau)\,dt\,dx\,d\tau\,dy\\
\to& T\|U-V\|_\infty\|f\|_{\mathrm{Lip}}\,\mathrm{TV}(u_0).
\end{aligned}$$

Using once more that $\int_{\mathbf{R}} h(y)\omega_\varepsilon'(x)dx = 0$, it follows that

$$\begin{aligned}
R_{32}(\varepsilon) =& \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T}(G-F)(u(x,t),v(y,\tau))V(y,\tau)\phi_x^\varepsilon(x-y,t-\tau)\,dt\,dx\,d\tau\,dy\\
=& \iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T}\phi_x^\varepsilon\big\{(G-F)(u(x,t),v(y,\tau))-(G-F)(u(y,t),v(y,\tau))\big\}V(y,\tau)\,dt\,dx\,d\tau\,dy\\
\leq& \|f-g\|_{\mathrm{Lip}}\|V\|_\infty\iint\limits_{\mathcal{D}_T}\iint\limits_{\mathcal{D}_T}|u(x,t)-u(y,t)|\,|\omega_\varepsilon(x-y)_x|\omega_\varepsilon(t-\tau)\,dt\,dx\,d\tau\,dy\\
\to& T\|f-g\|_{\mathrm{Lip}}\|V\|_\infty\,\mathrm{TV}(u_0).
\end{aligned}$$

All bounds can be summarized in the inequality

$$\|u(\cdot,T)-v(\cdot,T)\|_1 \leq \|u_0-v_0\|_1 + \gamma\int_0^T\|u(\cdot,t)-v(\cdot,t)\|_1\,dt + \lambda T\,\mathrm{TV}(u_0),$$

where

$$\begin{aligned}
\gamma =& 2\|U_x\|_\infty\|f\|_{\mathrm{Lip}} + \|V_x\|_\infty\|g\|_{\mathrm{Lip}},\\
\lambda =& \|f\|_{\mathrm{Lip}}\big(\|U-V\|_\infty+\nu_T(U)\big) + \|V\|_\infty\|f-g\|_{\mathrm{Lip}}.
\end{aligned}$$

As pointed out above, there are two ways of rearranging $R_3(\varepsilon)$, and then two ways of rearranging $R_{31}(\varepsilon)$. Altogether, this gives four combinations, of which we have computed one explicitly here. The other three give similar estimates. To get a sharper bound, one should therefore choose the minimum of the four.

Either way, the inequality is on the form $E(T) \leq E(0) + \gamma \int_0^T E(t)dt + B$. Applying a Gronwall type inequality gives $E(T) \leq e^{\gamma T}(E(0) + B)$, and we arrive at our final result concerning the continuous dependence

Theorem 3.1. *Let u and v denote two entropy solutions of* (7) *and* (8), *respectively. Suppose that U_x and V_x are bounded, and that f and g satisfy a Lipschitz condition. Then there are constants γ and λ such that*

$$\|u(\cdot,T) - v(\cdot,T)\|_1 \leq e^{\gamma T}\|u_0 - v_0\|_1 + \lambda T e^{\gamma T}\min\{\mathrm{TV}(u_0), \mathrm{TV}(v_0)\},$$

for every finite T.

The above result can be extended to velocities that are only Lipschitz continuous in x by using a smooth approximation. When $V(x,t) = U(x,t) \equiv 1$, γ is zero and λ is equal to $\|f - g\|_{\mathrm{Lip}}$, and the above inequality is simplified to that of Lucier [12, Theorem 3]. Moreover, by a tedious computation, the above stability result can be extended to multi-dimensional equations of the form

$$u_t + \sum_{i=1}^{d} V_i(x,t) f_i(u)_{x_i} = 0, \qquad u(x,0) = u_0(x). \tag{10}$$

The multi-dimensional counterpart of the above bound is,

$$\|u(\cdot,T) - v(\cdot,T)\|_1 \leq \|u_0 - v_0\|_1 + \gamma \int_0^T \|u(\cdot,t) - v(\cdot,t)\|_1 \, dt + \lambda C(T)\,\mathrm{TV}(u_0),$$

where

$$\gamma = \sum_{i=1}^{d} 2\|\partial_{x_i} U_i\|_\infty \|f_i\|_{\mathrm{Lip}} + \|\partial_{x_i} V_i\|_\infty \|g_i\|_{\mathrm{Lip}},$$

$$\lambda = \sum_{i=1}^{d} \|f_i\|_{\mathrm{Lip}}\big(\|U_i - V_i\|_\infty + \nu_T(U_i)\big) + \|V_i\|_\infty \|f_i - g_i\|_{\mathrm{Lip}}.$$

Note that these equations have bounded, but not necessarily diminishing total variation. Therefore the constant $T\,\mathrm{TV}(u_0)$ is replaced by $C(T)\,\mathrm{TV}(u_0)$.

4. A Front Tracking Method

In this section we will look further into how the construction from §2 can be developed into a numerical method. Recall that the shock paths satisfy the ordinary differential equation $\dot{x}(t) = sV(x,t)$, which can be solved explicitly if the approximation to V is chosen carefully (see §2). Suppose now that the velocity is given by $V(x,t) = (a_i x + b_i)c_j$ or $V(x,t) = a_i(c_j t + d_j)$ on the squares $[x_i, x_{i+1}] \times [t_j, t_{j+1})$, which are chosen such that V has constant sign on each square. Hence, the times $\{\tau_k\}$ where the algorithm is restarted (because V changes sign) forms a subset of $\{t_i\}$. Although a piecewise constant approximation in x and t is not covered by the analysis in §3, it is possible from a constructive point of view and is in fact used in the commercial reservoir simulator *Frontline* by Technical Software Consultants [1].

When implementing the front tracking method, all shock fronts are represented as data objects in a linked list [15, 10] given by their spatial ordering from left to right. Similarly, all possible collisions are kept in a list. The two lists are established initially by solving all Riemann problems given by the initial approximation and then computing the possible interactions. In the tracking part of the algorithm each step goes as follows: We find the next colliding fronts in the time list, remove them from the spatial list, and solve the corresponding Riemann problem. Possible new fronts are then inserted into the spatial list, and the collision list is updated.

The same algorithm will be used in our new method. However, there are two complicating factors. We see that a new calculation of the future path is required each time a shock enters another interval in V. So first, how do we compute the path of fronts passing through several spatial intervals in the velocity approximation? Somehow we must know when a front leaves a spatial interval and enters a new one. This is achieved by introducing artificial stationary 'fronts' to represent discontinuities in V or V_x. Thus, during the tracking of fronts there will be two kinds of collisions: (i) between two shock fronts, and (ii) between a shock front and a discontinuity in V_x of V. For type (ii) collisions we exchange places for the colliding fronts in the spatial list and update the collision list. Introducing artificial fronts is also advantageous if we were to consider boundary value problems.

The second problem is what to do when we reach a new temporal interval in the approximation of the velocity. The obvious solution would be to simply restart the algorithm at each time $t = t_j$. However, we can be more clever. After each update in the time list we check if the next collision occurs in the current time interval $[t_j, t_{j+1})$. If not, we advance the solution forward in time to $t = t_{j+1}$ and compute the new velocity approximation. If the velocity does not change sign anywhere, we need only reset the start positions of each front and update the collision list using the new velocity. Notice that we do not need to solve any Riemann problems. However, if t_{j+1} coincides with a τ_k, we have to restart the algorithm, by solving the Riemann problems described by the piecewise constant solution at time $t = t_{j+1}^-$.

In §2 we argued that there is a finite number of shock collisions if the number of initial jumps is finite. Assuming a finite number of intervals in the approximation of the velocity, the number of updates to front paths remains finite. The efficiency of front tracking is retained if the additional work introduced by these extra collisions amounts little to the total work during the tracking. Obviously this is not the case if the length of the velocity intervals is small compared to the distance each shock travels before a collision. It is therefore important that intervals in the velocity approximation are fairly large, reducing the number of type (ii) shocks.

So, what about the convergence rate of the new front tracking method? Assume that $V = V(x)$ and let u denote the entropy solution of (1) and $\tilde{u}$ the front tracking solution with flux function $\tilde{f}$, velocity $\tilde{V}$, and initial data $\tilde{u}_0$. Here Δ_f, Δ_V, and Δ_0 denotes the length of the respective intervals. Then it follows from Theorem 3.1 that

$$\|u(\cdot,T)-\tilde{u}(\cdot,T)\|_1 \le C_1\|u_0-\tilde{u}_0\|_1 + C_2\|f-\tilde{f}\|_{\mathrm{Lip}} + C_3\|V-\tilde{V}\|_\infty,$$

where the constants C_1, C_2, and C_3 are finite and independent of the parameters Δ_f, Δ_V, and Δ_0. By a proper choice of the initial approximation, for instance the average function on each interval, we can ensure that $\|u_0-\tilde{u}_0\|_1 \le C\Delta_0$. Assuming that f and V are Lipschitz continuous and piecewise C^2, we have that $\|f-\tilde{f}\|_{\mathrm{Lip}} \le \frac{1}{2}\Delta_f\|f''\|_\infty$ and $\|V-\tilde{V}\|_\infty \le C\Delta_V^2$. Thus, the error is dominated in norm as follows

$$\|u(\cdot,T)-\tilde{u}(\cdot,T)\|_1 \le C_1\Delta_0 + C_2\Delta_f + C_3\Delta_V^2.$$

Hence, the convergence of the front tracking method remains linear when extended to the variable coefficient equation. It is also interesting to note that the convergence is quadratic in the approximation parameters for the velocity. This means that the velocity intervals may be chosen large to ensure an efficient algorithm.

5. Numerical Experiments

In this section the behaviour of the front tracking method is illustrated by three simple examples. More examples can be found in Refs. [7, 8, 11].

Example 1. Consider Burgers' equation with a time dependent velocity field

$$u_t + V(t)\left(\tfrac{1}{2}u^2\right)_x = 0$$

and piecewise constant initial data $u_0(x) = I_{[0,\infty)}(x) - I_{[1/2,\infty)}(x)$. The velocity field is $V(t) = \cos(\pi t/T)$ for $t \in [0,T]$. The computational domain is $[-1.5, 1.5]$, the initial data is $u_0(x) = I_{[-0.5,0.5]}(x)$ (i.e, equal 1 for $x \in [-0.5, 0.5]$ and zero otherwise), and $T = 4.0$.

Figure 2 shows snapshots of the solution computed with a piecewise constant approximation of the velocity (on 8 time intervals) and exact initial data. Only ten breakpoints are used in the flux function to clearly demonstrate how rarefaction waves are approximated by the method. Compared with a reference solution generated by a Lax–Friedrichs scheme (CFL number 0.95), we see that the front tracking solution is very accurate, even for this relatively poor approximation of the velocity. The singular point $t = 2.0$ where $V(t) = 0$ represents a problem in the finite difference solver. Using too small time steps around $t = 2.0$ gives considerable numerical diffusion, whereas too large time steps gives a delayed propagation of the two rarefaction waves. In fact, we needed an inordinately large number of grid cells to get satisfactory resolution. The front tracking algorithm on the other hand has none of these difficulties.

Example 2. Next, consider a nonconvex flux function $f(u) = 4u^2(1-u^2)$ and strongly non-monotone initial data $u_0(x) = \sin(\pi x)I_{[-5,5]}(x)$ on the domain $x \in [-6, 6]$. The velocity $V(x) = 0.75\sin(\pi x/2)$ changes sign at $x_n = 2n$, $n \in \mathbf{Z}$, and these points will therefore be 'stagnation points'. Figure 3 shows the solution at time $t = 1.0$ computed with 240 intervals in the initial data, 60 breakpoints in the piecewise linear velocity, and 50 breakpoints in

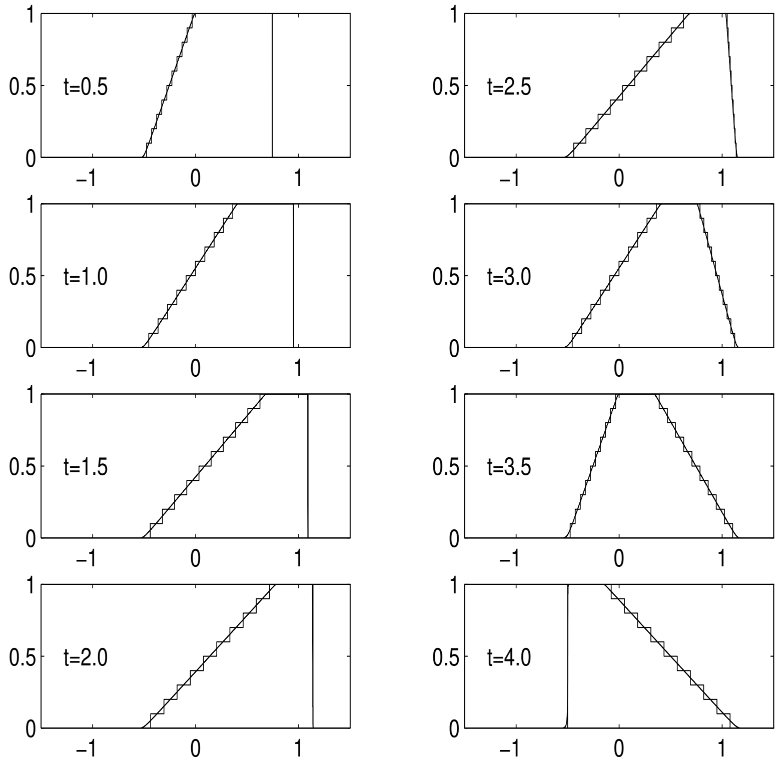

FIGURE 2. Example 1. Snapshots of the solution compared with a Lax-Friedrichs solution on a very fine grid (12 000 grid cells).

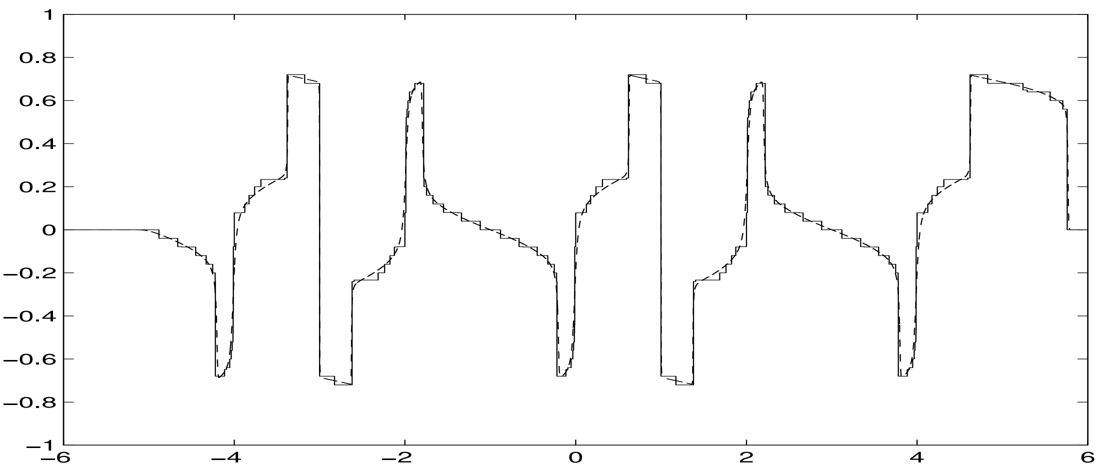

FIGURE 3. Example 2. Front tracking solution (solid) at time $t = 1.0$ compared with a Lax-Friedrichs solution (dashed) on a very fine grid (24 000 grid cells).

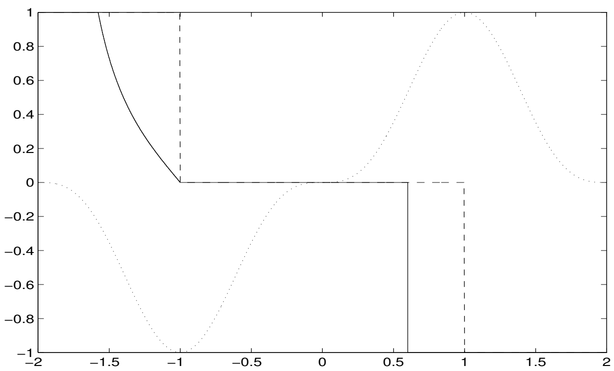

FIGURE 4. Example 3. Solution at time $t = 1.0$ (solid), initial data (dashed), and velocity field (dotted).

n	l^1-error	rate
3	5.39e-02	—
4	1.07e-02	2.33
5	2.74e-03	1.97
6	6.71e-04	2.03
7	1.68e-04	2.00
8	4.16e-05	2.01
9	1.01e-05	2.05

TABLE 1. Example 3. Observed error and convergence rates.

the flux function. The solution is compared with a fine grid solution computed by a Lax–Friedrichs scheme (CFL number 0.95). Once again, the front tracking solution is very accurate, especially at the shocks.

Example 3. We now wish to investigate the error contribution from the velocity approximation. To this end, we Burgers' equation with use exact initial data $u_0(x) = I_{(\infty,-1]}(x) - I_{[1,\infty)}(x)$ and a very fine approximation of the flux function (500 breakpoints). For the velocity field $V(x) = \sin^3(\pi x/2)$ we use a piecewise linear approximation with $2^n + 1$ breakpoints. The computational domain is $[-2, 2]$. Figure 4 shows the initial data, the velocity field, and the solution at time $t = 1.0$. The observed errors (with respect to a fine grid solution) and the corresponding convergence rates are shown in Table 1. We see that the observed rates agree with the second order convergence proved in §4.

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