COHOMOLOGY OF ORDERED BRATTELI DIAGRAMS

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Abstract. Given a simple ordered Bratteli diagram, B, with multiple maximal paths and minimal paths, and the Natural Extension system (X, T) generated by this diagram, there is a compact topological graph G which gives the following exact sequence:

$$0 \longrightarrow H^0(G) \longrightarrow K_0(B) \longrightarrow H^1(X, T) \longrightarrow H^1(G) \longrightarrow 0$$

where $H^*(G)$ is reduced Cech cohomology, $K_0(B)$ is the classical dimension group defined by B, and $H^1(X, T)$ is ordered continuous integer dynamical cohomology which is equated with the ordered K-theory $K_0(C(X) \times \mathbf{Z})$. The central arrow is a positive order embedding whose kernel is contained in the infinitessimals of $K_0(B)$, allowing a connection to be made with Putnam's exact sequence. A more detailed analysis determines effectively the order structure on $K_0(C(X) \times \mathbf{Z})$ and we deduce a positive order-embedding injection $\tilde{K}_0(B) \longrightarrow \tilde{K}_0(C(X) \times \mathbf{Z})$ which examples show is not always onto. The main technique used is an expression of ordered cohomology in terms of orientation and directed graphs due to Poon. This extends the existing work of the author and completes its application to substitution minimal systems. As a new example, we compute the ordered integer cohomology, in all cases, of minimal 0-1 valued generalised Morse sequences.

§1 **Introduction** The symbolic sequences, one-sided or two-sided, used to define many classical minimal expansive dynamical systems are often constructed or decomposed inductively. Among the many examples which can be described this way are substitution sequences [**Ma1**] [**Qu**], Toeplitz sequences [**JK**], Sturmian sequences [**HM**] and generalized Morse sequences [**Ke**] [**Ma2**] (this is only an incomplete list of primary references). Furstenberg gives a inductive view of the general recurrent symbolic sequence, [**F**] (Chapter 1 §3), which is close to the picture we have in mind here.

All such constructions describe inductively a finite concatenation of a collection of finite words, whose length and/or number grows, converging in the limit to an infinite sequence of symbols. The component words are often defined using a previous stage of the construction. At any stage we may stop the process and view what we have as the collection of words together with a description of how we concatenate them; thus the data necessary for the construction of the next inductive step are separated into two categories:

words and links. We may continue to separate these two categories of information as we proceed from step to step, and although the information about links at one stage is mostly absorbed into the information about words at the next stage, there can remain in the limit a residue of information exclusively about the links.

We illustrate this with the classical example of the Thue-Morse sequence [**Mo**]: Here we iterate the substitution $0 \mapsto 01$, $1 \mapsto 10$, starting, say, with the symbol 0, producng the infinite sequence 011010011001... whose combinatorics and symbolic dynamics have been studied and generalised widely, e.g. [**Ke**] [**Ma1**] [**Ma2**] [**Qu**]. At some advanced stage of iteration the result is a concatenation of smaller words, for example: 0110 1001 1001.... With this choice illustrated there are only two kinds of word-unit but all possible pairwise concatenations are found (there are still significant restrictions on the triples of words permitted). This description continues to hold when we pick units of twice the size to break up the original sequence: 01101001 10010110.... So we bound the number of word-units at the expense of allowing freedom with the links. And this freedom does not disappear in the limit; the word unit, $w = 01101001..0110$, of length 2^n, continues to be preceded or succeeded by either w *or* its complement no matter how large n is. On the other hand, other inductive decompositions can be found in which there are more words involved but greater restriction (in the limit) on the concatenations permitted.

The choice between those descriptions which restrict words and those which restrict links has been made each way: in the *adic systems* of Vershik [**V**] the choice favours restricting the words and, since the freedom of links can often be identified as a "measure zero problem", this choice has proved most effective in the description of measure-preserving dynamics (see [**LV**]): in the *ordered Bratteli diagrams* of Putnam etal. [**Pu**] [**HPS**], the choice has favoured restricting the links so much that there is, in the limit, no ambiguity possible in the concatenations of words; and this strictness seems necessary if we are to describe the topological dynamics faithfully.

Experience tells us that, as far as topological dynamics is concerned, the building blocks of the adic systems (the words) are described directly, forming an *improper Bratteli diagram*, but that we risk losing topological information by ignoring the links. On the other hand, the (*proper*) Bratteli diagrams represent the topological dynamics faithfully, but are sometimes difficult to construct.

This disparity appears when we look at orbital dynamics and the ordered cohomological or K-theoretical invariants of orbit equivalence, [**GPS**]. For example, the improper diagram for the Thue-Morse sequence has $\mathbf{Z}[1/2]$ as its group of coinvariants, whereas the proper diagram gives the true value: $\mathbf{Z}[1/2] \oplus \mathbf{Z}$.

It is the purpose of the present article to reunite these descriptions by showing how the information about links preserved in the limit augments the information from the improper Bratteli diagram, retrieving the dynamical orbit structure encoded in the proper

Bratteli diagram. This completes the work started in [**Fo**]. The aim is practical: we wish to compute the ordered (continuous integer) cohomology of a symbolic system using just its improper diagram and the link information.

The tactic of the argument is as follows: We encode the information about links in a compact topological space, G, constructed as an inverse limit of directed graphs, which we use to correct the comparison between the dimension group produced by the improper diagram, written $K_0(B)$, and the ordered cohomology or K-theory of the dynamical system, (X,S), written $H^1(X,S)$ or $K_0(C(X)\times_S \mathbf{Z})$ (see [**Po**] [**Pu**] for the proof of equivalence of these two ordered groups). For example, the following result, a generalisation of Theorem 9 of [**Fo**], determines the continuous integer dynamical cohomology, the state space and some of the order structure. The proof of this is a combination of Corollaries 5.8, 6.14 and Theorem 7.1.

Theorem 1.1 *Suppose that B is a simple ordered Bratteli diagram and that $(\Pi(B),S)$ is its Natural Extension System (see (2.8)), then there is a compact topological graph, G, and a long exact sequence*

$$H^0(G) \xrightarrow{\delta^*} K_0(B) \xrightarrow{q^*} K_0(C(\Pi(B))\times_S \mathbf{Z}) \longrightarrow H^1(G)$$

in which q^* is an order embedding whose kernel is contained in the infinitessimals of $K_0(B)$. Further, q^* induces an affine homeomorphism between the state spaces of $K_0(B)$ and $K_0(C(\Pi(B))\times_S \mathbf{Z})$ •

Descriptions of G and its cohomology are given in sections 4 and 5; and we can see now that the subgroup Q defined in [**Fo**] is better described as $H^0(G)$. By an elaboration of the argument, the order structure of the cohomology is completely decided in sections 6 and 7.

We note also that the exact sequence of Theorem 1.1 becomes, in the right context, Putnam's exact sequence [**Pu**]. Here we can use his construction of a Bratteli diagram for a minimal dynamical system, (X,S), whose set of maximal paths is the closed set $Y\subset X$. In this case the space G is built from a pair of disjoint closed subsets, $Y\cup Y'$, with $Y'=SY$, with intervals connecting $y\in Y$ with Sy. G is homeomorphic to $Y\times[0,1]$ and $H^*(G)=H^*(Y)$ therefore. So $H^1(G)=0$, since Y is a closed subset of a Cantor set, and $\mathbf{Z}\to C(Y:\mathbf{Z})\to H^0(Y)$ is a short exact sequence since we work with reduced cohomology. Theorem 1.1 implies

$$0\to \mathbf{Z}\longrightarrow C(Y:\mathbf{Z})\longrightarrow K_0(B)\longrightarrow K_0(C(X)\times_S \mathbf{Z})\to 0$$

with the order structure properties required.

The techniques employed here are mainly combinatorial and topological. We exploit a graphical description of the dynamics and its cohomology made originally by Parry and Tuncel [**PT**] and developed to account for ordered cohomology by Poon, [**Po**]: positive functions and ordered cohomology have topological counterparts in orientability and orientation preservation. This has been developed to more general settings by recent work of Boyle and Handelman, [**BH**], but the conditions assumed by Poon are sufficient here.

In this paper we keep the constructions at the level of topological spaces as long as possible and in this way our approach differs in detail from the foregoing references. The advantage here is that we may exploit classical exact sequences in order to shorten combinatorial arguments. Also we keep an intuitively pleasing picture of cohomology elements as maps from the space to the circle $\mathbf{T}$, equated by homotopy equivalence, and of homology elements as integrations around loops; correspondences shown by Poon to respect the orientation structure nicely (see Lemma 2.7). Such an overview puts the intricate calculations found in [**Fo**] onto a more canonical footing.

The sections are organised as follows: In section 2 we review the connections between ordered cohomology and the orientation of topological graphs, and we define the Natural Extension System of a Bratteli diagram. Section 3 reviews the oriented topological spaces that are constructed naturally from dynamical systems. Section 4 examines the construction of section 3 as applied to the Natural Extension System and makes rigorous the idea of separating the word information from the link information outlined at the beginning of this introduction. Sections 5 to 7 make progressively more detailed computation of the ordered cohomology and state space of the Natural Extension, compared always with the ordered dimension group $K_0(B)$. Section 8 completes the case of substitution minimal systems where we agree with Host [**Ho**] that the dimension group for a substitution system is determined most easily using the 2-symbol substitution constructed from it [**Qu**]. Section 9 describes in detail the calculations of the ordered cohomology for generalized Morse sequences and we show that whenever the dynamics is not uniquely ergodic it supports exactly two ergodic probability measures. These dynamical systems have, as yet, no general construction of a Proper Bratteli diagram.

This note assumes familiarity with basic algebraic topology and some definitions used in graph theory: see [**Sp**], [**Mn**] and [**B**]. More importantly we assume without comment the definitions and constructions associated with ordered Bratteli diagrams, see [**Pu**] and [**HPS**] for the original constructions, and [**Fo**], [**Ho**] and [**DHS**] for relevant overviews and further definitions. The last two references describe elegant algorithms which produce the ordered K-theory of substitution systems.

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§**2 Order and Topology** We start with some basic definitions and results about ordered groups, all to be found in [**E**] or [**Bl**]:

Definitions 2.1 Recall the definitions of *ordered group*, (A, A^+) with *order unit*, $\mathbf{1}$, in [**HPS**] or [**E**] for example. Namely: A is abelian and countable: $A^+ \cap (-A^+) = \{0\}$, $A^+ + A^+ \subset A^+$ and $A^+ - A^+ = A$: For all $a \in A$, there is an N such that $N\mathbf{1} - a \in A^+$ (written $N\mathbf{1} \geq a$).

A classical example is $\mathbf{Z}^d$ with the order: $(v_1, .., v_d) \geq 0$ if and only if $a_j \geq 0$ for all j. This we call the *quadrant order* on $\mathbf{Z}^d$. The unit is defined to be $(1, 1, ..1)$.

A homomorphism $h : A \longrightarrow B$ of ordered groups is *positive* if $a \geq 0$ implies $h(a) \geq 0$. It is *strictly positive* if the same holds with strict inequality. Note that a positive map, h, for which $h(\mathbf{1}_A) = 0$ must be zero.

A group which is a direct limit of ordered groups with unit $A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow ...$ with positive connecting maps has a canonical order which makes it an ordered group (the unit is $\mathbf{1}_1 \in A_1$).

If each of the groups in this limit A_n are $\mathbf{Z}^{d_n}$ with quadrant order and unit as above, then the limit is said to be a *dimension group*.

An ordered group is *simple* if, for each $a > 0$, there is an N such that $Na \geq \mathbf{1}$. It is *unperforated* if $na > 0$ implies that $a > 0$. It has the *strict Reisz interpolation property* if $a, b < c, d$ implies there is an $e \in A$ such that $a, b < e < c, d$.

A homomorphism $h : A \longrightarrow B$ is an *order embedding* if $h(a) > 0$ implies that there is an $a' \in A$ such that $a' > 0$ and $h(a') = h(a)$. This is not logically connected with positivity without further assumptions.

If A and B have order units, $\mathbf{1}_A$ and $\mathbf{1}_B$ respectively, then h is *unital* if $h(\mathbf{1}_A) = \mathbf{1}_B$.

The *infinitessimals* of A are defined $Inf(A) = \{a \in A : na \leq 1 \ \forall \ n \in \mathbf{Z}\}$. Write $\tilde{A}$ for the ordered group with unit $A/Inf(A)$.

A *state* on A is a positive unital homomorphism from A to $\mathbf{R}$ (with usual order and unit 1). The set of such, $\Sigma(A)$, is a closed Choquet simplex in the space of all homomorphisms $A \longrightarrow \mathbf{R}$ with weak* topology.

Lemma 2.2 *a/ (see* **[Bl]***) Suppose that A is a simple dimension group, then $a > 0$ if and only if, for all $s \in \Sigma(A)$, $s(a) > 0$. Moreover, $Inf(A) = \{a \in A : s(a) = 0 \ \forall \ s \in \Sigma(A)\}$.*

b/ **[EHS]** *A simple ordered group with unit, $A \neq 0, \mathbf{Z}$, is a dimension group if and only if it is unperforated and has the strict Reisz interpolation property.* •

Ordered cohomology The class of spaces which will concern us will be the topological realisation of finite directed graphs (with simple loops admitted, but multiple edges prohibited unless their sense is opposite), see **[B]**. In a CW complex description, each vertex is a 0-cell and each edge an oriented 1-cell, see **[Mn]**. Following closely Poon's definitions and analysis, **[Po]**, we adopt a notion of orientation which, in this low dimensional context, has its intuitive meaning and requires little further explanation.

Definitions 2.3 Given two finite directed graphs a *map*, $\phi : J \longrightarrow I$, is a continuous function between the topological realisations of the graphs. ϕ is a *cell map* if it sends each vertex (0-cell) of J to a vertex of I. ϕ is a *graph map* if it is a cell map for which the pre-image of a vertex of I is a union of vertices of J.

The circle, $\mathbf{T}$, has a canonical respresentation as a directed graph which consists of a single 0-cell and a single directed 1-cell.

A map $p : [0,1] \longrightarrow J$, to a directed graph, J, is *orientation preserving* if it represents a movement of a point along a portion of a sequence of directed edges in J always in the positive sense, but not necessarily strictly so.

A map $\phi : J \longrightarrow I$ between two directed graphs is *orientation preserving* if, for every orientation preserving map $p : [0,1] \longrightarrow J$, the map $\phi \circ p$ is orientation preserving to I. It is enough to check those p which represent motion at unit speed along single edges of J.

For any graphical space, J, there is a canonical orientation preserving graph map $\pi : J \longrightarrow \mathbf{T}$ which maps each 0-cell of J onto the 0-cell of $\mathbf{T}$ and which winds each 1-cell of J once positively around $\mathbf{T}$ at unit speed.

In the case that there is a map, $\phi : \mathbf{T} \longrightarrow J$ which is onto and orientation preserving, we say that ϕ is a *global orientation* on J. It is clear that a connected directed graph whose edges are covered by directed cycles (i.e. each edge is a member of a directed cycle) has a global orientation, and conversely.

Given a directed graph, J, the first cohomology group $H^1(J)$ is equated with the set (group) of homotopy classes of maps from J to $\mathbf{T}$. We distinguish carefully between maps and homotopy classes of maps: the homotopy class of a map $g : J \longrightarrow \mathbf{T}$ is written $\mathbf{g}$ and this latter only is considered an element of $H^1(J)$.

Those homotopy classes, elements of $H^1(J)$, which contain an orientation preserving representative will be termed the *positive elements of* $H^1(J)$. The set of such positive elements is written $H^1(J)^+$.

Lemma 2.4 *Each element of $H^1(J)$ has a representative which is a cell map $J \longrightarrow \mathbf{T}$.*

Each element of $H^1(J)^+$ has a representative which is an orientation-preserving cell map $J \longrightarrow \mathbf{T}$. •

The following can be extracted from [**Po**] and [**BH**].

Lemma 2.5 *Suppose that J is a directed graph with a global orientation and that $H^1(J)$ is its cohomology with positive elements $H^1(J)^+$, then*

i/ As a group $H^1(J) = \mathbf{Z}^d$ for some $d \in \mathbf{N}$, $d > 0$.

ii/ $(H^1(J), H^1(J)^+, \pi^*(1))$ is an unperforated ordered group with unit. •

Definition 2.6 The homology group $H_1(J)$ becomes naturally dual as an ordered group to $H^1(J)$ in the following way: $H_1(J)$ is equated with the abelianised fundamental group and so, for each element of $H_1(J)$, we find representatives among maps $\mathbf{T} \longrightarrow J$ (many of which are homotopy inequivalent of course). We note the homology classes which contain an orientation-preserving representative and call these the *positive elements of* $H_1(J)$: $H_1(J)^+$.

If $g : J \longrightarrow \mathbf{T}$ and $\gamma : \mathbf{T} \longrightarrow J$ are two maps, then we may compose them: $g \circ \gamma : \mathbf{T} \longrightarrow \mathbf{T}$. We write $[g, \gamma]$ for the homotopy class of $g \circ \gamma$; it is independent of the H^1 representative chosen for g and the H_1 representative chosen for γ. Thus we get a map $H^1 \otimes H_1 \longrightarrow [\mathbf{T}, \mathbf{T}] = \mathbf{Z}$, which we can write in inner-product notation $\mathbf{g} \otimes \gamma \mapsto [\mathbf{g}, \gamma]$.

If $h : J \longrightarrow \mathbf{T}$ is a cell map, then we write $||h||_\infty$ for the smallest number $r > 0$ such that h wraps each edge of J around $\mathbf{T}$ at most r times, positively or negatively. This is a function of maps and not of cohomology classes.

With such an h we may define $|h|$ as the cell map which wraps each edge of J around $\mathbf{T}$ the same number of times as did h, but always in the positive direction.

If $\gamma : \mathbf{T} \longrightarrow J$ is a map, then we define $||\gamma||_1 = \sup\{|[h, \gamma]|/||h||_\infty : \quad h : J \longrightarrow \mathbf{T}$ non-null cell map $\}$. This has an obvious interpretation in terms of the length of γ etc.

We draw the first part of the following lemma from [**Po**]. The second part is direct and together the two parts show the duality of homology and cohomology as ordered groups. The last part is by definition:

Lemma 2.7 *Suppose the data of Lemma 2.5, then*

i/ for $\mathbf{g} \in H^1(J)$,

a/ $\mathbf{g} = 0$ if and only if $[\mathbf{g}, \gamma] = 0$ for all $\gamma \in H_1(J)$, and

b/ $\mathbf{g} \in H^1(J)^+$ if and only if $[\mathbf{g}, \gamma] \geq 0$ for all $\gamma \in H_1(J)^+$.

ii/ for $\gamma \in H_1(J)$,

c/ $\gamma = 0$ if and only if $[\mathbf{g}, \gamma] = 0$ for all $\mathbf{g} \in H^1(J)$, and *d/* $\gamma \in H_1(J)^+$ *if and only if* $[\mathbf{g}, \gamma] \geq 0$ *for all* $\mathbf{g} \in H^1(J)^+$.

iii/ For $g : J \longrightarrow \mathbf{T}$, *a cell map, and* $\gamma : \mathbf{T} \longrightarrow J$, *we have* $||[g, \gamma]|| \leq ||g||_\infty ||\gamma||_1$. •

Bratteli diagrams We outline briefly the notation for the Bratteli diagrams needed in later sections, but see [**HPS**] for the details of the construction which will often be assumed.

Definitions 2.8 (from [**E**] [**HPS**] and [**Fo**]) A *Bratteli diagram* is an infinite locally finite graph whose vertices are partitioned into an infinite number of levels, W_n $n = 0, 1, 2, ...$, each of which is finite. W_0 is a single point. Edges, which can be multiple, can pass only between consecutive levels. Each edge from W_n to W_{n+1} has a *source* (its end in W_n) and a *range* (its end in W_{n+1}).

Write $K_0(B)$ for the dimension group classically defined from B.

An *ordered Bratteli diagram* is a Bratteli diagram with a partial order on the set of edges which compares two edges if and only if they share their range. This is sometimes called the *Vershik order* [**V**]. We may define therefore *maximal edges* and *minimal edges* and note that each vertex (except W_0) is the range of precisely one of each type. Define $Max(B)$ to be the union of all the maximal edges in B and $Min(B)$ to be the union of all the minimal edges. Let $Max_n(B)$ be the set of all finite paths in $Max(B)$ which start at level 0 and end at level n. Likewise define $Min_n(B)$.

This order defines a partial order on the (Cantor) space, $P(B)$, of all infinite paths; and on those elements of $P(B)$ which have an immediate successor, we may define a map, V, whose image is that successor: the *Vershik successor map.* Those elements of $P(B)$ on which V is not defined are called *maximal paths*, a set written $Max_\infty(B)$, and those elements on which V^{-1} is not defined are called *minimal paths*: $Min_\infty(B)$.

If $p \in P(B)$ is maximal and $q \in P(B)$ is minimal and there is a sequence of paths, $p_n \to p$ with $Vp_n \to q$, then we write $p \sim q$.

The *natural extension* of the ordered Bratteli diagram B is a dynamical system consisting of the space

$$\Pi(B) = \{(p_n) \in P(B)^{\mathbf{Z}} : p_{n+1} = Vp_n \text{ or } \ p_n \sim p_{n+1}\}$$

together with the coordinate shift, S.

An (ordered) Bratteli diagram is *simple* if it can be telescoped (see [**HPS**]) so that each vertex at each level is connected to every vertex in the next level. Further telescoping does not change this fact, and we shall often assume that a simple Bratteli diagram has been telescoped enough for this and other specified conditions to occur.

Lemma 2.9 *Suppose that* B *is an ordered Bratteli diagram, then* $Max(B)$ *is a tree which spans the vertices of* B. *The elements of* $P(B)$ *all of whose edges are contained in* $Max(B)$ *are precisely the elements of* $Max_\infty(B)$. *Likewise for minimal paths.*

Also, if B is simple, then $(\Pi(B), S)$ is a minimal Cantor system (see (3.1)). •

§3 **Dynamical Systems** We now turn to the directed graphs which arise naturally from dynamical systems. Much of what is mentioned here is to be found in [**Po**].

Definition 3.1 Suppose that X is a Cantor set and that S is a homeomorphism of X onto itself. The pair (X, S) is called a *Cantor system.* It is *minimal* if every orbit is dense.

Given a Cantor system (X, S), define the *mapping torus*:

$$M(X,S) = (X \times [0,1])/\langle (x,1) \simeq (Sx,0) \rangle$$

A sequence of clopen partitions of X, $\mathcal{P}_n$: $n = 0, 1, 2...$, *refines to points* if, for each $n \geq 1$, $\mathcal{P}_n$ refines $\mathcal{P}_{n-1}$ and the maximal diameter of elements of $\mathcal{P}_n$ tends to zero as n tends to infinity. We define $\mathcal{P}_0$ to be the trivial partition by convention.

Poon gives a useful construction of the mapping torus in general, which is also related closely to the *Graphe des mots* due originally to Rauzy. See [**Ho**], [**AR**].

Construction 3.2 Suppose (X, S) is a Cantor system and that $\mathcal{P}_n$ is a sequence of partitions which refines to points. For given n, define the directed graph, J_n, which has vertices the atoms of $\mathcal{P}_n$ and a directed edge from $U \in \mathcal{P}_n$ to $V \in \mathcal{P}_n$ if $SU \cap V \neq \emptyset$. Since the partitions form a refining sequence, there is a unique orientation preserving graph map $\alpha_n : J_{n+1} \to J_n$.

By convention, the graph J_0 is a single vertex with a single directed loop.

Lemma 3.3 *Given the construction above, the inverse limit* $\lim_\alpha J_n \equiv M(X, S)$*. Further, any different sequence of partitions produces an inverse limit sequence of oriented spaces which is intertwined with the J_n sequence by orientation preserving maps.*

Proof Define a map f_n from $X \times [0, 1]$ to J_n first in the following way: Suppose that $x \in U_0 \in \mathcal{P}_n$ and $Sx \in U_1 \in \mathcal{P}_n$. Let $f_n(x, 0) = U_0$ and $f_n(x, 1) = U_1$ considered as vertices in J_n. To determine $f_n(x, t)$ for $0 < t < 1$, we interpolate by travelling at unit speed along the edge directed from vertex U_0 to vertex U_1 (in the positive sense of direction).

These f_n are consistent with the inverse limiting procedure in the sense that the following diagram commutes:

$$\begin{array}{ccc} X \times [0,1] & = & X \times [0,1] \\ \downarrow f_n & & \downarrow f_{n+1} \\ J_n & \xleftarrow{\alpha} & J_{n+1} \end{array}$$

Furthermore, they are each continuous from $X \times [0,1]$ onto J_n. Thus the limit map to $\lim J_n$ is onto and it is clear that the quotient of the domain needed to obtain a 1-1 map is precisely the mapping torus construction. Hence we have the homeomorphism needed.

The intertwining of another sequence, $\mathcal{Q}_n$ say, follows by considering a sequence of refinements $\mathcal{P}_0 < \mathcal{Q}_{n_1} < \mathcal{P}_{m_1} < \mathcal{Q}_{n_2} <$ .. which can always be found, and taking the inverse limit which it defines. •

Definition 3.4 We note the phenomenon of *pro-homotopy equivalence.* This relates two inverse limits of compact topological spaces, $...A_n \longleftarrow A_{n+1}...$ and $...B_n \longleftarrow B_{n+1}...$, if there is a set of maps back and forth between the sequences of spaces,

$$\begin{array}{ccccccc} ... \leftarrow & A_{n_1} & \leftarrow ... \leftarrow & A_{n_2} & \leftarrow ... \leftarrow & A_{n_3} & \leftarrow ... \\ & \uparrow & & \downarrow & & \uparrow & \\ ... \leftarrow & B_{m_1} & \leftarrow ... \leftarrow & B_{m_2} & \leftarrow ... \leftarrow & B_{m_3} & \leftarrow ... \end{array}$$

such that any pair of map-compositions with same range and same domain are homotopy equivalent.

We say that two spaces are *pro-homotopy equivalent* if each can be described as an inverse limit and these limits are pro-homotopy equivalent in the first sense.

If all the spaces in the sequence are directed graphs and the intertwining maps are orientation preserving, then we say that the limit spaces are *oriented pro-homotopic.*

Lemma 3.5 *Suppose that* $... \leftarrow A_n \leftarrow A_{n+1} \leftarrow ...$ *and* $... \leftarrow B_n \leftarrow B_{n+1} \leftarrow ...$ *are two inverse limits which are pro-homotopy equivalent, then the direct limits* $H^*(A) = \lim H^*(A_n)$ *and* $H^*(B) = \lim H^*(B_n)$ *are isomorphic as groups.*

If, in addition, the A_n *and* B_n *are all finite directed graphs then* $H^1(A)$ *and* $H^1(B)$ *are isomorphic as ordered groups.* •

Definition 3.6 With the J_n above, note that the maps $\alpha_n^* : H^1(J_n) \to H^1(J_{n+1})$ send positive elements to positive elements. Lemmas 3.3 and 3.5 allow us to define $H^1(M(X,S)) = \lim_{\alpha^*} H^1(J_n)$ as an ordered group independently of the choice of partition decomposition. Furthermore, the element $\mathbf{1}$ in $H^1(J_0) = \mathbf{Z}$ is distinguished as the order unit, making $H^1(M(X,S))$ an ordered group with unit (Definition 2.1). Note that the equality as groups is automatic with Čech cohomology.

We extract the following results from [**Po**], [**Pu**] and [**HPS**]:

Theorem 3.7 *Suppose that* (X,S) *is a minimal Cantor dynamical system, then*

i/ $H^1(M(X;S))$ *is a simple dimension group with order unit* $\mathbf{1}$.

ii/ $K_0(C(X) \times_S \mathbf{Z}) = H^1(M(X,S)) = C(X;\mathbf{Z})/\langle f - Sf : f \in C(X;\mathbf{Z})\rangle$ *as ordered groups with unit, the first group being ordered C^* K-theory of the crossed product, the last group being the classical coinvariants of the continuous integer-valued functions on X.* •

Remarks 3.8 These results, modified appropriately, have been obtained in greater generality in [**BH**].

It is of some interest to note a particularly simple form of the graphs J_n in the case that we have a Kakutani-Rohlin tower decomposition for the dynamics, as is always the case for a minimal system (X,S). Using the constructions of [**Pu**] we have a sequence of partitions, $\mathcal{Q}_n$, which refine to points and whose atoms can be grouped into towers related by iterations of S, and with further essential properties which we shall assume here.

From this we form a slightly cruder partition, $\mathcal{P}_n$, by uniting into a single atom all those atoms of $\mathcal{Q}_n$ which are the base of a tower. If the construction has been arranged so that the limit of the base atom is a single point (as can be done with minimal systems [**Pu**]), then $\mathcal{P}_n$ is a sequence of partitions which refines to points. It is then clear that the graph , J_n, formed from each $\mathcal{P}_n$, is a collection of oriented cycles overlapping only at one point (the base atom). Thus $H^1(J_n)$ is $\mathbf{Z}^k$, where k is the number of loops, with quadrant order and unit (Definition 2.1).

The projection mapping between consecutive graphs is simply described in terms of oriented wrappings of individual loops. This makes clear the equality between the cohomology of the mapping torus and the dimension group of the Bratteli diagram defined in [**Pu**].

§**4 Three topological spaces** Here we adapt the results and constructions of the last section to the picture drawn in the introduction. The idea is to treat the maximal and minimal paths in the Bratteli diagram as the delimiters of the inductive building-blocks in the dynamics of the Natural Extension. The ways in which we may pass from maximal to minimal paths, by approximations of the Vershik map, give the information about links which we make precise in the following definitions.

Definitions 4.1 Let G_n be a topologically realised directed bipartite graph whose vertex set is the formal disjoint union of $Max_n(B)$ and $Min_n(B)$ and whose edges connect $p \in Max_n(B)$ with $q \in Min_n(B)$ if there are infinite paths, $p', q' \in P(B)$ which agree with p and q respectively up to level n, and which have $q' = Vp'$. Such an edges is directed from p to q.

Note that the map, τ, which transforms a finite path in B ending at level $n+1$ to a path ending at level n simply by omitting the last edge, is a graph map: $\tau : G_{n+1} \to G_n$, not necessarily onto.

Let G'_n be the graph with the same vertex set as G_n, but with an edge joining $p \in Max_n(B)$ to $q \in Min_n(B)$ if p and q share their end point in level n of B. Such an edge is directed from q to p.

Now let M_n be the directed graph with the same vertex set as G_n and whose set of directed edges is the union of the sets of directed edges of G_n and G'_n.

Lemma 4.2 *If the Bratteli diagram, B, is simple and sufficiently telescoped, then each M_n is connected and has an edge cover by directed cycles and so is a directed graph with global orientation.*

M_n decomposes into two bipartite graphs: G_n and its complement G'_n in which the direction of edges in each passes consistently from one part to the other. Moreover, G'_n is a disjoint union of edges (a matching). •

Definitions 4.3 Let $L_n = M_n/G_n$ and write $q : M_n \to L_n$ for the quotient map. Note that L_n is a collection of $|W_n|$ circles (recall that W_n is the set of vertices at level n in B) which are joined together at a point (a wedge) and that L_n inherits a directed structure from M_n. L_n may be considered as the graph G'_n with all its vertices equated.

In this case, the ordered group $H^1(L_n)$ is $\mathbf{Z}^{W_n}$ with quadrant order.

The ordered Bratteli diagram B defines, canonically up to homotopy, an orientation preserving map from L_n to L_{n-1} as follows: A vertex $v \in W_n$ defines a sequence of vertices in W_{n-1} by considering the ordered sequence of edges with range v and recording their source. Thus given a loop in L_n corresponding to vertex v in W_n, we can map it to the sequence of loops in L_{n-1} corresponding to this sequence of vertices. Call this map σ; it is a cell map.

Now we can state the main topological result:

Theorem 4.4 *Suppose that B is a simple ordered Bratteli diagram, sufficiently telescoped, and the notation above is assumed, then*

i/ In the sequence $G_n \xrightarrow{i} M_n \xrightarrow{q} L_n$, the inclusion, i, is a cofibration and q is natural quotient map.

ii/ The graph map τ and cell map σ can be mutually extended to give a cell map β such that the diagram

$$\begin{array}{ccccc} G_{n+1} & \xrightarrow{i} & M_{n+1} & \xrightarrow{q} & L_{n+1} \\ \downarrow \tau & & \downarrow \beta & & \downarrow \sigma \\ G_n & \xrightarrow{i} & M_n & \xrightarrow{q} & L_n \end{array}$$

commutes. Furthermore τ, σ and β are orientation preserving with respect to the defined orientations.

iii/ $G = \lim_\tau G_n$ *is a compact topological space which, if the number of maximal and minimal paths in* B *is finite, is homeomorphic to the max-to-min graph of* [**Fo**].

iv/ $\lim_\beta M_n$ *is oriented pro-homotopy equivalent to* $M(\Pi(B), S)$.

Proof Parts i/ to iii/ are uncomplicated from definitions (see [**Fo**] for part iii/ and [**Sp**] and [**Mn**] for part i/) and the observations above. For part ii/ we give a combinatorial description of β which will be used later:

We know that β restricts to τ on G_{n+1} so we now define it on edges of G'_{n+1}. Suppose that (q, p) is an edge of G'_{n+1}, so that $p \in Max_{n+1}(B)$ and $q \in Min_{n+1}(B)$ are finite paths in B which share an end vertex, v say, in level $n+1$. The edges for which v is a range are ordered consecutively by the Vershik order, $e_1, ..., e_r$, having source vertices in level n: $w_1, ..., w_r$ respectively. From each w_j there is exactly one element $q_j \in Min_n(B)$ and one element $p_j \in Max_n(B)$ which have w_j as their end vertex. Therefore, (q_j, p_j) is a directed edge in M_n.

Furthermore, for $1 \leq j \leq r-1$, (p_j, q_{j+1}) is a directed edge in M_n (indeed it is in G_n) for the following reason: Take an infinite path, p_o say, starting at v (as above) in level $n+1$ and proceeding arbitrarily through all higher levels. Let p' be the path $p_j \to e_j \to p_o$ and q' the path $q_{j+1} \to e_{j+1} \to p_o$, both elements of $P(B)$. By construction $Vp' = q'$ and we have the connection required by the definition of G_n.

Now map (p, q) onto the sequence of directed edges in M_n in order:

$$(q_1, p_1), (p_1, q_2), (q_2, p_2), .., (p_{r-1}, q_r), (q_r, p_r)$$

The precise rate at which these edges are covered is immaterial for the homotopy calculations ahead, but there are canonical uniform speed constructions which make the map definite if necessary. By construction, this β factors through q to the map σ as required.

iv/ A proof demands a description of the direct limit which gives $M(\Pi(B), S)$:

Given an infinite path, p, define p^n as the finite path which is the initial segment of p up to level n.

Suppose that $p = (p_n) \in \Pi(B)$ and define $R_n(p) = \{k \in \mathbf{Z} : p^n_k \in Min_n(B)\ \}$: By simplicity, this is a syndetic set whose maximal size of gap, a_n, and minimal size of gap, b_n, are independent of the choice of p. Also, by simplicity of the diagram, $b_n \to \infty$.

$\Pi(B)$ is a subset of $P^{\mathbf{Z}}$. The clopen partition, $\mathcal{P}_n$, of P, which discriminates according to the value of p^n, can be imposed on the 0th. coordinate of $P^{\mathbf{Z}}$ and is denoted by the same symbol with no confusion. So we define $\mathcal{Q}_n = \vee_{-c_n \leq j \leq c_n} S^j \mathcal{P}_n$, a clopen partition of $\Pi(B)$, where $c_n = [b_n/4]$. Since $\mathcal{P}_n$ refine to points on P and the $c_n \to \infty$, so $\mathcal{Q}_n$ refine to points in $\Pi(B)$.

We now apply Construction 3.2 to the sequence of partitions $\mathcal{Q}_n$.

There are three kinds of atom in $\mathcal{P}_n$:

(A+) those which correspond to an element of $Max_n(B)$:

(A–) those which correspond to an element of $Min_n(B)$:

(A0) and the rest.

This classification passes to $\mathcal{Q}_n$, where the construction of c_n and $\Pi(B)$ helps to simplify matters: Each atom in $\mathcal{Q}_n$ has the form $\cap_{-c_n \leq j \leq c_n} S^j U_j$ where the U_j are atoms of $\mathcal{P}_n$. There are four kinds of atom in $\mathcal{Q}_n$:

(C0) those where all U_j are all (A0):

(C+) those for which U_{c_n} is (A+) and all others (A0):

(C–) those for which U_{-c_n} is (A–) and all others (A0):

(C+–) and those in which there is a $j : -c_n \leq j < c_n$ so that U_j is (A+), U_{j+1} is (A–), and all others (A0).

In the graph J_n, the atoms in class (C0) are linked in directed paths of length at least $2c_n - 1$. Likewise the atoms in class (C+–). The directed paths of class (C0) are bounded by atoms of class (C–) at the beginning and class (C+) at the end. The directed paths of class (C+–) are bounded the other way about. So we see that the graph J_n decomposes into a collection of directed paths of type (C0) or (C+–) linked only at atoms of class (C+) or (C–).

An atom of class (C+) corresponds naturally with an element of $Max_n(B)$; take the element of $Max_n(B)$ which corresponds to U_{c_n}. Likewise an atom of class (C–) corresponds to an a element of $Min_n(B)$. So we have a 1-1 correspondence from the union of these two classes of vertices of J_n to the vertices of M_n.

Moreover, a directed path of atoms of class (C+–) corresponds naturally to an edge of G_n: namely take the edge in $G_n \subset M_n$ from $Max_n(B)$ to $Min_n(B)$ implied by the atoms U_j and U_{j+1} in the definition. Also a directed path of atoms of class (C0) corresponds naturally with the other edges of M_n: namely an element of class (C–) bounds the start of the path, and an element of class (C+) bounds the end; so take the edge in M_n from $Min_n(B)$ to $Max_n(B)$ implied by this.

This relates the graphs J_n to M_n by an orientation preserving homeomrphism: $\kappa : M_n \to J_n$ for every $n \geq 1$.

The two natural projections $\alpha : J_{n+1} \to J_n$ and β (of part ii/) do not intertwine κ perfectly, but rather only up to homotopy equivalence. Therefore κ is a oriented pro-homotopy equivalence as required. •

Finally we note:

Lemma 4.5 *Suppose that B is a simple ordered Bratteli diagram, sufficiently telescoped, then, for every n, M_n is connected.* •

§5 **Computing the cohomology groups** This section computes the unordered cohomology of the various spaces defined in the last section and in the process builds exact sequences which will be used in the next section to give the order structure.

Definition 5.1 In what follows we abbreviate $M(\Pi(B), S)$ as M, $\lim_\sigma L_n$ as L, and $\lim_\tau G_n$ as G.

Lemma 5.2 *Suppose that B is an ordered simple Bratteli diagram, then*

i/ If we define the unit in $H^1(L_0) = \mathbf{Z}$ as the element 1, then $\lim_{\sigma^*} H^1(L_n) = K_0(B)$ as an ordered group, with 1 as unit.

ii/ If we define the unit in $H^1(M_0) = \mathbf{Z}$ as the element 1, then $\lim_{\beta^*} H^1(M_n) = H^1(M(\Pi(B), S))$ as ordered groups with unit.

Consequently, $H^1(M)$ and $K_0(B)$ are simple unperforated dimension groups.

Proof i/ is direct from definition.

ii/ From Theorem 4.4 iv/, the map κ^* is the ordered group isomorphism.

At level 0, the spaces J_0 and M_0 are both circles and their order units coincide. The natural orientation preserving homeomorphism, $\kappa : J_0 \longrightarrow M_0$ fits in with the prevailing pro-homotopy equivalences defined in part iv/ of Theorem 4.4 and we deduce that κ^* is unital.

The final statement comes by combining Lemma 2.9 and Theorem 3.7. •

Lemma 5.3 *Suppose that B is an ordered simple Bratteli diagram, sufficiently telescoped. Then, for each n, there is a long exact sequence*

$$0 \longrightarrow H^0(G_n) \xrightarrow{\delta^*} H^1(L_n) \xrightarrow{q^*} H^1(M_n) \xrightarrow{i^*} H^1(G_n) \longrightarrow 0$$

in which we use reduced cohomology in dimension 0. Further, as n increments, this sequence intertwines the direct limit maps, τ^*, β^* and σ^*.

Proof The first part follows from the classical cofibration long exact sequence for cohomology (see [**Mn**]) applied to Theorem 4.4 i/. We note that, by Lemma 4.5, M_n is connected and so, in reduced cohomology, we have $H^0(M_n) = 0$ giving the left hand stop to the sequence claimed. The right hand stop is given by the fact that L_n is a wedge of circles having trivial second cohomology. The second part follows directly from Theorem 4.4 ii/. •

Remark 5.4 We note the particular form of the map $\delta^* : H^0(G_n) \longrightarrow H^1(L_n)$. Suppose that C is a connected component of G_n with vertices $C_1 \cup C_2$ the union representing the decomposition into elements of $Max_n(B)$ and $Min_n(B)$ respectively. Let D_1 be the collection of edges in G'_n which include an element of C_1 and let D_2 be similis for C_2. Let D'_j be the corresponding sets of edges in L_n. Let $d_C : L_n \longrightarrow \mathbf{T}$ be the map which wraps each element of $D'_1 \setminus D'_2$ once around $\mathbf{T}$ in a positive direction, and each element of $D'_2 \setminus D'_1$ once around $\mathbf{T}$ in the negative direction. So we define an element, $\mathbf{d}_C$ of $H^1(L_n)$.

The map $C \mapsto \mathbf{d}_C$ induces a map from unreduced cohomology of G_n to $H^1(L_n)$ therefore. It is not hard to see that the kernel of this map is a complimented copy of $\mathbf{Z}$ and so we deduce a 1-1 map δ^* from reduced cohomology which fits into the diagram above.

Construction 5.5 Here we adopt a good choice of basis for the group $H^1(G_n)$. Using this we will compute $H^*(G)$ and identify a map from $H^1(G_n)$ into $H^1(M_n)$ which will be used in the next section. The basic construction is to be found in [**B**].

Suppose that, for each n, F_n is a maximal spanning forest of G_n chosen in such a way that $F_n \subset \tau^{-1}(F_{n-1})$. This may be done inductively since τ is a graph map. Let F'_n be the collection of edges in $G_n \setminus F_n$.

Each element, e, of F'_n defines a map $h_e : G_n \longrightarrow \mathbf{T}$ which wraps the edge once positively around $\mathbf{T}$ and everything else to the base point. e also defines and is contained in a canonical cycle in G_n, $\gamma_e : \mathbf{T} \longrightarrow G_n$, all of whose edges are contained in F_n but for e itself. Note that $[h_e, \gamma_{e'}] = 1$ if $e = e'$, and 0 otherwise. Therefore $\{\mathbf{h}_e : e \in F'_n\}$ is a free set in $H^1(G_n)$. Since their number equals the rank of $H^1(G_n)$ (see [**B**]), the group they generate has finite index in G_n. However, the use of the orthogonality with the loops γ_e defined above, quickly shows that this index must equal 1.

Lemma 5.6 *As n increases there is a sequence of generating sets for $H^*(G_n)$ consistent with the maps τ^*, giving in the limit a finite or countable free abelian generating set for both $H^0(G)$ and $H^1(G)$.*

Such generators define, for each n a canonical injection $j_n^* : H^1(G_n) \longrightarrow H^1(M_n)$, a back map for $H^1(M_n) \xrightarrow{i^*} H^1(G_n)$ of Lemma 5.3.

Proof To deal with H^0 first: The maps τ are continuous and so define a natural connection between the components of G_n and the components of G_{n+1}: Each component of G_{n+1} is mapped to only one component of G_n. This describes the components of G effectively for calculation of unreduced $H^0(G)$: it is a limit of $\mathbf{Z}^k$ groups in which the connecting maps are represented by $0-1$ matrices with exactly one 1 in each column. This gives the required free limit in cohomology, reduced or unreduced.

For H^1 we use the basis for each G_n defined above: $\mathbf{h}_e : e \in F'_n$. By construction of the sets F_n, we see that if $e \in F'_n$ and $e' \in G_{n+1}$ is such that $\tau(e') = e$, then $e' \in F'_{n+1}$. Thus we may build a representative of the class $\tau^*\mathbf{h}_e$:

$$\tau' h_e = \sum_{e' \in G_{n+1} : \tau(e') = e} h_{e'}$$

This is a sum over e' chosen with multiplicity 1 from F'_{n+1}; and the indexing of these sums is disjoint for different e. Thus the matrix which, with respect to the bases defined above, records the map $H^1(G_n) \xrightarrow{\tau^*} H^1(G_{n+1})$ is zero everywhere but for at most one 1 in each column. This gives the desired result for $H^1(G)$.

We note that, if $e \in F'_n$, then, by injection of G_n into M_n, the map h_e may be extended as a map $j_n h_e : M_n \longrightarrow \mathbf{T}$, defining an element, $j_n^* \mathbf{h}_e$ of $H^1(M_n)$. So we define a map $H^1(G_n) \xrightarrow{j_n^*} H^1(M_n)$. It is clear that $i^* j_n^* = id$ and we deduce that j_n^* is injective, the kernel of i^* is a compliment of $Im\ j_n^*$ in $H^1(M_n)$. •

Corollary 5.7 *For each n, the following sequence is short exact*

$$H^0(G_n) \xrightarrow{\delta^* \oplus 0} H^1(L_n) \oplus H^1(G_n) \xrightarrow{q^* \oplus j_n^*} H^1(M_n)$$

•

The following was proved in [**Fo**] for the case that G was a finite graph and completes the theory of unordered K-theory for Natural Extensions of improper Bratteli diagrams:

Corollary 5.8 *As groups*

$$K_0(C(X) \times \mathbf{Z}) \equiv (K_0(B)/\delta^* H^0(G)) \oplus H^1(G)$$

Proof This combines the results of Corollary 5.7, Lemma 5.2 and Theorems 3.7 and 4.4 ii/. •

Remarks 5.9 Note that the maps j_n^* do not intertwine the operations of β^* and τ^*. This is because, if $g : G_n \longrightarrow \mathbf{T}$ is a cell map and is interpreted as a map $M_n \longrightarrow \mathbf{T}$, then the map $g \circ \beta$, from $M_{n+1} \longrightarrow \mathbf{T}$, has some support outside of G_{n+1}. Indeed this is the nature of the next definition:

Definition 5.10 Let $\overline{\sigma}_n^* : H^1(G_n) \longrightarrow H^1(L_{n+1})$ be defined on the generators of $H^1(G_n)$ as follows:

Let $e \in F'_n$ and consider $j_n h_e : M_n \longrightarrow \mathbf{T}$ the map extending h_e. Then $j_n h_e \circ \beta = \tau' h_e + g_e$ is a decomposition of the left hand side into two functions of disjoint support, where $\tau' h_e$ is defined in the proof of Lemma 5.6 and where g_e is supported on G'_{n+1}. Thus g_e is of the form $g'_e \circ q$ with $g'_e : L_{n+1} \longrightarrow \mathbf{T}$ and this defines a class, $\mathbf{g}'_e$, of $H^1(L_{n+1})$ which we shall define as the image of the element $\mathbf{h}_e$: $\overline{\sigma}_n^*(\mathbf{h}_e)$.

Since the $\mathbf{h}_e$ are free generators of $H^1(G_n)$, $\overline{\sigma}_n^*$ extends as a homomorphism:

Remark 5.11 We note that this construction depends somewhat on the particular generators for $H^1(G_n)$ used. This is also true of the maps j_n^* for exactly the same reason. However, we can retrieve a canonical expression:

Definition 5.12 Define $\theta_n^* : H^1(L_n) \oplus H^1(G_n) \longrightarrow H^1(L_{n+1}) \oplus H^1(G_{n+1})$ by the matrix

$$\begin{pmatrix} \tau^* & \overline{\sigma}_n^* \\ 0 & \sigma^* \end{pmatrix} : \mathbf{g} \oplus \mathbf{h} \mapsto (\overline{\sigma}_n^* \mathbf{h} + \sigma^* \mathbf{g}) \oplus \tau^* \mathbf{h}$$

Lemma 5.13 *The following diagram of short exact rows commutes:*

$$\begin{array}{ccccc} H^0(G_n) & \stackrel{\delta^*\oplus 0}{\longrightarrow} & H^1(L_n) \oplus H^1(G_n) & \stackrel{q^*\oplus j_n^*}{\longrightarrow} & H^1(M_n) \\ \downarrow \tau^* & & \downarrow \theta_n^* & & \downarrow \beta^* \\ H^0(G_{n+1}) & \stackrel{\delta^*\oplus 0}{\longrightarrow} & H^1(L_{n+1}) \oplus H^1(G_{n+1}) & \stackrel{q^*\oplus j_n^*}{\longrightarrow} & H^1(M_{n+1}) \end{array}$$

Proof This follows directly from the construction of $\overline{\sigma}^*$. •

Before we go on to describe the ordered group structure of this diagram we note a computational device which often simplifies the construction of $H^1(G)$.

Definition 5.14 Consider $G_n^\infty = \cap_{m \geq n} \tau_m^n(G_m)$ and note that τ restricts to give a graph map $\tau^\infty : G_{n+1}^\infty \longrightarrow G_n^\infty$ which is onto.

Lemma 5.15 *For each n, there is an m such that $G_n^\infty = \tau_m^n(G_m)$ and so the sequence $...G_n^\infty \stackrel{\tau^\infty}{\longleftarrow} G_{n+1}^\infty...$ is orientably pro-equivalent to the sequence for G as defined in (5.1). Moreover τ^∞ is onto and so we may compute $H^1(G) = \lim H^1(G_n^\infty)$ as an increasing union of groups.* •

For each n, G_n^∞ can be described in the following way: Recall the set of infinite maximal paths, $Max_\infty(B)$, and the set of infinite minimal paths, $Min_\infty(B)$. Consider the collection of finite paths in B which start at level 0 and end at level n and which are the intial segment of some eleent of $Max_\infty(B)$. This is a subset of $Max_n(B)$ which we shall write $Max_n'(B)$. Likewise, define $Min_n'(B)$ in terms of the initial segements of $Min_\infty(B)$.

Suppose that $p' \in Max_n'(B)$ and $q' \in Min_n'(B)$, initial segements of $p \in Max_\infty(B)$ and $q \in Min_\infty(B)$ respectively, then connect p' to q' if there is a sequence of infinite paths $p_n \to p$ on which the Vershik map is defined and for which $Vp_n \to q$.

Then the graph G_n^∞ is precisely the bipartite graph with vertices $Min'_n(B)$ and $Max'_n(B)$ and with these connections defined. This can be a much smaller and more easily described graph than G_n. (In **[Fo]**, in the context of substitution sequences, the graph G was labelled G' and the graph G^∞ was labelled G).

Without its topology therefore, G is a bipartite graph with vertices $Max_\infty(B) \cup Min_\infty(B)$ and connections as defined above. In the case that the graph is finite, then the topology can be recovered from a drawing, otherwise we should be careful of the nature of the limiting process.

§**6 The order structure** This section compares the properties of the product $[.,.]$ defined on $H^1(L)$ and on $H^1(M)$. These will give the order structure on $H^1(M)$ in terms of $H^1(L)$. First we adopt a useful notation.

Definition 6.1 Recall the maps $\beta : M_n \longrightarrow M_{n-1}$, τ and σ defined before with no confusion by the omission of the indexing variable. Here, for $m > n$, we write β_n^m for the compositions of the β maps which take $M_m \longrightarrow M_n$. Likewise we write $\tau_n^m : G_m \longrightarrow G_n$ and $\sigma_n^m : L_m \longrightarrow L_n$.

The maps $\beta_n^{m*} : H^1(M_n) \longrightarrow H^1(M_m)$ are defined in the same way, as are τ_n^{m*}, σ_n^{m*}, $\overline{\sigma}_n^{m*}$, θ_n^{m*}.

Lemma 6.2 *Suppose that B is an ordered simple Bratteli diagram, then there is a telescoping of B so that, for each n, the map $\beta : M_{n+1} \longrightarrow M_n$ has the following properties:*

i/ Each edge of G'_{n+1} is mapped by β onto the whole of M_n, indeed covering M_n at least 2 times.

ii/ Each edge of G_{n+1} is mapped by β (which is equal to τ in this case) 1-1 and onto a single edge of G_n

Moreover, futher telescoping does not affect these properties.

Proof Part ii/ is true by construction in any case, so we concentrate on part i/. We prepare B first of all by telescoping enough so that, for each $n \geq 1$, between each vertex at level n and each vertex in level $n-1$ there are at least 2 edges. This is a condition which is not altered by further telescoping.

To do this, we induct on n and note that the case $n = 0$ is trivial. So suppose that we have achieved the desired result to level n by telescoping B sufficiently.

Suppose that $\{p, q\}$ is an edge of G_n, where $p \in Max_n(B)$ and $q \in Min_n(B)$ and there are infinite paths $p', q' \in P(B)$ such that $q' = Vp'$ and p' and q' agree with p and q respectively up to level n. We note that the requirement $q' = Vp'$ implies that p' and q' are cofinal and we let $k(p, q)$ be the level $> n$ in B from which they continue to agree (the construction of $k(p,q)$ is not canonical, but we fix a choice to be definite). Let k be the maximum value of $k(p,q)$ where $\{p,q\}$ are chosen among the edges of G_n.

Suppose that $\{a,b\}$ is an edge in G'_{k+1}. Thus, refering to the original construction (4.1) and (4.3), a and b are finite paths in B, with $a \in Min_{k+1}(B)$ and $b \in Max_{k+1}(B)$, which share their end-point, v say, in level $k+1$. Suppose also that $\{p,q\}$ is an edge of G_n as in the previous paragraph. Then p' and q' share a vertex, w say, in level k to which v is attached by at least 2 edges. Using the description of β_n^{k+1} in the proof of Theorem 4.4 i/, we see that $\{a,b\}$ is mapped by β_n^{k+1} to cover $\{p,q\}$ at least 2 times.

A similar and quicker argument deals with the covering of edges of G'_n.

Now telescope B between levels n and $k+1$ to a single level, forming a new diagram, B', and note that this does not affect the truth, in B', of the inductive hypothesis previously held in B up to level n. However, B' does satisfy the hypothesis up to level $n+1$ now. •

Definition 6.3 Write $\mathbf{1}_0$ for the map $M_0 \longrightarrow \mathbf{T}$ whose class produces the order unit in $H^1(M_0)$. For $n \geq 1$, let $\mathbf{1}_n = \mathbf{1}_0 \circ \beta_0^n : M_n \longrightarrow \mathbf{T}$ be a representative of the class of the order unit $H^1(M_n)$.

Similarily define $\mathbf{1}'_n : L_n \longrightarrow \mathbf{T}$ a sequence of canonical represntatives of the order unit class as it appears in $H^1(L_n)$.

Thus for $m > n$, $\mathbf{1}_n \circ \beta_n^m = \mathbf{1}_m$ and $\mathbf{1}'_n \circ \sigma_n^m = \mathbf{1}'_m$. Also, by construction, $\mathbf{1}'_m \circ q = \mathbf{1}_m$ for all $m \geq 0$.

Recall also the maps $\pi_n : M_n \longrightarrow \mathbf{T}$ which map each edge of M_n once around $\mathbf{T}$ in a positive sense. Note the distinction $\pi_0 \simeq 2\mathbf{1}_0$ since M_0 has two edges.

Define π_n^G as the map $M_n \longrightarrow \mathbf{T}$ which wraps each edge of G_n once around $\mathbf{T}$ and sends all else to the base point; and define π_n^L as the map $: L_n \longrightarrow \mathbf{T}$ which wraps each loop of L_n once around $\mathbf{T}$.

Suppose that $g : M_n \longrightarrow \mathbf{T}$ and $C > 0$, then we write $g \geq C\pi_n$ to mean the phrase: g wraps each edge of M_n around $\mathbf{T}$ at least C times in a positive direction. Note that this is a comparison of maps which implies the corresponding comparison of cohomology classes: $\mathbf{g} \geq C\pi_n$ in $H^1(M_n)$.

Lemma 6.4 *Suppose that B has been telescoped according to the conclusion of Lemma 6.2, and that $\gamma : \mathbf{T} \longrightarrow M_n$ is an orientation preserving map. Then, as we trace around γ, the edges of G_n contained in γ alternate with the edges of G'_n contained in γ. Their number is the same therefore and $||\gamma||_1 = [\pi_n, \gamma] = 2[\pi_n^L \circ q, \gamma] = 2[\pi_n^G, \gamma]$.*

If $g : M_{n-1} \longrightarrow \mathbf{T}$ is orientation preserving, then $g \circ \beta_n^m \geq (2^{m-n} - 1)\pi_m$. In particular, $2^n\pi_n \leq \mathbf{1}_n$.

Proof The first part is immediate from Lemma 4.2. The second part comes from Lemma 6.2 i/. •

Constuction 6.5 We define a spacial realisation of the map, θ_m^*: Suppose that $f : M_n \longrightarrow \mathbf{T}$ is a cell map. For each $m \geq n$, write $f_m = f \circ \beta_n^m : M_m \longrightarrow \mathbf{T}$. Let g_m^o be the part of f_m supported on G'_m and h_m the part of f_m supported on G_m. Write $g_m^o = g_m \circ q$.

In cohomology, this construction is described alternatively using Lemma 5.13 via the formula: $\theta_m^*(\mathbf{g}_m \oplus \mathbf{h}_m) = \mathbf{g}_{m+1} \oplus \mathbf{h}_{m+1}$.

Lemma 6.6 *With the construction above, for all* $m \geq n$, $||h_m||_\infty \leq ||h_n||_\infty \leq ||f||_\infty$.

Proof Direct from Lemma 6.2 part ii/. $\bullet$

Construction 6.7 Suppose that $n < m$. Let $\gamma : \mathbf{T} \longrightarrow L_m$ be a single loop in L_m. Let $\gamma' : \mathbf{T} \longrightarrow L_n$ be the composition $\beta_n^m \circ \gamma$. Let $\phi_0 : [0,1] \longrightarrow M_n$ be the orientation-preserving lifting of γ' so that the diagram

$$\begin{array}{ccc} [0,1] & \xrightarrow{\phi_0} & M_n \\ \downarrow {}^{0=1} & & \downarrow q \\ \mathbf{T} & \xrightarrow{\gamma'} & L_n \end{array}$$

commutes. By Lemma 6.2i/ the path ϕ_0 passes over every edge of M_n at least 2^{m-n} times.

Now connect the two end points of the path ϕ_0 by a directed path so that we form a directed loop: This may be done by using a global orientation of M_n, and in such a way that each edge of M_n is traversed by this connecting path at most once. Call this newly constructed loop $\phi_\gamma : \mathbf{T} \longrightarrow M_n$.

Lemma 6.8 *Suppose the construction above and that* $m > n$. *Suppose that* $g : L_n \longrightarrow \mathbf{T}$ *is a map. Then* $|[g \circ \sigma_n^m, \gamma] - [g \circ q, \phi_\gamma]| \leq 2^{n-m} \min\{[|g| \circ q, \phi_\gamma], [|g| \circ \sigma_n^m, \gamma]\}$.

Proof We assume first that g is orientation preserving. The general result follows by splitting into positive and negative parts and adding.

By the construction of ϕ_γ, $[g \circ q, \phi_\gamma]$ is decomposed into two parts: the part over ϕ_0, contributing a_0, and the part over the newly constructed connecting path contributing a_1. Both a_j are positive and a_1/a_0 is at most 2^{n-m} by construction of Lemma 6.2i/. Also, by construction, $a_0 = [g, \gamma'] = [g \circ \sigma_n^m, \gamma]$. These observations give directly the inequality above. $\bullet$

Construction 6.7 ctd Recall the construction of ϕ_γ, a global orientation in M_n, from a single loop, γ in L_m. Suppose rather that we are given a map $\gamma : \mathbf{T} \longrightarrow L_m$. By taking the individual loops covered by γ in turn, we may piece together the correspoding

ϕ maps, accounting carefully for orientation preservation or reversal, and define a map $\phi_\gamma : \mathbf{T} \longrightarrow M_n$. We note that in this case $[f, \phi_\gamma] = \sum_\lambda a_\lambda [f, \phi_\lambda]$ for all $f \in H^1(M_n)$, where the sum is taken over all single loops, λ, of L_n and the integer a_λ represents the aggregate winding number of γ around λ.

Lemma 6.9 *Suppose that $g : L_n \longrightarrow \mathbf{T}$ and that $\gamma : \mathbf{T} \longrightarrow L_m$ is orientation preserving, then $|[g \circ \sigma_n^m, \gamma] - [g \circ q, \phi_\gamma]| \le 2^{-m+n} \min\{[|g| \circ q, \phi_\gamma], [|g| \circ \sigma_n^m, \gamma]\}$*

If $a \in H^0(G_n)$ and $\mathbf{g} = \delta^*(a) \in H^1(L_n)$, then, $|[g \circ \sigma_n^m, \gamma]| \le 2^{-m+n} C_a ||\gamma||_1$ where C_a is a constant depending only on a.

Proof The first part is direct from (6.8). In the second, we know that $g \circ q = 0$ and so the first part gives the bound required. •

Proposition 6.10 *Suppose that $f : M_n \longrightarrow \mathbf{T}$ is a cell map. Then, in $H^1(M)$, $\mathbf{f} > 0$ if and only if, for all $C > 0$, there is an $m > n$ so that $\mathbf{g}_m > C\pi_m^L$ as an element of $H^1(L_m)$ (notation from (6.3)).*

Proof Suppose that $\gamma : \mathbf{T} \longrightarrow M_m$ is an orientation-preserving map. Then $[f_m, \gamma] = [g_m \circ q, \gamma] + [h_m, \gamma]$. From Lemmas 6.4 and 6.6 we know that $|[h_m, \gamma]| \le ||f||_\infty [\pi_m^G, \gamma] = ||f||_\infty ||\gamma||_1/2$, uniformly in m.

First we prove the reverse implication: Let $C = 2||f||_\infty$ and, by hypothesis, choose m so that $g'_m > 2^{m-n}\pi_m^L$. Therefore

$$[f_m, \gamma] \ge C[\pi_m^L \circ q, \gamma] - ||f||_\infty ||\gamma||_1/2 \ge (C - ||f||_\infty)||\gamma||_1/2$$

which, by the choice of C, is positive independently of the choice of γ. Therefore, by Lemma 2.7, $\mathbf{f}_m \ge 0$ in $H^1(M_m)$ for such m.

Conversely, if $f_n : M_n \longrightarrow \mathbf{T}$ is a cell map such that $\mathbf{f}_n > 0$ in $H^1(M_n)$, then we choose (by Lemma 2.4) an orientation preserving cell map representative, $f'_n : M_n \longrightarrow \mathbf{T}$, of the class $\mathbf{f}_n$. We may form the maps g'_m and h'_m from $f'_m = f'_n \circ \beta_n^m$ so that $f'_m = g'_m \circ q + h_m$ as in Construction 6.5. By Lemma 6.4, $f'_m > (2^{m-n} - 1)\pi_m$ and so $g'_m \ge (2^{m-n} - 1)\pi_m^L$.

However, by Lemma 5.3, $\mathbf{g}'_n - \mathbf{g}_n = \delta^*(a)$ for some $a \in H^0(G_n)$ and $\mathbf{g}'_m - \mathbf{g}_m = \delta^*(\tau_n^{m*}a) = \sigma_n^{m*}\delta^*(a)$ for all $m \ge n$.

Therefore, if $C > 0$, then $[g_m, \gamma] - [C\pi_m^L, \gamma] \ge [g'_m, \gamma] - |[\sigma_n^{m*}\delta^*(a), \gamma]| - C||\gamma||_1/2 \ge (2^{m-n} - 1)||\gamma||_1/2 - 2^{n-m}C_a||\gamma||_1 - C||\gamma||_1/2$ which, for m chosen sufficiently large, independently of γ, is strictly positive. Thus, by Lemma 2.7, $\mathbf{g}_m > C\pi_m^L$ as required. •

Definition 6.11 The previous Proposition compels the following definition: Define the group $H^1(P) = \lim_{\theta_n^*} H^1(L_n) \oplus H^1(G_n)$ define an element $\mathbf{g} \oplus \mathbf{h}$, $\mathbf{g} \in H^1(L_n)$ and $\mathbf{h} \in$

$H^1(G_n)$ to be positive ($\mathbf{g} \oplus \mathbf{h} >_P 0$) if, for all $C > 0$, there is an m such that, writing $\mathbf{g}_m \oplus \mathbf{h}_m = \theta_n^{m*}(\mathbf{g} \oplus \mathbf{h})$, we have $\mathbf{g}_m > C\pi_m^L$ in $H^1(L_m)$.

Lemma 6.12 *With this order and a unit $\mathbf{1}^P = \mathbf{1}_0 \oplus 0 \in H^1(L_0) \oplus H^1(G_0)$, $H^1(P)$ becomes a simple ordered group with unit.* •

Corollary 6.13 of Proposition 6.10 *With the order $<_P$ on $H^1(P)$, the map $q^* \oplus j^* : H^1(P) \longrightarrow H^1(M)$, produced in the limit from the diagram of Lemma 5.13, is a unital strict order embedding whose kernel, $\delta^* H^0(G) \oplus 0$, is contained in $Inf(H^1(L)) \oplus 0$.*

Proof The first part follows directly from the Proposition. The identification of the kernel is due to Lemma 5.3. If $a \in H^0(G_n)$ and we write $\mathbf{g}_m \oplus 0 = \theta_n^{m*}(\delta^*(a) \oplus 0)$, more shortly $\mathbf{g}_m = \sigma_n^{m*}\delta^*(a)$, then for all N and all $C > 0$, we have, by Lemma 6.9,

$$[N\mathbf{g}_m, \gamma] \leq N2^{n-m}C_a||\gamma||_1 \leq (NC_a 2^{n-m} + C/2)||\gamma||_1 - C[\pi_m^L, \gamma]$$

$$\leq (NC_a 2^{n-m} + C/2)2^{-m+1}[\mathbf{1}'_m, \gamma] - C[\pi_m^L, \gamma]$$

the last inequality by Lemma 6.4. So, for m chosen sufficiently large independently of γ, $[N\mathbf{g}_m, \gamma] \leq [\mathbf{1}'_m, \gamma] - C[\pi_m^L, \gamma]$ and so by definition, that $N(\delta^*(a) \oplus 0) < \mathbf{1}'_m \oplus 0 = \theta_0^{m*}\mathbf{1}^P$: the unit in $H^1(P)$. Also, we have more directly, $N\delta^*(a) < \mathbf{1}'$ in $H^1(L)$. Thus the image of $\delta^* \oplus 0$ is contained in $Inf(H^1(L)) \oplus 0$ which in turn is contained in $Inf(H^1(P))$ as required. •

Corollary 6.14 *The natural map $H^1(L) \longrightarrow H^1(P)$ is a unital order embedding.* •

§**7 States** In this section, the last one concerned with general theory, we characterise the states on the ordered group $H^1(M)$ in terms of the states on $H^1(L)$. Since the groups we produce are all dimension groups, this, by Lemma 2.2, is an efficient way to find the order structure of $H^1(M)$.

We note the following natural maps induced by the order embeddings found in the previous section: $\Sigma(H^1(M)) \xrightarrow{q \oplus j} \Sigma(H^1(P))$ and $\Sigma(H^1(P)) \longrightarrow \Sigma(H^1(L))$. The composition $\Sigma(H^1(M)) \longrightarrow \Sigma(H^1(L))$ is the natural map induced from q.

Also we note that since $\delta^*(H^0(G))$ is contained in the infinitessimals of $H^1(L)$, the states on $H^1(L)/\delta^* H^0(G)$ are in natural 1-1 correspondence with the states on $H^1(L)$ and we sometimes transfer from one group to the other without confusion.

The main theorem of this section is

Theorem 7.1 *The natural map $\Sigma(H^1(M)) \longrightarrow \Sigma(H^1(L))$ is a bijection, indeed an affine homeomorphism. As a consequence, both the maps above, $\Sigma(H^1(M)) \longrightarrow \Sigma(H^1(P))$ and $\Sigma(H^1(P)) \longrightarrow \Sigma(H^1(L))$, are affine homeomorphisms.*

Proof We shall construct a map $\Sigma(H^1(L)) \longrightarrow \Sigma(H^1(M))$ which will invert the composition of the two maps above. Since each of these maps is clearly onto, the theorem is proved.

Fix an element $s \in \Sigma(H^1(L))$. Suppose that $f : M_n \longrightarrow \mathbf{T}$ is the representative of a class $\mathbf{f} \in H^1(M_n)$. For each $m \geq n$ we have the decomposition $f \circ \beta_n^m = g_m \circ q + h_m$ as in (6.5). In particular we have $\mathbf{g}_m = \sigma^* \mathbf{g}_{m-1} + \overline{\sigma}^* \tau_n^{(m-1)*} \mathbf{h}_n$. From this we deduce, using also the fact that s is a state, the equation

$$s(\mathbf{g}_m) - s(\mathbf{g}_k) = \sum_{j=k}^{m-1} s(\overline{\sigma}^* \tau_n^{j*} \mathbf{h}_n) \qquad (*)$$

valid for all $n < k < m$. However, $||h_n \circ \tau_n^j||_\infty \leq ||f||_\infty$ by Lemma 6.6, and so an upper bound on the function $\overline{\sigma}^* \tau_n^{j*} \mathbf{h}_n$ is $||f||_\infty \sigma^* \pi_j^L \leq 2^{-j} ||f||_\infty \sigma^* \mathbf{1}'_j = 2^{-j} ||f||_\infty \mathbf{1}'_{j+1}$, the middle inequality by Lemma 6.4. But $s(\mathbf{1}'_j) = 1$ and so we have $|s(\mathbf{g}_m) - s(\mathbf{g}_k)| \leq ||f||_\infty 2^{-k+2}$ and $s(\mathbf{g}_m)$ forms a Cauchy sequence therefore, converging to a value which we'll write $\psi(s)(f)$.

If f and f' are homotopy equivalent maps $M_n \longrightarrow \mathbf{T}$, then g_m and g'_m, the sequences produced respectively by Construction 6.5, differ, for each $m \geq n$, by an element of $\delta^* H^0(G)$; this is the argument of (6.13). Also by Corollary 6.13, $\delta^* H^0(G) \subset Inf(H^1(L))$, and so $\mathbf{g}_m - \mathbf{g}'_m$ is mapped by s to 0 (Lemma 2.2). So, the terms of the defining sequences being the same, the limits $\psi(s)(f)$ and $\psi(s)(f')$ agree. Thus the correspondence $\mathbf{f} \mapsto \psi(s)(f)$ is well-defined on $H^1(M)$.

Since all the maps, σ^*, τ^* etc, involved in the definition are group homomorphisms, it is clear that $\psi(s)$ is a homomorphism: $H^1(M) \longrightarrow \mathbf{R}$.

Suppose that $f : M_n \longrightarrow \mathbf{T}$ is an orientation preserving non-trivial map. From Proposition 6.10, this implies there is an m such that $\mathbf{g}_m > \pi_m^L$. Since f is orientation preserving, so also is h_n and hence $\tau_n^{j*} \mathbf{h}_n \geq 0$ for each $j \geq n$. Thus, by equation (*), $s(\mathbf{g}_k) : k > n$ is an increasing sequence and so $\psi(s)(\mathbf{f}) \geq s(\mathbf{g}_m) > 0$. Thus $\psi(s)$ is a strictly positive homomorphism.

Furthermore, the equality $\mathbf{1}_m = \mathbf{1}'_m \circ q$ is the outcome of Construction 6.5 applied to the unit, and this shows that $\psi(s)(\mathbf{1}) = s(\mathbf{1}') = 1$. Thus $\psi(s) \in \Sigma(H^1(M))$.

The continuous affine dependence of $\psi(s)$ on s is clear from the definition.

Now write $\psi(s) = t$ and suppose that t' is a state on $H^1(M) = H^1(L)/H^0(G) \oplus H^1(G)$ such that $t'(\mathbf{g} \oplus 0) = s(\mathbf{g})$ for all $\mathbf{g} \in H^1(L)$. Suppose that $f : M_n \longrightarrow \mathbf{T}$ is an orientation preserving map and that $f \circ \beta_n^m = g_m \circ q + h_m$ is its familiar decomposition. Then, by definition of t, $|t(\mathbf{f}) - t'(\mathbf{f})| \leq |s(\mathbf{g}_m) - t'(\mathbf{f})| + o(1)$ where the Landau notation refers to the condition $m \to \infty$. However, $|s(\mathbf{g}_m) - t'(\mathbf{f})| = |t'(\mathbf{h}_m)| \leq ||f||_\infty t'(\pi_m^G) \leq 2^{-m} ||f||_\infty t'(\mathbf{1}_m) = 2^{-m} ||f||_\infty$ which also tends to zero as m tends to infinity. Thus $t = t'$ on the positive cone of $H^1(M)$ and therefore we have equality everywhere.

In particular $s \mapsto \psi(s)$ is an inverse to the map $t \mapsto t \circ q^*$ and we have the desired bijection. •

Corollary 7.2 *The ordered group with unit, $H^1(P)$, is an unperforated simple dimension group.*

Proof By the work before $H^0(G) \longrightarrow H^1(P) \overset{q^*\oplus j^*}{\longrightarrow} H^1(M)$ is a short exact sequence, in which the first map has range in the infinitessimals of $H^1(P)$ (Corollary 6.13) and the second map is a strictly positive strict order embedding (6.13) which induces, by Theorem 7.1, a 1-1 correspondence between the states of $H^1(M)$ and $H^1(P)$. Further, $H^1(M)$ is a dimension group (Lemma 5.2).

Thus $\mathbf{f} \in H^1(P)$ is strictly positive if and only if $q^* \oplus j^*(\mathbf{f})$ is strictly positive. This shows that the unperforation and the strict Reisz property, which holds in $H^1(M)$, passes back to $H^1(P)$ (we use the fact here that $H^1(P)$ is clearly torsion free). For the same reason the positive cone of $H^1(P)$ is defined by the states. •

We redraft the statement and proof of Theorem in the form it will be most useful:

Corollary 7.3 *If $s \in \Sigma(H^1(L))$, then there is a $t \in \Sigma(H^1(M))$ such that $t \circ q^* = s$. Indeed this t is unique and there is an affine map $\psi : \Sigma(H^1(L)) \longrightarrow Hom_{\mathbf{Z}}(H^1(G), \mathbf{R})$ such that: $t(\mathbf{g} \oplus \mathbf{h}) = s(\mathbf{g}) + \psi(s)(\mathbf{h})$ (using the decomposition of Corollary 5.8).*

Thus the invariant probability measures in The Natural Extension $(\Pi(B), S)$ are in natural 1-1 correspondence with the states on $K_0(B)$.

There is a positive order embedding injection $\tilde{K}_0(B) \longrightarrow \tilde{K}_0(C(\Pi(B)) \times \mathbf{Z})$.

In particular, the Natural Extension $(\Pi(B), S)$ is uniquely ergodic if and only $K_0(B)$ has only one state. If, in this case, s is the unique state on $K_0(B)$, then there is a homomorphism $\eta : H^1(G) \longrightarrow \mathbf{R}$ such that $\tilde{K}_0(C(X) \times \mathbf{Z}) = s(K_0(B)) + \eta(H^1(G))$ a subgroup of the reals.

Proof The first statement is immediate from the proof of Theorem 7.1 upon noting that the states on $H^1(M)$ are in 1-1 correspondence with the invariant probability measures on $(\Pi(B), S)$ (using Lemma 5.2 and [**HPS**]), and that $K_0(B) = H^1(L)$.

The map in the third statement is the map $H^1(L) \overset{q^*}{\longrightarrow} H^1(M)$ factored through the quotient by infinitessimals. Since the kernel of this map is in the infinitessimals, this factoring does indeed work (see Lemma 2.2). If $t \circ q^*(a) = t \circ q^*(b)$ for every $t \in \Sigma(H^1(M))$, then, since the map $t \mapsto t \circ q^*$ is onto (Theorem 7.1), $s(a) = s(b)$ for every $s \in \Sigma(H^1(L))$ and so we deduce that the factored map is 1-1.

The positivity and order embedding properties are deduced by similar arguments, using the fact that the order is determined by the states (Lemma 2.2).

The final statement comes by setting $\eta(\mathbf{h}) = \psi(s)(0 \oplus \mathbf{h})$. •

§**8 Substitution systems** As with [**Ho**], [**Fo**] and [**DHS**], the techniques are applied to the case of a substitution minimal system and we refer to [**Qu**], [**Ho**], [**Fo**] and [**DHS**] for definitions in this context. The following is Lemma 11 in [**Fo**].

Lemma 8.1 *Suppose that B is the improper ordered Bratteli diagram for a given primitive aperiodic substitution, σ; then the minimal two sided shift system, (X_σ, S), constructed from a two sided fixed point of the substitution, is topologically conjugate to* $(\Pi(B), S)$ *.* •

Given a primitive aperiodic substitution, σ, with finite alphabet, a symbol, a, is maximal if there are infiitely many n such that a is the last letter $\sigma^n(a)$. Likewise minimal with respect to the first letter. The space G^∞ defined in section 3 above can be recovered as the bipartite graph, $G(\sigma)$, whose vertices are a formal disjoint union of maximal symbols a and minimal symbols b with an edge from a to b if and only if ab appears in the language of the substitution.

The following theorem which allows the computation of the ordered K-theory of an arbitrary primitive substitution system and it completes the calculations from [**Fo**] and complements the results in [**DHS**]. Its proof is an uncomplicated application of Theorems 4.4 and Corollary 6.13 (or indeed Theorem 13, and the comment after Corollary 10 of [**Fo**]), together with a particular analysis of the structure of G^∞.

Theorem 8.2: *Suppose that σ is a primitive aperiodic substitution, with improper Bratteli diagram B, and (X_σ, S) the minimal Cantor system which it defines. If it happens that $G(\sigma)$ is a cycle free graph, then $K_0(C(X_\sigma) \times_S \mathbf{Z}) = K_0(B)/H^0(G(\sigma))$ as ordered groups with unit.*

In any case, construct σ_2: the 2-symbol substitution defined by Queffélec [**Qu**] *and let B_2 be its improper Bratteli diagram. Then $G(\sigma_2)$ is cycle free and $K_0(C(X_\sigma) \times_S \mathbf{Z}) = K_0(B_2)/H^0(G(\sigma_2))$ as ordered groups with unit.*

As a consequence, $\tilde{K}_0(B_2) = \tilde{K}_0(C(X_\sigma) \times \mathbf{Z})$ as ordered groups with unit.

Proof We use Lemma 5.15 to equate $H^*(G^\infty)$ with $H^*(G)$ and then note that the two equations above are special cases of Corollary 6.13. We confirm that the graph $G(\sigma_2)$ is always cycle free straight from Queffelec's construction. •

§**9 Generalised Morse Sequences** The construction of the Morse-Thue sequence [**Mo**] has been generalised in several ways. The following procedure, due to Keane [**Ke**], gives a one sided infinite sequence of 0s and 1s.

Construction 9.1 Given two finite words of 0s and 1s, 1101 and 010 say, define a product of the two: 1101×010 by considering the word on the right to act on the word on the left, concatenating copies of the left word or its compliment according to the parity and sequence of the digits on the right. Thus $1101 \times 010 = 1101\ 0010\ 1101$ (the gaps are usually closed).

This binary operation is associative but not commutative. We may iterate the procedure multiplying by whatever words (*factors*)on the right we wish and obtain at each stage a word of 0s and 1s: $1101 \times 010 \times 11 \times 001 \times \ldots$. By bracketing together suitable combinations of factors (*telescoping*) we may assume without loss of generality that each factor, except perhaps the left-most, begins and ends with the symbol 0: $(1101 \times 010 \times 11) \times 010 \times (\ldots =$ $(001011010010001011010010) \times 010\ldots$ In this case the growing words produced (we assume each factor has length at least 2) stabilise in the limit defining an infinite one-sided sequence of 0s and 1s - a *generalised Morse sequence*.

Lemma 9.2 [**Ke**] *A generalised Morse sequence is uniformly recurrent if and only if in each factor (having telescoped further if necessary) there is the appearance of at least one* 1 *and at least one* 0.

In this case the classical Natural Extension (see [**Pe**] *for example) of the one-sided shift is minimal.* •

Construction 9.1 ctd It is straightforward to define an improper Bratteli diagram from this multiplicative construction of a generalised Morse sequence. At each level $n \geq 1$ there are two vertices, labelled 0_n and 1_n. For $n \geq 1$, the edges between levels n and $n+1$ are given naturally by the nth factor: for example, if this factor is 0111010, then there are 7 edges with 0_{n+1} as their range, 4 with 1_n as their source and 3 with 0_n as their source. The edges with source 1_n are ordered $2, 3, 4, 6$ and those with source 0_n are ordered $1, 5, 7$. Among the edges with 1_{n+1} as their range we have complimentary pattern: those with source 0_n (there are 4 of them) are ordered $2, 3, 4, 6$, and those with source 1_n (3 in number) are ordered $1, 5, 7$.

The single vertex at level 0 is attached to each of the two vertices of level 1 by one edge.

If ω_n is the nth factor producing the sequence in question, we record the following parameters:

Definition 9.3 Let $s_n =$ the length of ω_n: $a_n =$ (number of 0s in ω_n): $b_n =$ (number of 1s in ω_n): $d_n = a_n - b_n$: $t_n =$ number of 00s in ω_n (overlaps admitted): $u_n =$ number of 11s in ω_n (overlaps admitted).

Define $j_0 = \sup\{0, j \geq 1 : d_j = 0\}$ (this may be infinite or 0).

Lemma 9.4 *If $j_0 < \infty$, then with telescoping if necessary, we may assume that $d_n > 0 : n > 1$ and that $j_0 = 0$ or 1 therefore.*

Proof We pair consecutively the indices $n > j_0$ for which $d_n < 0$, leaving at most one index unpaired which we may choose to be the smallest index. For each index pair, $n_1 < n_2$ say, we telescope from levels n_1 to n_2 inclusive, i.e. we group the factors $\omega_{n_1} \times \omega_{n_1+1} \times \ldots \times \omega_{n_2-1} \times \omega_{n_2} = \omega'$ and obtain a word, ω', with more 0s than 1s. This word becomes a component part of a new factorisation of the original sequence ω.

If the smallest index giving $d_n < 0$ is unpaired, then we telescope levels 1 to n_0 inclusive.

If all the indices giving $d_n < 0$ are paired,then we telescope levels 1 to j_0 inclusive.

The resulting new factorisation of ω has the properties requred.

Note that this procedure does not change the assumptions about factors beginning and ending with the symbol 0. •

Definition 9.5 Thus we may assume without loss of generality that $j_0 = 0, 1$ or ∞ and that $d_n \geq 0$ for all $n > 1$. In the case $j_0 < \infty$ we adopt a more natural division into three cases: $d_1 > 0$ (*positive*): $d_1 < 0$ (*negative*): $d_1 = 0$ (*null*).

Let $S_n = \Pi_{1 \leq j \leq n} s_j$ and $D_n = \Pi_{j_0 < j \leq n} |d_j|$, the former defined for all $n \geq 1$ and the latter defined for $n > j_0$. If $n \leq j_0$ (and this includes the case $j_0 = \infty$), then we define $D_n = 1$.

Let
$$\lambda = \lim_n \frac{D_n}{S_n}$$
which, since the sequence is monotone decreasing and positive, does converge.

Write $\mathbf{Z}[S]$ for the subgroup of $\mathbf{Q}$ generated by the numbers $1/S_n : n \geq 1$; this inherits a canonical order from its embedding in $\mathbf{R}$. Write $\mathbf{Z}[D]$ for the ordered subgroup of $\mathbf{Q}$ generated by the numbers $1/D_n : n \geq 1$, also with the canonical order.

The following gives the structure of $H^1(L) = K_0(B)$ (Lemma 5.2) a necesasary ingredient for the constructions of sections 5,6 and 7.

Lemma 9.6 *The dimension group, $K_0(B)$, for the improper diagram constructed for the generalised Morse sequence with parameters above can be described in several cases as follows:*

a/ If $j_0 = \infty$, then
$$K_0(B) = \mathbf{Z}[S]$$
with canonical order and order unit 2. There are on infinitessimals.

b/ If $j_0 < \infty$, then

$$K_0(B) = \mathbf{Z}[S] \oplus \mathbf{Z}[D]$$

with order unit $2 \oplus 0$ and an order which is one of two types:

i/ If $\lambda = 0$, then $w \oplus v > 0 \oplus 0$ if and only if $w > 0$ in the canonical order of $\mathbf{Z}[S]$. The infinitessimals are the elements of the form $0 \oplus v : v \in \mathbf{Z}[D]$.

ii/ If $\lambda > 0$, then $w \oplus v > 0 \oplus 0$ if and only if $-\lambda v < w < \lambda v$, w and v considered canonically as real numbers with usual ordering. There are no non-zero infintessimals.

In the cases a/ and bi/, there is a unique normalised state: the map $s : w \mapsto w/2$ in the first case, and $s : w \oplus v \mapsto w/2$ in the second.

In case bii/, the space of normalised states is homeomorphic to the unit interval and its two extreme points are the states: $s^+ : w \oplus v \mapsto w/2 + \lambda v$ and $s^- : w \oplus v \mapsto w/2 - \lambda v$.

Proof In the limit of $\mathbf{Z}^2$ groups, the connecting matrices are

$$J_n = \begin{pmatrix} a_n & b_n \\ b_n & a_n \end{pmatrix} : H^1(L_n) \longrightarrow H^1(L_{n+1})$$

which have eigen-vectors $(1,1)^T$ and $(1,-1)^T$, with values s_n and d_n respectively. The independence of these vectors from the index n gives the splitting of the group at each level: the second group is generated by $(1,-1)^T$ and, with due attention, the first is generated by $(1,0)^T \equiv \frac{1}{2}(1,1)^T \bmod (1,-1)^T$.

We represent $H^1(L_n)$ as $(\mathbf{Z}^2/\langle(1,-1)^T\rangle) \oplus \langle(1,-1)^T\rangle$. For notational purposes we write an element in this splitting with direct sum notation $w \oplus v$. With respect to this decomposition, J_n takes the form

$$\tilde{J}_n = \begin{pmatrix} s_n & 0 \\ 0 & d_n \end{pmatrix} : H^1(L_n) \longrightarrow H^1(L_{n+1})$$

from which we deduce the structure of the groups as claimed immediately.

For example, at level 1, $(1,1)^T$ is the canonical order unit which is represented as $2 \oplus 0$ at level 1 in the group decomposition above. This is transformed by applications of $\tilde{J}$ to $2S_{n-1} \oplus 0$ at level n. From this we deduce the expression for the order unit claimed.

The order and state structure claimed is automatic in the case $j_0 = \infty$, so we concentrate on the case $j_0 < \infty$.

Note that at level j_0+1, the element $(1,-1)^T$ is non-zero in the limit, and it represents the element $0 \oplus 1$ in the group description above.

Suppose that s is a normalised state on $K_0(B)$ and that (x_n, y_n) is the row vector at level n which represents s: i.e. if $(k,l)^T \in \mathbf{Z}^2$ is an element of $H^1(L_n)$, then $s((k,l)^T) = x_n k + y_n l$. Note that $x_n, y_n \geq 0$ and $(x_{n+1}, y_{n+1}) = (x_n, y_n) J_n$ for all $n \geq 1$.

By normalisation, $1 = s((1,1)^T) = (x_n, y_n)J_{n-1}J_{n-2}...J_1(1,1)^T = S_{n-1}(x_n + y_n)$ for all $n \geq 1$.

Also we extract the value of the state on the element $(1,-1)^T$ at level $j_0 + 1$: $\lambda_1 = s(J_{n-1}J_{n-2}...J_{j_o+1}(1,-1)^T) = D_{n-1}(x_n - y_n)$ for $n \geq j_0 + 2$.

Using these we may retrieve an expression for s on the direct sum description of the group, via the formulae: $s(2 \oplus 0) = 1$ and $s(0 \oplus 1) = \lambda_1$: and so $s(w \oplus v) = w/2 + \lambda_1 v$.

By definition we have

$$|\lambda_1| = \frac{D_{n-1}|x_n - y_n|}{S_{n-1}(x_n + y_n)} \leq \frac{D_{n-1}}{S_{n-1}}$$

for every $n \geq j_0 + 1$. Since the right-hand side tends to the parameter λ we deduce that, in the case $\lambda = 0$, $\lambda_1 = 0$ automatically. This gives the order and state structure for part bi/ immediately. The description of the infinitessimals comes from Lemma 2.2

In the case $\lambda > 0$, λ_1 has wider latitude and for each choice of $|\lambda_1| \leq \lambda$ we obtain a different state. With only one degree of freedom, there are exactly two extreme states, with parameters $\lambda_1 = \pm\lambda$. •

The following can be proved in a way similar to **[Fo]** Lemma 11:

Lemma 9.7 *Suppose that B is the improper Bratteli diagram constructed for an aperiodic uniformly recurrent generalised Morse sequence, ω. Then the dynamical system $(\Pi(B), S)$ is conjugate to the classical Natural Extension of the one-sided shift defined by ω.* •

And this corollary of 9.7 and Theorem 7.1 reproduces Lemma 3 of **[Ke]**:

Corollary 9.8 *Under the conditions $j_0 = \infty$ or $\lambda = 0$, the classical Natural Extension of the one-sided shift defined by ω is uniquely ergodic. Otherwise, it supports exactly two ergodic probability measures.* •

The symmetry of the construction ensures the following structure:

Lemma 9.9 *The improper Bratteli diagram, B, constructed for an aperiodic uniformly recurrent generalised Morse sequence $\omega_1 \times \omega_2 \times ...$, has two maximal paths and two minimal paths. For each $n \geq 2$, the graph G_n is a 4 cycle with alternating orientations on the edges, and the map τ is the identity. Thus $H^1(G) = \mathbf{Z}$ and $H^0(G) = 0$.* •

Combining the results of Lemmas 9.9 and 5.8 we have therefore:

Corollary 9.10 *As a group, the first cohomology of the generalised Morse sequence is $\mathbf{Z}[S] \oplus \mathbf{Z}[D] \oplus \mathbf{Z}$ (in the case $j_0 < \infty$) or $\mathbf{Z}[S] \oplus \mathbf{Z}$ (in the case $j_0 = \infty$) (notation from Definitions 9.3 and 9.5).* •

To find more precisely the order on this group we pursue the constructions of Sections 6 and 7 in some detail: The graph M_n is G_n with two edges added as shown below:

The map from M_{n+1} to M_n maps the graph G_{n+1} identically onto G_n, sends e_0 in M_{n+1} positively around e_0 (a_n times) and e_1 (b_n times) in M_n in the order dictated by ω_n and sends e_1 in M_{n+1} positively around e_0 (b_n times) and e_1 (a_n times) in M_n in the order dictated by the complement of ω_n.

As in Lemma 5.6, we select the edge in G_n which corresponds to the transition 00 to support the generator of $H^1(G_n)$. This is the emboldened edge in the diagram. With this choice, the map θ_n takes the form:

$$\begin{pmatrix} a_n & b_n & t_n \\ b_n & a_n & u_n \\ 0 & 0 & 1 \end{pmatrix} : \mathbf{Z}^2 \oplus \mathbf{Z} \longrightarrow \mathbf{Z}^2 \oplus \mathbf{Z}$$

We use the decomposition of the second summand into eigen spaces (as in Lemma 9.4) to reinterpret θ_n as the map

$$\tilde{\theta}_n : \begin{pmatrix} s_n & 0 & t_n + u_n \\ 0 & d_n & t_n - u_n \\ 0 & 0 & 1 \end{pmatrix} : H^1(L_n) \oplus \mathbf{Z} \longrightarrow H^1(L_{n+1}) \oplus \mathbf{Z}$$

and we note that for purely combinatorial reasons $t_n - u_n = d_n - 1$ for each $n \geq 2$ since each such ω_n begins and ends with 0.

Let

$$\eta = \sum_{n=1}^{\infty} \frac{(t_n + u_n)}{S_n}$$

which, being less than $\sum_n \frac{s_n}{S_n} \leq \sum_{n \geq 0} 2^{-n} = 2$ and monotone increasing converges.

Assuming the special form of Lemma 9.4 and Definition 9.5, let λ' be defined according to the values of d_1: $\lambda' = \lambda$ in the null case: $\lambda' = \lambda |(t_1 - u_1 + 1)/d_1|$ in the positive case: and $\lambda' = -\lambda |(t_1 - u_1 + 1)/d_1|$ in the negative case.

Lemma 9.11 *With the data above and the special form of Lemma 9.4 assumed, consider the following triple of iterative equations:*

$$\alpha_1 = 0, \quad \alpha_n = s_{n-1}\alpha_{n-1} + t_{n-1} + u_{n-1} \quad : n \geq 2$$

$$\beta_{j_0+1} = 0, \qquad \beta_n = d_{n-1}\beta_{n-1} + t_{n-1} - u_{n-1} \quad : n \geq j_0 + 2$$

(if $j_0 = \infty$, then the second set of equations is not defined). Then α_n/S_{n-1} is a convergent monotone increasing sequence, tending to η. The sequence $\beta_n/S_{n-1} : n \geq j_0+1$ converges to λ'.

Proof With the substitution, $\alpha'_n = \alpha_n/S_{n-1}$ the first equation becomes $\alpha'_n = \alpha'_{n-1} + (t_{n-1}+u_{n-1})/S_{n-1}$ with initial condition: $\alpha'_1 = 0$. This gives the limit η for α'_n as required.

With the substitution $\beta'_n = |(\beta_n+1)/D_{n-1}|$, the second equation becomes $\beta'_n = \beta'_{n-1} : n \geq 3$ because, as noted before, we have $t_{n-1} - u_{n-1} = d_{n-1} - 1$ for $n \geq 3$. The initial conditions are best recalculated at β'_2 therefore, and we have: $\beta'_2 = 1$ in the null case: $\beta'_2 = |(t_1 - u_1 + 1)/d_1|$ in the non-null cases. Thus $|(\beta_n + 1)/S_{n-1}| = \beta'_n D_{n-1}/S_{n-1}$ and, since $1/S_{n-1} \to 0$ certainly, we have the limit of β_n/S_{n-1}: λ in the null case: $\lambda|(t_1 - u_1 + 1)/d_1|$ in the positive case: and $-\lambda|(t_1 - u_1 + 1)/d_1|$ in the negative case, as required. •

Theorem 9.12 *Suppose that (X,T) is the two-sided symbolic dynamical system generated by a generalised Morse sequence, with data as above and as in Lemma 9.4, and that η, λ and λ' are as defined above, then the T-invariant measures, equated with states on $H^1(X,T) = K_0(B) \oplus \mathbf{Z}$, are organised into three cases:*

a/ $j_0 = \infty$: In this case $H^1(X,T) = \mathbf{Z}[S] \oplus \mathbf{Z}$, on which there is a unique state $s_o(w \oplus z) = (\eta z + w)/2$.

b/ $j_0 < \infty$: In this case $H^1(X,T) = \mathbf{Z}[S] \oplus \mathbf{Z}[D] \oplus \mathbf{Z}$ and we have two possibilities:

i/ $\lambda = 0$: there is a unique state $s_o(w \oplus v \oplus z) = (\eta z + w)/2$.

ii/ $\lambda > 0$: the space of states is homeomorphic to the unit interval and the two extreme states are given by $s_o^+(w \oplus v \oplus z) = (\eta/2 + \lambda\lambda')z + w/2 + \lambda v$ and $s_o^-(w \oplus v \oplus z) = (\eta/2 - \lambda\lambda')z + w/2 - \lambda v$.

Proof From Theorem 7.1 we know the form of the state spaces are as given above. It remains to confirm the formulae:

a/ The unique state, s_o, on $H^1(X,T) = \mathbf{Z}[S] \oplus \mathbf{Z}$ is equal to s, the unique state on $\mathbf{Z}[S]$, when restricted to the first summand. Moreover the proof of Theorem 7.1 gives the method of calculating the value of $s_o(0 \oplus 1)$: In the notation given there, $\mathbf{g}_m \oplus \mathbf{h}_m = \theta_1^{m*}(0 \oplus 1)$ and $s_o(0 \oplus 1) = \lim_m s(\mathbf{g}_m)$, therefore. We note that in the splitting used to form $\tilde{\theta}$ above, $\mathbf{h}_m = 1$ and $\mathbf{g}_m = \alpha_m$ where $\tilde{\theta}_m(\mathbf{g}_m \oplus \mathbf{h}_m) = \mathbf{g}_{m+1} \oplus \mathbf{h}_{m+1}$. So $\alpha_{m+1} = s_m\alpha_m + (t_m + u_m)$. Furthermore, the unit is $2 \oplus 0$ at level 1 and hence $2S_{m-1} \oplus 0$ at level m. Thus $s(\mathbf{g}_m) = \alpha_m/(2S_{m-1}) \to \eta/2$ by Lemma 9.11 and this is the value of $s_o(0 \oplus 1)$ therefore.

b/ An identical construction applies to this case where, given a state, s, on $H^1(L)$, we must compute $s_o(0 \oplus 0 \oplus 1)$. This value is given by $\lim_m s(\mathbf{g}_m)$ where $\mathbf{g}_m \oplus \mathbf{h}_m =$

$\theta_1^{m*}(0 \oplus 0 \oplus 1)$ as above and $\mathbf{g}_m$, an element of $H^1(L_m)$, is represented as $\alpha_m \oplus \beta_m$. From the equation $\tilde{\theta}_m(\mathbf{g}_m \oplus \mathbf{h}_m) = \mathbf{g}_{m+1} \oplus \mathbf{h}_{m+1}$, we see that $\mathbf{h}_m = 1$ for all m and $\alpha_m \oplus \beta_m$ obey the equations of Lemma 9.11.

i/ Here there is one state, and on $H^1(L_m)$ it takes the form $w \oplus v \mapsto w/2$. Using the value of state on the unit to normalise the result, we get: $s_o(0 \oplus 0 \oplus 1) = \lim_m \alpha_m/(2S_{m-1}) = \eta/2$.

ii/ Here there are two states, s^+ and s^-, and, by Lemma 9.6, s^+ takes the form: $w \oplus v \mapsto w/2 + \lambda v$ on $H^1(L_m)$. From this we deduce $s_o^+(0 \oplus 0 \oplus 1) = \lim_m(\alpha_m/2 + \lambda\beta_m)/S_{m-1} = \eta/2 + \lambda\lambda'$ by Theorem 7.1 and Lemma 9.11. A similar formula with change of sign applies to s^-.

From these three computations therefore, we deduce the formulae for the states as above. •

By using the results of Lemma 2.2 and the Theorem above, we determine the order on $H^1(X,T) = K_0(C(X) \times_T \mathbf{Z})$ for the generalised Morse sequence.

Examples 9.13 a/ Thus it is quite straightforward to give examples where $\lambda = 0$ and $\eta \in \mathbf{Z}[S]$ and so we get $Inf(H^1(X,T)) = \{w \oplus v \oplus z : (\eta z + w)/2 = 0\} \equiv \mathbf{Z} \oplus \mathbf{Z}[D]$ a complemented subgroup of $H^1(X,T)$.

b/ If the group $\mathbf{Z}[S]$ is not the whole of $\mathbf{Q}$, may also choose t_n and u_n so as to give examples where $\lambda = 0$ and $k\eta \in \mathbf{Z}[S]$ where $k > 1$ is chosen minimally. So we get $Inf(H^1(X,T)) = \{w \oplus v \oplus z : z \in k\mathbf{Z},\ (k\eta z + w)/2 = 0\} \equiv \mathbf{Z} \oplus \mathbf{Z}[D]$, but this is no longer a complemented subgroup of $H^1(X,T)$.

c/ In any case, we may choose t_n and u_n to give examples where $\lambda = 0$ and $k\eta \notin \mathbf{Z}[S]$ for any choice of k, and so $Inf(H^1(X,T)) \equiv \mathbf{Z}[D]$ and $\tilde{H}^1(X,T) = \mathbf{Z}[S] \oplus \mathbf{Z}$.

d/ We may have the same sorts of examples with $\lambda > 0$.

Remark 9.14 The case c/ above shows that the positive order embedding injection $\tilde{H}^1(L) \longrightarrow \tilde{H}^1(M) = H^1(X,T)$ of Corollary 7.3 need not be onto, even in the uniquely ergodic case.

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