A Front Tracking Method for Conservation Laws with Boundary Conditions

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Abstract. A front tracking method is used to construct weak solutions to scalar conservation laws with two kinds of boundary conditions — Dirichlet conditions and a novel zero flux (or no-flow) condition. The construction leads to an efficient numerical method. The main feature of the scheme is that there is no stability condition correlating the spatial and temporal discretization parameters. The analysis uses the traditional method of proving compactness via Helly's theorem as well as the more modern concept of measure valued solutions. Three numerical examples are presented.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with piecewise regular boundary $\partial\Omega$ and outward unit normal n. Let $x = (x_1, \ldots, x_d)$ and $f = (f_1, \ldots, f_d)$. We study scalar conservation laws

$$u_t + \nabla \cdot f(u) \equiv u_t + \sum_{i=1}^{d} f_i(u)_{x_i} = 0, \qquad u(x,0) = u_0(x), \tag{1}$$

for $(x,t) \in \Omega \times \langle 0, T]$ with either prescribed Dirichlet boundary data

$$u(x,t) = r(x,t), \qquad x \in \partial\Omega, t > 0 \tag{2}$$

or a zero flux (no-flow) boundary condition

$$f(u(x,t)) \cdot n = 0, \qquad x \in \partial\Omega, t > 0. \tag{3}$$

Approximate solutions are constructed by dimensional splitting, using front tracking for the one-dimensional equations. The resulting method is an extension of an unconditionally stable method proposed by Holden and Risebro [5], see also [6].

2. Dirichlet Boundary Condition

Consider (1) with prescribed boundary condition (2). For this problem, a singular solution may develop in finite time even for smooth initial data. Moreover, the hyperbolic boundary value problem (1) and (2) (even when (1) is linear) is usually not well-posed when the boundary condition is required to hold in the strong sense. For these reasons, the problem has been to find a physically reasonable framework which incorporates discontinuous solutions as well as a correct mathematical formulation of the boundary condition.

A common approach is the *vanishing viscosity* method, in which a small viscosity term $\varepsilon\Delta u$ is added to regularize the problem. Then the correct solution is chosen as the L^1 limit of solutions to the corresponding parabolic problem as ε tends to zero. This approach leads to the following Kruzkov type *entropy condition* [1]: We call a function $u(x,t) \in L^\infty \cap \mathrm{BV}$ an entropy weak solution to (1) and (2) provided that for all $k \in \mathbb{R}$ and suitable test functions $\phi \geq 0$ with $\phi|_{t=T} = 0$, the following (entropy) inequality holds

$$\begin{aligned} \mathcal{L}_\phi(u) := &\int_\Omega \int_0^T \Big(|u-k|\phi_t + \operatorname{sgn}(u-k)\big(f(u)-f(k)\big)\cdot\nabla\phi \Big)\,dt\,dx \\ &- \int_{\partial\Omega}\int_0^T \operatorname{sgn}(r(s,t)-k)\big(f(\gamma u(s,t)) - f(k)\big)\cdot n(s)\phi(s,t)\,dt\,ds \\ &+ \int_\Omega |u_0(x)-k|\phi(x,0)\,dx \geq 0. \end{aligned} \tag{4}$$

Here γu denotes the L^1 trace. Thus, the boundary condition is satisfied in the following sense

$$\operatorname{sgn}(\gamma u(x,t)-k)(f(\gamma u(x,t)) - f(k))\cdot n(x) \geq 0, \qquad \forall k \in I(r(x,t),\gamma u(x,t)),$$

for a.e. $(x,t) \in \partial\Omega \times \langle 0,T]$, where $I(\alpha,\beta)$ denotes the closed interval with bounds α and β. Furthermore, the entropy solution is unique and depends L^1 continuously on the initial and boundary data [1, 7].

2.1. One Spatial Dimension

Consider first the one-dimensional equation

$$v_t + f(v)_x = 0, \qquad (x,t) \in \langle a,b\rangle \times \langle 0,T] \tag{5}$$

with initial and boundary data

$$\begin{aligned} &v(x,0) = v_0(x), && x \in \langle a,b\rangle, \\ &v(a,t) = v_a, \quad v(b,t) = v_b, && t \in \langle 0,T]. \end{aligned} \tag{6}$$

We construct approximate solutions by front tracking [3]. Assume that the initial data is a step function, i.e., a series of initial Riemann problems. If the flux function f is piecewise linear then all Riemann problems have solutions within the class of step functions. At the left boundary ($x = a$) we have a Riemann problem with initial and boundary conditions

$$\begin{aligned} &v(x,0) = v_0(a^+), && x > a \\ &v(a,t) = v_a, && t \in \langle 0,T]. \end{aligned}$$

The solution of this problem is defined by restricting the solution of the Riemann problem with left state v_a and right state $v_0(a^+)$ to $x > a$. A similar construction can be used at the right boundary ($x = b$). All wave interactions lead to new Riemann problems, either in the interior of the domain or at the boundary. Therefore a global solution to the approximate problem can be obtained by tracking discontinuities and solving Riemann problems. This construction is well-defined in the sense that the number of wave interactions and interactions with the boundaries is finite even in infinite time; the proof is similar as for the initial value problem [3]. By construction, the front tracking solution satisfies the entropy condition for the approximate equation. Furthermore, the front tracking method is well-defined also when the boundary data are step functions, provided the step functions have a finite number of discontinuities. In fact, we have

Lemma 2.1. *Suppose that $v_0(x)$, $v_a(t)$, and $v_b(t)$ are piecewise constant functions with a finite number of discontinuities, have compact support and are bounded within $[m, M]$. Let f be a Lipschitz continuous, piecewise linear function with a finite number of breakpoints in $[m, M]$. Then the Dirichlet problem (5) and (6) has an entropy weak solution $v(x,t)$ that is piecewise constant in x for each fixed $t > 0$ and takes values in the set $\{v_0(x)\} \cup \{v_a(t), v_b(t)\} \cup \{$the breakpoints of $f\}$. Furthermore,*

$$\begin{aligned} &\|v(x,t)\|_\infty \leq \max\bigl(\|v_0\|_\infty, \|v_a\|_\infty, \|v_b\|_\infty\bigr), \\ &\mathrm{TV}\bigl(v(\cdot,t)\bigr) \leq \mathrm{TV}(v_0) + |v_a - v_0(a^+)| + |v_b - v(b^-)|, \\ &\|v(\cdot,t_2) - v(\cdot,t_1)\|_1 \leq C|t_2 - t_1|. \end{aligned} \tag{7}$$

The solution can be constructed by front tracking in a finite number of steps for any $t > 0$ and is stable with respect to initial and boundary data,

$$\|v^1(\cdot,t) - v^2(\cdot,t)\|_1 \leq \|v_0^1 - v_0^2\|_1 + t\|f\|_{Lip}\bigl(\|v_a^1 - v_a^2\|_\infty + \|v_b^1 - v_b^2\|_\infty\bigr). \tag{8}$$

We can use front tracking to construct approximate solutions to a general equation, by approximating the initial and boundary data by step functions and the flux function by a piecewise linear function. Then it follows by a standard compactness argument using (7) that the approximate solutions converge to a solution of (5) and (6). Moreover, the limit is an entropy weak solution satisfying the Kruzkov inequality (4).

Theorem 2.2. *Suppose that $v_0(x)$, $v_a(t)$, and $v_b(t)$ are $L^\infty \cap BV$ and that f is Lipschitz continuous. Then the Dirichlet problem (5) and (6) has an entropy weak solution $v(x,t)$ which can be constructed as a limit of front tracking solutions.*

2.2. Arbitrary Spatial Dimension

We construct approximate solutions of the Dirichlet problem (1) and (2) only in two dimensions as generalizations to multidimensions are straightforward. Assume that the domain Ω is covered by a Cartesian grid $\{i\Delta x, j\Delta x\}$, and that $\partial\Omega$ is represented by a piecewise constant curve on this grid. Let $S^{f_i}(t)$ denote the solution operator associated with (5) and (6) in the ith direction. Then the solution $u(x,t)$ of (1) and (2) is approximated by

$$u(x, n\Delta t) \approx u^n(x) = \bigl[\pi S^{f_2^\delta}(\Delta t)\pi S^{f_1^\delta}(\Delta t)\bigr]^n \pi u_0(x). \tag{9}$$

Here π denotes a grid cell average operator and f_i^δ is a piecewise linear approximation to f_i. The boundary function r is approximated by a function that is constant in x and piecewise constant in t along each cell at the boundary.

To investigate convergence of a sequence of approximations, we choose an appropriate time interpolation [6]

$$u_\eta(x,t) = \begin{cases} S^{f_1^\delta}(2(t-t_n))u^n, & t \in \langle t_n, t_{n+1/2}], \\ S^{f_2^\delta}(2(t-t_{n+1/2}))u^{n+1/2}, & t \in \langle t_{n+1/2}, t_{n+1}], \end{cases} \tag{10}$$

where $t_n = n\Delta t$, $\eta = (\Delta t, \Delta x, \delta)$, and $u^{n+1/2} = \pi S^{f_1^\delta}(\Delta t)u^n$. Then we can prove the following result:

Lemma 2.3. *Suppose that $\Omega \subset \mathbb{R}^d$ is rectangular. Then the front tracking approximations defined by* (9) *and* (10) *satisfy the following estimates*

$$\begin{aligned} &\|u_\eta(\cdot,t)\|_\infty \leq C_1, \\ &\mathrm{TV}(u_\eta(\cdot,t)) \leq C_2(T), \\ &\|u_\eta(\cdot,t_2) - u_\eta(\cdot,t_1)\|_1 \leq C_3(T)|t_2-t_1|, \end{aligned} \tag{11}$$

where C_1, $C_2(T)$ and $C_3(T)$ are positive constants not depending on η.

Proof. The first inequality follows from (7) and $\|\pi v\|_\infty \leq \|v\|_\infty$. To prove the second, we enlarge the grid by a set of ghost cells outside the domain on which the solution is given by the boundary condition. This way, contributions from the boundary to the total variation are easily included and the second inequality can be proved for the auxiliary sequence on the enlarged grid; the proof is similar to the proof in [5] using the estimates in (7) and the stability inequality (8). The third result follows from the second and the finite speed of propagation of all waves. □

A standard compactness argument (Helly's theorem) gives that the sequence $\{u_\eta\}$ is convergent, and as for the one-dimensional case we can show that the limit is an entropy solution in the Kruzkov sense (4).

Theorem 2.4. *Suppose that u_0 and r are $L^\infty \cap BV$ and that the flux functions $f = (f_1, \ldots, f_d)$ are locally Lipschitz continuous. Let $\Omega \subset \mathbb{R}^d$ be rectangular. Then the Dirichlet problem* (1) *and* (2) *has an entropy weak solution $v(x,t)$ which is the limit of $\{u_\eta\}$ as $\eta \to 0$.*

The assumption that the boundary function r should be of bounded variation can be relaxed. Let us for the moment assume $d = 2$ and that $r : [a,b] \subset \mathbb{R} \to \mathbb{R}$ does not depend on t. Recall that $r \in BV$ if and only if for each $\Delta s > 0$,

$$\int_a^b |r(s+\Delta s) - r(s)|\, ds \leq C\Delta s,$$

for some constant $C > 0$. We introduce the spaces BV_α consisting of integrable functions on $[a,b]$ satisfying

$$\int_a^b |r(s+\Delta s) - r(s)|\, ds \leq C(\Delta s)^\alpha, \qquad \alpha \in \langle 0, 1].$$

Clearly, $BV \equiv BV_1 \subset BV_\alpha$. Assuming that $r \in BV_\alpha$ for some $\alpha \in \langle 0,1]$ will in an essential way affect the variation bound in (11). Recall that this bound relies on the stability result (8), so that in the present situation we can only show that

$$\mathrm{TV}(u_\eta(\cdot,t)) \le C(T)(\Delta x)^{\alpha-1}. \tag{12}$$

This estimate is in general not sufficient to ensure compactness of $\{u_\eta\}$. Instead we need to use the concept of measure valued (mv) solutions. Since $\{u_\eta\}$ is uniformly bounded, by Young's theorem we can infer the existence of a subsequence, still denoted by $\{u_\eta\}$, and a family of compactly supported probability (Young) measures $\nu_{x,t}$ such that the L^∞ weak-star limit $g(u_\eta) \rightharpoonup \bar{g}$ exists for any continuous function g, where the limit $\bar{g}$ is given by

$$\bar{g}(x,t) = \int_{\mathbb{R}} g(\lambda)\,d\nu_{x,t}(\lambda) =: \langle \nu_{x,t}, g\rangle, \qquad \text{for almost all } x,t.$$

Furthermore, the sequence $\{u_\eta\}$ converges strongly to u in $L^1(x,t)$ if and only if $\nu_{x,t}$ reduces to a Dirac measure located at $u(x,t)$. Following Benharbit, Chalabi, and Vila [2], we call $\nu_{x,t}$ an entropy mv solution of the Dirichlet problem (1) and (2) if for all $k \in \mathbb{R}$ and suitable test functions $\phi \ge 0$ with $\phi|_{t=T} = 0$,

$$\begin{aligned}
&\int_\Omega \int_0^T \Big(\langle \nu_{x,t}, |\lambda - k|\rangle \phi_t + \langle \nu_{x,t}, \big(\mathrm{sgn}(\lambda - k)(f(\lambda) - f(k))\big)\rangle \cdot \nabla\phi \Big)\,dt\,dx \\
&\quad - \int_{\partial\Omega} \int_0^T \mathrm{sgn}(r(s,t) - k)\langle \gamma\nu_{s,t}, f(\lambda) - f(k)\rangle \cdot n(s)\phi(s,t)\,dt\,ds \\
&\quad + \int_\Omega |u_0(x) - k|\phi(x,0)\,dx \ge 0,
\end{aligned} \tag{13}$$

where $\gamma\nu$ is the trace associated with ν. In general, the trace $\gamma\nu$ is associated with the Young measure ν in a non-unique way. However, it is shown in [8] that the expectation $\langle \gamma\nu_{s,t}, f(\lambda)\rangle$ appearing in (13) is uniquely defined. The definition (13) used here and in [2] is slightly different from the one used in [8].

Now, equipped with the (weaker) time estimate

$$\int_\Omega |u_\eta(x,t+\Delta t) - u_\eta(x,t)|\,dx = \mathcal{O}(1)\Delta t(\Delta x)^{\alpha-1} = \mathcal{O}(1)(\Delta t)^\alpha, \tag{14}$$

which follows from the space estimate (12), we can show that the Young measure $\nu_{x,t}$ associated with the sequence $\{u_\eta\}$ is in fact an entropy mv solution of (1) and (2) in the sense of (13). Because of the uniqueness result for entropy mv solutions of Szepessy [8] (see also [2]), we can conclude that $\nu_{x,t}$ reduces to a Dirac measure located at $u(x,t)$, where $u(x,t)$ denotes the unique BV entropy weak solution of (1) and (2). Consequently, we have the following theorem (which, of course, holds for any number of space dimensions $d \ge 1$):

Theorem 2.5. *Suppose that u_0 and r are in $L^\infty \cap BV_\alpha$ for some $\alpha \in \langle 0,1]$, and that the flux functions $f = (f_1,\dots,f_d)$ are locally Lipschitz continuous. Let $\Omega \subset \mathbb{R}^d$ be rectangular. Then, as $\eta \to 0$, the sequence $\{u_\eta\}$ converges in $L^1(x,t)$ to the unique entropy weak solution of the Dirichlet problem (1) and (2).*

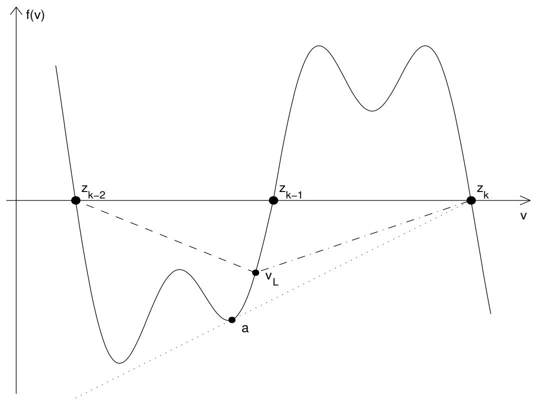

FIGURE 1. Illustration of a Riemann problem at the boundary.

3. Zero Flux Boundary Condition

Consider (1) with prescribed initial and boundary data of the form (3). The flux function f must have compact support. Without loss of generality we assume that u_0 takes values in $[0,1]$ and that $f(0) = f(1) = 0$. We seek weak solutions of (1) and (3) that satisfy

$$\int_\Omega \int_0^T (u\phi_t + f(u)\cdot\nabla\phi)\,dt\,dx + \int_\Omega u_0(x)\phi(x,0)\,dx = 0.$$

Note that there are no boundary terms due to the no-flow boundary condition.

3.1. One Spatial Dimension

Consider now the one-dimensional problem (5) with initial and boundary data

$$\begin{aligned} &v(x,0) = v_0(x), && x \in \langle a,b\rangle, \\ &f(v(a,t)) = f(v(b,t)) = 0, && t \in \langle 0,T]. \end{aligned} \tag{15}$$

As above, we use front tracking as a means of analysis. To this end we must approximate f by a piecewise linear function f^δ and the initial data by a step function v_0^Δ. Assume that f has a finite number of zeros $z_1 = 0 < z_2 \cdots < z_N = 1$ and that f^δ is piecewise linear on each interval $I_k = [z_k, z_{k+1}]$ with $f^\delta(z_k) = f^\delta(z_{k+1}) = 0$. (In the following we drop the superscripts on f and v_0). Without loss of generality, we can assume that all the intervals I_k have length ℓ. This is not a severe restriction and can easily be achieved by reparametrising the v-line.

We first discuss how to solve Riemann problems at the boundary. Consider a discontinuity hitting the right boundary at $x = b$. Assume that z_k is the boundary value at $x = b^-$ and that $f'(z_k) < 0$. The state v_L colliding with z_k must have $f(v_L) < 0$, see Fig. 1. Accordingly, we have that $a \le v_L < z_{k-1}$, where a is the (unique) largest state such that $f'(a) = -f(a)/(a - z_k)$. After the collision all waves must move left, and the new state adjacent to $x = b^-$ is z_{k-2}. This is indicated in Fig. 1: the right moving discontinuity before the collision (dash-dotted line) is replaced by a left moving discontinuity (dashed line). The picture is similar if the rightmost value is z_{k-2} and $v_L > z_{k-1}$. At the left boundary $x = a$ and at both boundaries initially we use a similar construction. Note that the boundary values initially are *constructed* from the initial data $v_0(a^+)$ and $v_0(b^-)$. We can prove that the front tracking algorithm has a finite number of steps by proving that a certain interaction functional is decreasing; see [3] for a similar proof. In fact, we can prove the result:

Lemma 3.1. *The front tracking solutions satisfy*

$$\begin{aligned} &\underline{v} \leq v(x,t) \leq \bar{v}, \\ &\mathrm{TV}(v(\cdot,t)) \leq \mathrm{TV}(v_0) + 2\ell, \\ &\|v(\cdot,t_2) - v(\cdot,t_1)\|_1 \leq C|t_2 - t_1|, \end{aligned} \tag{16}$$

where

$$\underline{v} = \max_k \{z_k | z_k \leq v_0(x), x \in \langle a,b \rangle\}, \quad \bar{v} = \min_k \{z_k | z_k \geq v_0(x), x \in \langle a,b \rangle\}.$$

Moreover, the solution is stable with respect to the initial data.

Proof. It follows immediately from the above construction that the solution is bounded in L^∞. If $v_0(a)$ and $v_0(b)$ are zeros in the flux function, a careful case analysis of wave interactions shows that $\mathrm{TV}(v(\cdot,t)) \leq \mathrm{TV}(v_0)$. The result depends on the fact that all intervals I_k have equal length ℓ. Assume next that $z_{k-1} < v_0(a) < z_k$ such that $f(v_0(a)) \neq 0$. Then the boundary value $v(a,0)$ must be either z_{k-1} or z_k. Similarly at $x = b$. Once the boundary values are determined, the zero flux problem can be reformulated as a Dirichlet problem and the second estimate follows from (7). The third estimate is a result of bounded total variation and finite speed of propagation.

To prove stability wrt. initial data we proceed in three steps. Let $w(x,t)$ and $v(x,t)$ denote the front tracking solution with initial data $w_0(x)$ and $v_0(x)$, respectively. Before the first interaction with the boundary, the zero flux problem can be reformulated as a Dirichlet problem and we have from (8) that

$$\|w(\cdot,t) - v(\cdot,t)\|_1 \leq \|w_0 - v_0\|_1 + tA\big(w_0(a), v_0(a), w_0(b), v_0(b); f\big).$$

Here A is positive and bounded above by $2\|f\|_{\mathrm{Lip}}$, but is not continuous with respect to its arguments, due to the construction at the boundary. However, a refined analysis shows that $A(\cdot) \to 0$ as $\|w_0(a) - v_0(a)\|_\infty$ and $\|w_0(b) - v_0(b)\|_\infty$ tend to zero. Then by introducing the concept of pseudopolygonals (see e.g., the monograph [4]), we prove that $\|w(\cdot,t) - v(\cdot,t)\|$ is non-increasing for every interaction with the boundaries and the stability result follows. ☐

Thus we have constructed a sequence $\{v_\eta\}$ that can be shown to be compact by standard arguments. Each v_η is an exact solution of the approximate problem, and hence the limit is a weak solution of (5) and (15).

Theorem 3.2. *The zero flux problem* (5) *and* (15) *has a unique weak solution that can be constructed as the $L^1(x,t)$ limit of front tracking solutions.*

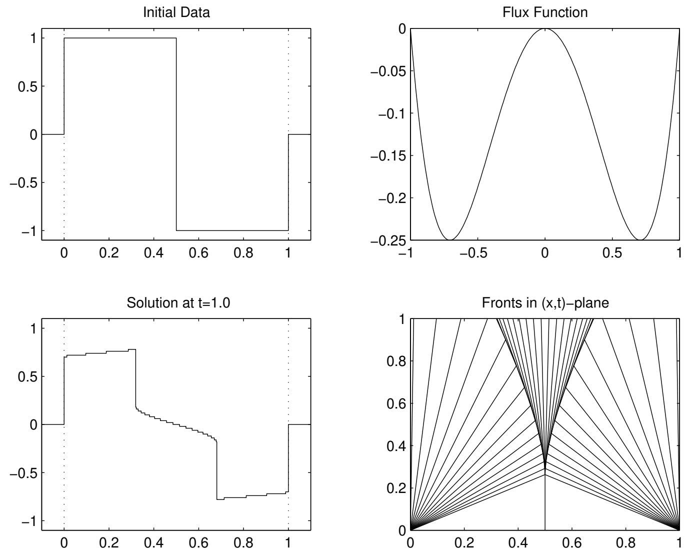

FIGURE 2. Example 1.

3.2. Arbitrary Spatial Dimension

As for the Dirichlet problem we can use dimensional splitting to construct solutions in two or higher space dimensions for (1) and (3). We cover the domain Ω by a Cartesian grid and make a piecewise constant approximation of the boundary $\partial\Omega$. For piecewise linear flux functions $f = (f_1, f_2)$ the approximate solution u_η can be defined as in (9) and (10).

The constructed solution is bounded; this follow from (15) and the properties of the projection operator. However, establishing a bound on the total variation is more difficult, since we have no control over the boundary trace *constructed* initially in each step. This is opposed to the Dirichlet case, where one could assume a certain regularity of the *prescribed* boundary condition.

4. Numerical Examples

EXAMPLE 1 The first example is a one-dimensional Dirichlet problem with prescribed boundary values equal zero at $x = 0$ and $x = 1$, as given in Fig. 2. Initially there are three Riemann problems; from each boundary we get rarefaction waves that propagate into the domain and collide simultaneously with the stationary shock from the Riemann problem at $x = 0.5$. This produces a symmetric rarefaction wave propagating out towards the boundaries.

EXAMPLE 2 The next example is a one-dimensional zero flux problem, see Fig. 3. Each Riemann problem gives a rarefaction wave propagating to the left and two nearly aligned shocks that propagates right. A complex wave pattern develops as the waves reflect from the boundaries and interact with each other. Note that

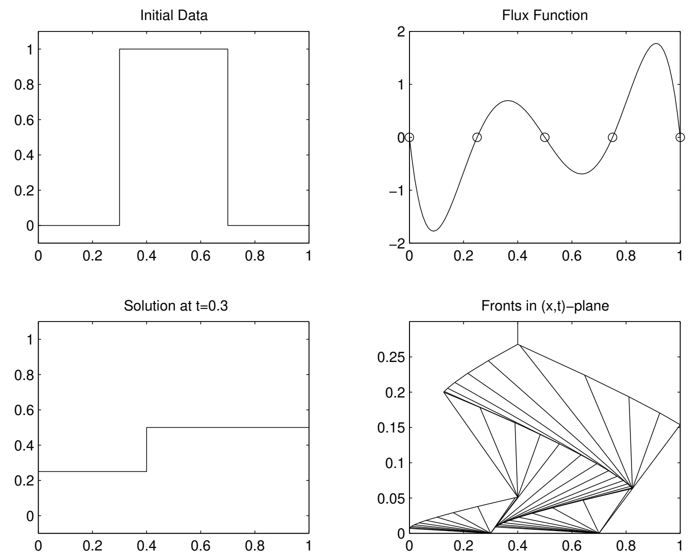

FIGURE 3. Example 2.

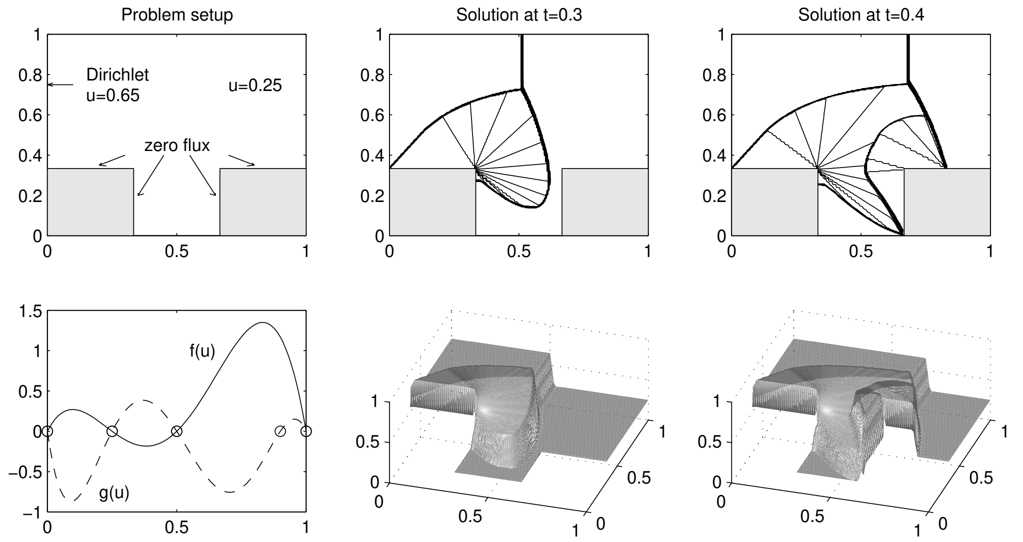

FIGURE 4. Example 3.

the boundary values change as waves are reflected at the boundaries. After time $t = 0.27$ the solution is stationary and consists of two zeros in the flux function.

EXAMPLE 3 Fig. 4 (left) describes the setup for the next problem. At the unspecified boundaries we impose absorbing boundary conditions; i.e., waves are allowed to pass out of the boundary with no effects on them. The Dirichlet boundary condition at the left boundary gives a shock wave that propagates into the domain.

The post-shock value corresponds to a nonzero flux in the y-direction and a complex wave pattern forms, consisting of rarefaction waves and three shocks meeting in a triple point. The rightmost shock propagates faster than the incident shock wave and is diffracted around the corner (Fig. 4, middle). Then the leading shock is reflected at the right vertical wall (Fig. 4, right). The reflected wave propagates backward and upward to meet with the triple point.

References

[1] C. Bardos, A. Y. LeRoux, and J. Nedelec. First order quasilinear equations with boundary conditions. *Comm. in Partial Differential Equations*, **4 (9)** (1979), 1017–1034.

[2] S. Benharbit, A. Chalabi, and J. P. Vila. Numerical viscosity and convergence of finite volume methods for conservation laws with boundary conditions *SIAM J. Numer. Anal.*, **32** (1995), 775–796.

[3] H. Holden, L. Holden, and R. Høegh-Krohn. A numerical method for first order nonlinear scalar conservation laws in one-dimension. *Comput. Math. Applic.*, **15 (6–8)** (1988) 595–602.

[4] H. Holden and N. H. Risebro. Front tracking for conservation laws. Department of Mathematics, Norwegian University of Science and Technology. Lecture Notes.

[5] H. Holden and N. H. Risebro. A method of fractional steps for scalar conservation laws without the CFL condition. *Math. Comp.*, **60 (201)** (1993), 221–232.

[6] K.-A. Lie, V. Haugse, and K. H. Karlsen. Dimensional splitting with front tracking and adaptive grid refinement. *Numer. Methods Partial Differential Equations*. To appear.

[7] J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs*, volume 13 of *Applied Mathematics and Mathematical Computation*. Chapman & Hall, London, 1996.

[8] A. Szepessy Measure valued solutions to scalar conservation laws with boundary conditions. *Arch. Rational Mech. Anal*, **107** (1989), 181–193.

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