On dynamic iteration for delay differential equations

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September 30, 1994

Abstract

In this paper we study dynamic iteration techniques for systems of nonlinear delay differential equations. After pointing out a close connection to the 'truncated infinite embedding', as proposed by Feldstein, Iserles, and Levin, we give a proof of the superlinear convergence of the simple dynamic iteration scheme. Then we propose a more general scheme that in addition allows for a decoupling of the equations into disjoint subsystems, just like what we are used to from dynamic iteration schemes for ODEs. This scheme is also shown to converge superlinearly.

Mathematics Subject Classification (1991): 65L05, 34K99
Keywords: delay differential equations, dynamic iteration, waveform relaxation

1 Introduction

Since the early eighties there has been a considerable interest in a class of iterative methods for ODE initial value problems called dynamic iteration or waveform relaxation methods. The basic idea of the techniques date back to the work by Picard and Lindelöf in the end of the last century, but the actual numerical implementation of such techniques were never a real issue until 100 years later. At this time, fast computing facilities with large amounts of memory made it reasonable to consider the practical exploitation of dynamic iteration methods, and it was in the electrical engineering community that the method was first proposed in its recent form by Lelarasmee *et.al.* [2]. With the growth of parallel computers, the methods have become even more interesting.

The idea is basically to simplify the dependency structure in a system of differential equations by introducing an iteration, in which some of the couplings in the right hand side are read from the previous iteration. More precisely, given the initial value problem

$$\dot{y} = f(y) \ , \ \ y(0) = y_0 \ , \tag{1}$$

we define the iteration

$$\dot{y}^{(k+1)} = \tilde{f}(y^{(k+1)}, y^{(k)}) \ , \ \ y^{(k+1)}(0) = y_0 \ , \tag{2}$$

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$k = 0, 1, 2, \ldots$; where $\tilde{f}$ must satisfy $\tilde{f}(y, y) = f(y)$. Here, and in the remainder of this paper, we work in $\mathbb{R}^m$, equipped with the Euclidean norm. The initial function $y^{(0)}(t)$ can be anything as long as it satisfies the initial condition, but it is in practice most often taken to be constant. It is well known that the iteration converges superlinearly on finite intervals (see e.g. [3]). If $\tilde{f}$ is chosen such that the system appearing in the iteration (2) consists of independent subsystems of $y^{(k+1)}$, $\tilde{f}$ is called a block-Jacobi splitting of the system; needless to say, this is an ideal situation from a parallel point of view.

For systems of ODEs it is the difficulties introduced by the couplings between the different components of the system that motivates the use of dynamic iteration. For the scalar delay equation

$$\dot{y}(t) = F(t, y(t), y(\theta(t))) \ \ , \ \ y(0) = y_0 \ \ , \ \ \theta(0) = 0 \tag{3}$$

however, it is the coupling between $y(t)$ and $y(\theta(t))$ that introduces difficulties. This coupling can be relaxed by defining a dynamic iteration scheme in which the delay term is read from the previous iteration. In other words, instead of (3), we introduce the scheme

$$\dot{y}^{(k+1)}(t) = F(t, y^{(k+1)}(t), y^{(k)}(\theta(t))) \ \ , \ \ y^{(k+1)}(0) = y_0 \ , \tag{4}$$

where $k = 0, 1, 2, \ldots$ and $y^{(0)}(t)$ is a given function. A natural choice is $y^{(0)}(t) \equiv y_0$. In this way, we are left with a 'normal' ODE to solve in each iteration. This dynamic iteration scheme is what we intend to investigate below.

2 Relation to the 'standard embedding'

It is interesting to see that the sequence of iterates $y^{(k+1)}(t)$, defined by (4), is actually nothing but scaled and dilated versions of the components in the infinite-dimensional system of ODEs into which (3) can be embedded, as described by Feldstein, Iserles, and Levin [1]. This system is given by

$$\dot{x}_m(t) = r^m \dot{\theta}_m(t) F(\theta_m(t), r^{-m} x_m(t), r^{-m-1} x_{m+1}(t)) \ \ , \ \ x_m(0) = r^m y_0 \tag{5}$$

for $m \in \mathbb{Z}^+$ and $r \in (0, 1)$; where $\theta_m(t) = \theta \circ \cdots \circ \theta(t)$ (m times), and $\theta_0(t) = t$. The 'standard embedding' of (3) follows from the fact that $x_0(t) \equiv y(t)$. We can state the following result.

THEOREM 2.1 *For $N \geq 0$, let $y^{(0)}(t) \equiv y_0$, $x_{N+1}(t) \equiv r^{N+1} y_0$, and consider the sequences $\{y^{(k)}\}_{k=0}^{N+1}$ and $\{x_m\}_{m=0}^{N+1}$. Then we have the following relation*

$$x_{N+1-j}(t) = r^{N+1-j} y^{(j)}(\theta_{N+1-j}(t)) \ \ , \ \ j = 0, \ldots, N+1 \ .$$

Proof. Let $\omega^j(t) = x_{N+1-j}(t)/r^{N+1-j}$, $j = 0, \ldots, N+1$. Then we have, for $j = 1, \ldots, N+1$,

$$\begin{aligned}
\dot{\omega}^j(t) &= \frac{\dot{x}_{N+1-j}(t)}{r^{N+1-j}} \\
&= \dot{\theta}_{N+1-j}(t) F(\theta_{N+1-j}(t), r^{-(N+1-j)} x_{N+1-j}(t), r^{-(N+1-j)-1} x_{(N+1-j)+1}(t)) \\
&= \dot{\theta}_{N+1-j}(t) F(\theta_{N+1-j}(t), \omega^j(t), \omega^{j-1}(t)) \ .
\end{aligned}$$

Let $\eta^j(t) = y^{(j)}(\theta_{N+1-j}(t))$, $j = 0, \ldots, N+1$. Then we have, for $j = 1, \ldots, N+1$,

$$\begin{aligned}
\dot{\eta}^j(t) &= F(\theta_{N+1-j}(t), y^{(j)}(\theta_{N+1-j}(t)), y^{(j-1)}(\theta_{N+1-(j-1)}(t))) \cdot \dot{\theta}_{N+1-j}(t) \\
&= \dot{\theta}_{N+1-j}(t) F(\theta_{N+1-j}(t), \eta^j(t), \eta^{j-1}(t)) \ .
\end{aligned}$$

Since $\omega^j(0) = \eta^j(0) = y_0$, it follows that $\omega^j(t) = \eta^j(t)$ if $\omega^{j-1}(t) = \eta^{j-1}(t)$. But as $\omega^0(t) = \eta^0(t)$ by assumption, it follows by induction that $\omega^j(t) = \eta^j(t)$ for $j = 0, \ldots, N+1$. In other words,

$$\frac{x_{N+1-j}(t)}{r^{N+1-j}} = y^{(j)}(\theta_{N+1-j}(t))$$

for $j = 0, \ldots, N+1$. □

Hence, the numerical method one gets by truncating (5) to a system with $N+1$ components, and successively solving for $x_m(t)$, $m = N, \ldots, 0$ as proposed in [1], is essentially the same as calculating the $y^{(k)}$, $k = 1, \ldots, N+1$ in the dynamic iteration scheme. In other words, this means that the choice of N in the former method is not critical, as passing to negative m just corresponds to exceeding N iterations in the dynamic iteration scheme. Of course, the results for negative m would need some rescaling in order to be appropriate approximations to $y(t)$. Note that we easily can state a more general version of the theorem.

Corollary 2.2 *The conclusion of Theorem 2.1 holds unchanged also if the assumptions on $y^{(0)}(t)$ and $x_{N+1}(t)$ are changed to*

$$y^{(0)}(\theta_{N+1}(t)) = r^{-(N+1)} x_{N+1}(t) \, .$$

Proof. The proof of Theorem 2.1 holds verbatim also in this case. □

3 Convergence properties

We now look at the convergence properties of the dynamic iteration scheme (4) for an autonomous problem, namely

$$\dot{y}^{(k+1)}(t) = F(y^{(k+1)}(t), y^{(k)}(\theta(t))) \quad , \quad y^{(k+1)}(0) = y_0 \, . \tag{6}$$

We assume in the following that F is Lipschitz continuous in its second argument, and satisfies a one-sided Lipschitz condition with respect to its first argument; i.e.,

$$(F(v_1, w) - F(v_2, w), v_1 - v_2) \le \mu \|v_1 - v_2\|^2 \, , \tag{7}$$

$$\|F(v, w_1) - F(v, w_2)\| \le \nu \|w_1 - w_2\| \, . \tag{8}$$

We also assume

$$\theta(t) \le t \quad , \quad \theta'(t) > 0 \, , \tag{9}$$

which is natural when we talk of delay equations. Now, let $e_k(t) = y^{(k)}(t) - y(t)$; then we have

$$\begin{aligned} (\dot{e}_{k+1}(t), e_{k+1}(t)) &= \Big(F(y^{(k+1)}(t), y^{(k)}(\theta(t))) - F(y(t), y(\theta(t))), y^{(k+1)}(t) - y(t)\Big) \\ &= \Big(F(y^{(k+1)}(t), y^{(k)}(\theta(t))) - F(y(t), y^{(k)}(\theta(t))), y^{(k+1)}(t) - y(t)\Big) \\ &\quad + \Big(F(y(t), y^{(k)}(\theta(t))) - F(y(t), y(\theta(t))), y^{(k+1)}(t) - y(t)\Big) \\ &\le \mu \|e_{k+1}(t)\|^2 + \nu \|e_k(\theta(t))\| \|e_{k+1}(t)\| \, . \end{aligned} \tag{10}$$

Since

$$(\dot{e}_{k+1}(t), e_{k+1}(t)) = \frac{1}{2}\frac{d}{dt}\|e_{k+1}(t)\|^2 = \|e_{k+1}(t)\| \cdot \frac{d}{dt}\|e_{k+1}(t)\| \, ; \tag{11}$$

we have, when $\|e_{k+1}(t)\| \neq 0$ the inequality

$$\frac{d}{dt}\|e_{k+1}(t)\| \leq \mu\|e_{k+1}(t)\| + \nu\|e_k(\theta(t))\| \,. \tag{12}$$

For those t where $\|e_{k+1}(t)\|$ happens to vanish it is probably not differentiable, even if $e_{k+1}(t)$ is C^1 there due to the conditions on F, so in these points (12) cease to be valid. However, the derivations below only depend on the Gronwall lemma, which still holds by applying it successively on the intervals between the zeroes of $\|e_{k+1}(t)\|$. Now we can state the following lemma.

LEMMA 3.1 *Given the assumptions (7), (8), and (9), we have*

$$\begin{aligned} \|e_{k+1}(t)\| \quad \leq \quad & \nu^{k+1}\int_0^t\int_0^{s_{k+1}}\cdots\int_0^{s_2}\|e_0(\theta_{k+1}(s_1))\| \times \\ & \times e^{\mu(\theta_k(s_2)-\theta_k(s_1))}\cdots e^{\mu(\theta(s_{k+1})-\theta(s_k))}e^{\mu(t-s_{k+1})} \times \\ & \times \theta'_k(s_1)\cdots\theta'_2(s_{k-1})\theta'(s_k)\,ds_1\cdots ds_k\,ds_{k+1}\,. \end{aligned} \tag{13}$$

Proof. First note that the inequality (12) implies

$$\|e_{k+1}(t)\| \leq \nu\int_0^t\|e_k(\theta(s))\|e^{\mu(t-s)}\,ds \tag{14}$$

by Gronwall's lemma. The statement of the lemma is now proved by induction. We note that (13) is true for $k=0$ by (14), and if it is assumed that (13) is true when $k+1$ is replaced by k we have

$$\begin{aligned} \|e_{k+1}(t)\| \quad \leq \quad & \nu\int_0^t\|e_k(\theta(s_{k+1}))\|e^{\mu(t-s_{k+1})}\,ds_{k+1} \\ \leq \quad & \nu\int_0^t\Bigg[\nu^k\int_0^{\theta(s_{k+1})}\int_0^{z_k}\cdots\int_0^{z_2}\|e_0(\theta_k(z_1))\| \times \\ & \times e^{\mu(\theta_{k-1}(z_2)-\theta_{k-1}(z_1))}\cdots e^{\mu(\theta(z_k)-\theta(z_{k-1}))}e^{\mu(\theta(s_{k+1})-z_k)} \times \\ & \times \theta'_{k-1}(z_1)\cdots\theta'_2(z_{k-2})\theta'(z_{k-1})\,dz_1\cdots dz_{k-1}\,dz_k\Bigg]\,e^{\mu(t-s_{k+1})}\,ds_{k+1}\,. \end{aligned}$$

Now, introduce the new variables

$$s_j = \theta^{-1}(z_j)\,,\ j = 1,\ldots,k\,;$$

i.e., we get the substitution rules

$$z_j = \theta(s_j) \ \text{ and } \ dz_j = \theta'(s_j)ds_j\,.$$

Performing the change of variables in the integral above yields

$$\begin{aligned} \|e_{k+1}(t)\| \quad \leq \quad & \nu\int_0^t\Bigg[\nu^k\int_0^{s_{k+1}}\int_0^{s_k}\cdots\int_0^{s_2}\|e_0(\theta_{k+1}(s_1))\| \times \\ & \times e^{\mu(\theta_k(s_2)-\theta_k(s_1))}\cdots e^{\mu(\theta_2(s_k)-\theta_2(s_{k-1}))}e^{\mu(\theta(s_{k+1})-\theta(s_k))} \times \\ & \times \theta'_{k-1}(\theta(s_1))\cdots\theta'_2(\theta(s_{k-2}))\theta'(\theta(s_{k-1})) \times \\ & \times\ \theta'(s_1)ds_1\cdots\theta'(s_{k-1})ds_{k-1}\,\theta'(s_k)ds_k\Bigg]\,e^{\mu(t-s_{k+1})}\,ds_{k+1}\,. \end{aligned}$$

By noting that

$$\theta_i'(\theta(s_j)) \cdot \theta'(s_j) = \theta_{i+1}'(s_j) \ , \tag{15}$$

the above expression reduces to (13), and hence the induction is completed. □
We now proceed to get a bound for this integral. Note first the relation

$$\theta_{i+1}'(s_j) = \prod_{\ell=0}^{i} \theta'(\theta_\ell(s_j)) \ ,$$

which is a consequence of (15). This means in particular that

$$\begin{aligned} \theta_k'(s_1) \cdots \theta_2'(s_{k-1}) \theta'(s_k) &= \prod_{n=1}^{k} \theta_n'(s_{k+1-n}) \\ &= \prod_{n=1}^{k} \prod_{\ell=0}^{n-1} \theta'(\theta_\ell(s_{k+1-n})) \\ &= \prod_{\ell=0}^{k-1} \prod_{n=\ell+1}^{k} \theta'(\theta_\ell(s_{k+1-n})) \ . \end{aligned} \tag{16}$$

When it comes to the exponential terms in the integrand, they can be estimated according to the following lemma.

Lemma 3.2 *Let* $\mu' = \max\{0, \mu\}$. *Then*

$$e^{\mu(\theta_k(s_2) - \theta_k(s_1))} \cdots e^{\mu(\theta(s_{k+1}) - \theta(s_k))} e^{\mu(t - s_{k+1})} \le e^{\mu' t} \ . \tag{17}$$

Proof. For $\mu = 0$ the statement is trivial. Let $\mu > 0$, and observe that we can write the exponent in the left hand side of (17) as

$$\mu \left(\sum_{j=1}^{k} [\theta_j(s_{k-j+2}) - \theta_{j-1}(s_{k-j+2})] + t - \theta_k(s_1) \right) \ =: \ \mu \cdot a_1 \ .$$

Using the fact that $\theta_j(s_i) - \theta_{j-1}(s_i) \le 0$, it then follows that

$$e^{\mu \cdot a_1} \le e^{\mu(t - \theta_k(s_1))} \le e^{\mu t} \ .$$

For $\mu < 0$, we write the expression as

$$\mu \left(\sum_{j=1}^{k} [\theta_j(s_{k-j+2}) - \theta_j(s_{k-j+1})] + t - s_{k+1} \right) \ =: \ \mu \cdot a_2 \ .$$

Now using the fact that $\theta_j(s_i) - \theta_j(s_{i-1}) \ge 0$ we see that

$$e^{\mu \cdot a_2} \le e^{\mu(t - s_{k+1})} \le 1 \ .$$

This completes the proof of the lemma. □
So we have the following corollary of Lemma 3.1.

COROLLARY 3.3 *Given the assumptions (7), (8), and (9), we have*

$$\begin{aligned}\|e_{k+1}(t)\| &\leq \nu^{k+1} e^{\mu t} \int_0^t \int_0^{s_{k+1}} \cdots \int_0^{s_2} \|e_0(\theta_{k+1}(s_1))\| \times \\ &\quad \times \prod_{\ell=0}^{k-1} \prod_{n=\ell+1}^{k} \theta'(\theta_\ell(s_{k+1-n}))\, ds_1 \cdots ds_k\, ds_{k+1} .\end{aligned}$$

Using this result, we can state the following theorem.

THEOREM 3.4 *Given the assumptions (7), (8), and (9), we have*

$$\sup_{t\in[0,T]} \|e_{k+1}(t)\| \leq e^{\mu' T} \frac{(\nu T)^{k+1}}{(k+1)!} \prod_{\ell=0}^{k-1} \left(\sup_{t\in[0,\theta_\ell(T)]} \theta'(t) \right)^{k-\ell} \sup_{t\in[0,\theta_{k+1}(T)]} \|e_0(t)\| .$$

Proof. First we note that all factors in the integrand are positive. The product term comes from taking the supremum of (16) over $[0,T]$, and similarly for the term containing the initial error. After taking these two terms outside the integral, we get an iterated integral in which the integrand is 1. Calculating this integral yields $T^{k+1}/((k+1)!)$, and the proof is complete. □

We see that if $\theta'(t)$ is bounded by a constant, say $C(\theta)$, we have

$$\prod_{\ell=0}^{k-1} \left(\sup_{t\in[0,\theta_\ell(T)]} \theta'(t) \right)^{k-\ell} \leq \prod_{\ell=0}^{k-1} C(\theta)^{k-\ell} = C(\theta)^{\sum_{\ell=0}^{k-1}(k-\ell)} = C(\theta)^{k(k+1)/2} .$$

This leads to the following main convergence theorem.

THEOREM 3.5 *Assume $C(\theta) = q \leq 1$. Then, given the assumptions (7), (8), and (9), we have*

$$\sup_{t\in[0,T]} \|e_{k+1}(t)\| \leq e^{\mu' T} \frac{(\nu T)^{k+1}}{(k+1)!} q^{k(k+1)/2} \sup_{t\in[0,\theta_{k+1}(T)]} \|e_0(t)\| .$$

In other words, the iteration converges on all finite time intervals, since the first two terms are equal to the terms predicting the superlinear convergence of dynamic iteration for ODEs, and the remaining term consists of a term converging to zero as $q^{k(k+1)/2}$ if $q < 1$, and a term converging to zero as $\mathcal{O}(\theta_{k+1}(T))$ if $\theta(t) < t$.

Proof. The only thing that needs clarification is the last statement, which follows by looking at the Taylor expansion of $e_0(t)$ around $t = 0$, taking into account that $y_0(0) = y(0)$; i.e., we have

$$e_0(t) = t(\dot{y}(0) - \dot{y}_0(0)) + \mathcal{O}(t^2) ,$$

and the result follows. □

REMARK 3.6 It can be noted that if $\theta(t) = t$, this proof reduces to the proof for the superlinear convergence of dynamic iterations for ODEs (2), with $\tilde{f}$ replaced by F.

EXAMPLE 1 *Let us look at the scalar pantograph equation $\dot{y}(t) = ay(t) + by(qt)$, where $0 < q < 1$. We have $\mu' = \max\{a, 0\}$, $\nu = |b|$ and $C(\theta) = C(q) = q$. We then get the following bound on the error in the iteration for $a \geq 0$:*

$$\sup_{t\in[0,T]} |y^{(k+1)}(t) - y(t)| \leq e^{aT} \frac{(|b|T)^{k+1}}{(k+1)!} q^{k(k+1)/2} \sup_{t\in[0,q^{k+1}T]} |y^{(0)}(t) - y(t)| ,$$

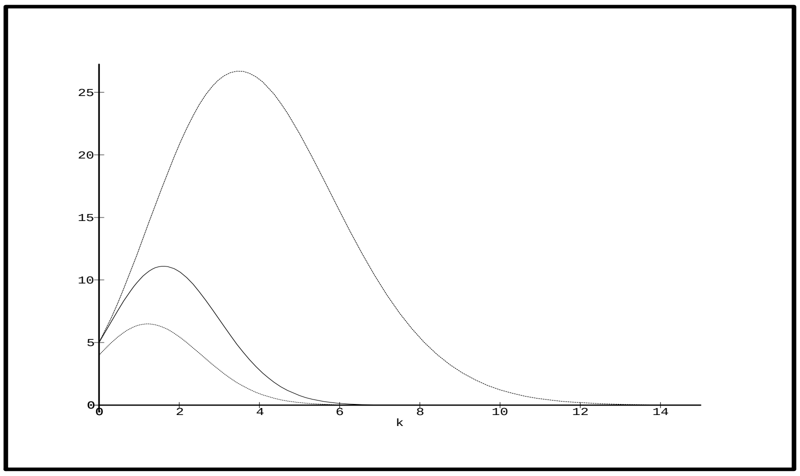

Figure 1: The functions f_1, f_2 and f_3 with $A = 5$, $q = 0.8$

and for $a < 0$ we just replace e^{aT} by 1. To illustrate the speed with which the error decays, we have in Figure 1 plotted the three functions (where A is a real constant)

$$f_1(k) = \frac{A^{k+1}}{(k+1)!} \,,$$

$$f_2(k) = \frac{A^{k+1}}{(k+1)!}\, q^{k(k+1)/2} \,,$$

$$f_3(k) = \frac{A^{k+1}}{(k+1)!}\, q^{k(k+1)/2}\, q^{k+1} \,;$$

for the special case with $A = 5$ and $q = 0.8$. The first function is often appearing when one wants to show the superlinear convergence of dynamic iteration for ODEs, the second includes the term showing the influence of the delay, and the last includes the diminishing effect of the error in the initial guess. In the language of truncated infinite embedding, this bound can be written as

$$\begin{aligned}
\sup_{t\in[0,T]} |x_0(t) - y(t)| &= \sup_{t\in[0,T]} |y^{(N+1)}(t) - y(t)| \\
&\leq e^{aT} \frac{(|b|T)^{N+1}}{(N+1)!}\, q^{N(N+1)/2} \sup_{t\in[0,q^{N+1}T]} |y^{(0)}(t) - y(t)| \\
&= e^{aT} \frac{(|b|T)^{N+1}}{(N+1)!}\, q^{N(N+1)/2} \sup_{t\in[0,q^{N+1}T]} |r^{-N-1} x_{N+1}(t/q^{N+1}) - y(t)| \\
&= e^{aT} \frac{(|b|T)^{N+1}}{(N+1)!}\, q^{N(N+1)/2} \sup_{t\in[0,T]} |r^{-N-1} x_{N+1}(t) - y(q^{N+1}t)| \,.
\end{aligned}$$

Remember that in this last approach, $x_{N+1}(t)$ is a fixed function taken as our initial guess, and $x_0(t)$ is the approximation to $y(t)$ produced by the infinite embedding truncated to a system of $N+1$ equations.

4 A more general scheme

One of the motivations for using the dynamic iteration approach for systems of ODE initial value problems, is the potential for parallelism it offers. In their paper, Feldstein, Iserles, and

Levin propose a parallel implementation of the truncated scheme (5). Their algorithm seems especially suitable for massively parallel machines, and works for scalar equations too. For systems of equations, however, we would also like the splitting introduced by the dynamic iteration to give us the possibility of integrating different subsystems independently. As this is not the case in the scheme (4), we introduce a more general scheme, namely (in autonomous form)

$$\dot{y}^{(k+1)}(t) = G(y^{(k+1)}(t), y^{(k)}(t), y^{(k)}(\theta(t))) \ , \ \ y^{(k+1)}(0) = y_0 \ , \tag{18}$$

where $k = 0, 1, 2, \ldots$ and $y^{(0)}(t)$ is, as always, a given function. In practice we would choose $G(u, v, w)$ such that the Jacobian $\partial G/\partial u$ is block diagonal. This means that the system decouples into subsystems which can be solved independently of each other in each iteration. The function G must clearly satisfy

$$G(y(t), y(t), y(\theta(t))) = F(y(t), y(\theta(t))) \, . \tag{19}$$

If we want to prove convergence of this scheme, we must place some restrictions on G. We will use the same tools as in the previous section, and we therefore assume that

$$(G(u_1, v, w) - G(u_2, v, w), u_1 - u_2) \leq \mu \|u_1 - u_2\|^2 \ , \tag{20}$$

$$\|G(u, v_1, w) - G(u, v_2, w)\| \leq \nu_1 \|v_1 - v_2\| \ , \tag{21}$$

$$\|G(u, v, w_1) - G(u, v, w_2)\| \leq \nu_2 \|w_1 - w_2\| \ . \tag{22}$$

Note that these restrictions on G implicitly impose restrictions on F, through (19). As before, we use the notation $e_k(t) = y^{(k)}(t) - y(t)$; and by calculations completely analogous to the ones in (10) and (11) we get when $\|e_{k+1}(t)\| \neq 0$ the inequality

$$\frac{d}{dt}\|e_{k+1}(t)\| \leq \mu \|e_{k+1}(t)\| + \nu_1 \|e_k(t)\| + \nu_2 \|e_k(\theta(t))\| \ .$$

Now, Gronwall's lemma yields

$$\|e_{k+1}(t)\| \leq \int_0^t (\nu_1 \|e_k(s)\| + \nu_2 \|e_k(\theta(s))\|) \, e^{\mu(t-s)} \, ds \ . \tag{23}$$

We have the following lemma.

Lemma 4.1 *Given the assumptions (19), (20), (21), (22), and the following relaxed version of (9)*

$$0 \leq \theta(t) \leq t \ , \tag{24}$$

we have

$$\begin{aligned} \|e_{k+1}(t)\| \ \leq \ & \int_0^t \int_0^{s_{k+1}} H_{s_{k+1}}(s_k) \int_0^{s_k} H_{s_k}(s_{k-1}) \cdots \int_0^{s_2} H_{s_2}(s_1) \ \times \\ & \times \ (\nu_1 \|e_0(s_1)\| + \nu_2 \|e_0(\theta(s_1))\|) \, e^{\mu'(t-s_1)} \, ds_1 \cdots ds_k \, ds_{k+1} \ , \end{aligned} \tag{25}$$

where μ' is as before, and

$$H_u(v) = \begin{cases} \nu_1 + \nu_2 & , \quad 0 \leq v \leq \theta(u) \\ \nu_1 & , \quad \theta(u) < v \leq u \end{cases}$$

Proof. We prove the statement by induction. For $k = 0$ (25) is immediately satisfied by (23). Assume (25) is true, with $k+1$ replaced by k. We then have, from (23),

$$
\begin{aligned}
\|e_{k+1}(t)\| &\leq \int_0^t (\nu_1 \|e_k(s_{k+1})\| + \nu_2 \|e_k(\theta(s_{k+1}))\|)\, e^{\mu'(t-s_{k+1})}\, ds_{k+1} \\
&\leq \int_0^t \Bigg(\nu_1 \Bigg[\int_0^{s_{k+1}} \int_0^{s_k} H_{s_k}(s_{k-1}) \cdots \int_0^{s_2} H_{s_2}(s_1) \times \\
&\qquad\qquad \times (\cdots)\, e^{\mu'(s_{k+1}-s_1)}\, ds_1 \cdots ds_{k-1}\, ds_k \Bigg] + \\
&\qquad + \nu_2 \Bigg[\int_0^{\theta(s_{k+1})} \int_0^{s_k} H_{s_k}(s_{k-1}) \cdots \int_0^{s_2} H_{s_2}(s_1) \times \\
&\qquad\qquad \times (\cdots)\, e^{\mu'(\theta(s_{k+1})-s_1)}\, ds_1 \cdots ds_{k-1}\, ds_k \Bigg] \Bigg)\, e^{\mu'(t-s_{k+1})}\, ds_{k+1} \\
&\leq \int_0^t \int_0^{s_{k+1}} H_{s_{k+1}}(s_k) \int_0^{s_k} H_{s_k}(s_{k-1}) \cdots \int_0^{s_2} H_{s_2}(s_1) \times \\
&\qquad \times (\nu_1 \|e_0(s_1)\| + \nu_2 \|e_0(\theta(s_1))\|)\, e^{\mu'(t-s_1)}\, ds_1 \cdots ds_k\, ds_{k+1} \;;
\end{aligned}
$$

where the last inequality is possible by noting that $e^{\mu'(\theta(s_{k+1})-s_1)} \leq e^{\mu'(s_{k+1}-s_1)}$, and splitting the first of the two integrals in square brackets in two parts. Hence the lemma is proved. □
We can now show that the scheme (18) converges superlinearly, just by making a rather crude estimate of the expression for $\|e_{k+1}(t)\|$ in the lemma.

THEOREM 4.2 *Given the assumptions (19), (20), (21), (22), and (24), we have*

$$
\sup_{t\in[0,T]} \|e_{k+1}(t)\| \leq \frac{[(\nu_1+\nu_2)T]^{k+1}}{(k+1)!} e^{\mu' T} \sup_{t\in[0,T]} \|e_0(t)\| \,,
$$

i.e., the iteration converges superlinearly on all finite time intervals as $k \to \infty$.

Proof. The proof is clear by the following observations about the factors involved in the integrand of (25). Firstly, we have $\sup_{s\in[0,T]} H_r(s) \leq \nu_1 + \nu_2$. Secondly, $\sup_{s\in[0,T]}(\nu_1\|e_0(s)\| + \nu_2\|e_0(\theta(s))\|) \leq (\nu_1+\nu_2)\sup_{s\in[0,T]} \|e_0(s)\|$. Thirdly, $\sup_{s,t\in[0,T]} e^{\mu'(t-s)} \leq e^{\mu' T}$. Taking the supremum over $[0, T]$ in (25) and introducing these estimates, we are left to integrate the unit function. Evaluating these $k+1$ iterated integrals yields the result. □
It would be natural to try to obtain other results for the convergence by introducing better estimates than the ones used to prove the theorem. We will do this below. However, we seem to have lost some information already in Lemma 4.1 when we introduced μ' instead of μ, as this results in a loss of information about the possible dissipative nature of the system. Hence one should maybe not expect too much when deriving convergence estimates from the result of the lemma.

Let us try to keep the exponentials under the integral, but estimating the other terms as in the proof of Theorem 4.2. This leads to the problem of estimating an integral of the form

$$
\int_0^T \int_0^{s_{k+1}} \int_0^{s_k} \cdots \int_0^{s_2} e^{\mu'(t-s_1)}\, ds_1 \cdots ds_k\, ds_{k+1} \,.
$$

We can calculate this integral exactly to get, when $\mu' \neq 0$,

$$
(-\mu')^{-(k+1)}\, e^{\mu' T} \sum_{j=k+1}^{\infty} \frac{(-\mu' T)^j}{j!} =: (-\mu')^{-(k+1)}\, \Gamma_{k+1}(\mu') \,.
$$

When $\mu' > 0$ the sign of the function $\Gamma_k(\mu')$ is $(-1)^k$, and we have the bound

$$|\Gamma_k(\mu')| < e^{\mu' T} \ .$$

We have then proved the following result.

THEOREM 4.3 *Under the same assumptions as before, we have when* $\mu' > 0$

$$\sup_{t\in[0,T]} \|e_{k+1}(t)\| \leq \left(\frac{\nu_1+\nu_2}{-\mu'}\right)^{k+1} \Gamma_{k+1}(\mu') \sup_{t\in[0,T]} \|e_0(t)\| \ . \tag{26}$$

This result would have been more interesting if we could replace μ' by μ, since for negative μ the function $\Gamma_k(\mu)$ has the property

$$0 < \Gamma_{k+1}(\mu) < \Gamma_k(\mu) < 1 \ .$$

As remarked above, this is not possible in our case, since we take Lemma 4.1 as our point of departure. However, in the case where $\mu' > (\nu_1 + \nu_2)$, the result (26) is still interesting in the sense that it assures that the error will be bounded by a factor $e^{\mu' T}$ times the initial error for all k. Such estimates are crucial to assure the numerical stability of the iteration when implemented on a computer.

References

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