A note on the convergence of discretized dynamic iteration

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Abstract

In this note we prove convergence results, including error estimates, for the dynamic iteration scheme where the forward Euler and backward Euler method are used to compute the iterates. The proofs are interesting in that they are exact analogues of the proof for the continuous case, using discrete versions of Gronwall's inequality.

We wish to approach the solution $y : \mathbb{R}^+ \to \mathbb{R}^m$ of the initial value problem

$$\dot{y} = f(y) \ \ , \ \ y(0) = y_0 \ ,$$

where f is assumed to be globally Lipschitz continuous, by successively calculating the solutions $y^{(k)}$ of

$$\dot{y}^{(k)} = \tilde{f}(y^{(k)}, y^{(k-1)}) \ \ , \ \ y^{(k)}(0) = y_0 \ ,$$

for $k = 1, 2, 3, \ldots$ with $y^{(0)}(t) \equiv y_0$. Here $\tilde{f}(v, w)$ is a splitting of f; i.e., $\tilde{f}$ is a function satisfying $\tilde{f}(y, y) = f(y)$. We recall the convergence theorem for the continuous case.

THEOREM 1 *Let $T > 0$. If $\tilde{f}$ is globally Lipschitz continuous with constants K_1 and K_2, then $y^{(k)}$ converges uniformly to y on $[0, T]$ as $k \to \infty$. The asymptotic convergence rate is superlinear, which follows from the error estimate* (4) *below.*

Proof. The proof goes as follows: First we establish a control of the growth of the error $\|e^{(k)}(t)\| := \|y(t) - y^{(k)}(t)\|$ by bounding its derivative. Here $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^m$. Then we use Gronwall's lemma to convert this into a bound on $\|e^{(k)}(t)\|$ itself. By successive application, we are able to express the bound in terms of an iterated integral of the initial error $\|e^{(0)}(t)\|$. An estimate of this integral yields the result.

The bound of the derivative is found by

$$\begin{aligned} \|e^{(k)}\| \cdot \frac{d}{dt}\|e^{(k)}\| &= (\dot{e}^{(k)}, e^{(k)}) \\ &= (\tilde{f}(y, y) - \tilde{f}(y^{(k)}, y^{(k-1)}), y - y^{(k)}) \\ &\leq K_1\|y - y^{(k)}\|^2 + K_2\|y - y^{(k-1)}\| \cdot \|y - y^{(k)}\| \ , \end{aligned}$$

where $K_1, K_2 > 0$ are the Lipschitz constants of $\tilde{f}$. So, for $\|e^{(k)}(t)\| \neq 0$ we have the bound

$$\frac{d}{dt}\|e^{(k)}(t)\| \leq K_1\|e^{(k)}(t)\| + K_2\|e^{(k-1)}(t)\| \ . \tag{1}$$

Using Gronwall's lemma on the intervals where $\|e^{(k)}(t)\| \neq 0$, we get the bound

$$\|e^{(k)}(t)\| \leq K_2 \int_0^t e^{K_1(t-\tau)} \|e^{(k-1)}(\tau)\| d\tau \ . \tag{2}$$

Successive application of this bound yields

$$\|e^{(k)}(t)\| \leq K_2^k \int_0^t \int_0^{s_1} \cdots \int_0^{s_{k-1}} e^{K_1(t-s_k)} \|e^{(0)}(s_k)\| \, ds_k \cdots ds_2 \, ds_1 \ . \tag{3}$$

Taking the supremum, and estimating the iterated integrals by pulling everything out yields

$$\sup_{0\leq t\leq T} \|e^{(k)}(t)\| \leq \frac{(K_2 T)^k}{k!} e^{K_1 T} \sup_{0\leq t\leq T} \|e^{(0)}(t)\| \ . \tag{4}$$

The asymptotic linear convergence rate is given by

$$\lim_{k\to\infty} \left[\frac{(K_2 T)^k}{k!} e^{K_1 T} \right]^{1/k} = 0 \ .$$

This completes the proof. ■

Remark 2 If we in the proof rather let K_1 be the one-sided Lipschitz constant of $\tilde{f}$ with respect to its first argument, we still get the bound (3); however now K_1 can be negative. Following Zennaro [7], we can calculate the integrals leaving the exponential inside to get the following stronger estimate to replace (4) when $K_1 \neq 0$:

$$\sup_{0\leq t\leq T} \|e^{(k)}(t)\| \leq \left(\frac{K_2}{-K_1} \right)^k \left(\sum_{j=k}^{\infty} \frac{(-K_1 T)^j}{j!} \right) e^{K_1 T} \sup_{0\leq t\leq T} \|e^{(0)}(t)\| \ . \tag{5}$$

This estimate gives more insight than (4) into how the choice of the splitting (characterized by K_1 and K_2) affects the convergence. A brief discussion of (5) can be found in [2]. It should also be noted that in the linear case, it coincides when $K_1 < 0$ with an estimate derived by Nevanlinna (cf. [6], equation (2.21)).

Remark 3 Similar convergence results for the dynamic iteration as a method for solving delay differential equations have been derived by Bjørhus [3].

Now, assume that we want to apply a Runge-Kutta method with step size h in order to compute numerical approximations $U_n^{(k)} \approx y^{(k)}(nh)$. We assume that this is implemented by introducing the splitting $\tilde{f}$ of f directly into the Runge-Kutta formulae. This means that to compute the approximation in iteration k, we need the stage values and approximations computed in iteration $k-1$.

Remark 4 Another way to implement this is discussed by Bellen and Zennaro [1]. They use Runge-Kutta methods with continuous interpolants which enables the treatment of the previous iterate as a continuous forcing term. Contrary to our approach, the limit solution is generally not given by a Runge-Kutta method.

We want to compare $U_n^{(k)}$ to Y_n, where $Y_n \approx y(nh)$ is computed with the same Runge-Kutta method. For this purpose, we introduce $z(N, \mathbb{R}^m)$, the space of all sequences $Z = \{Z_0, Z_1, \ldots, Z_N\}$ where $Z_i \in \mathbb{R}^m$. This is a Banach space when equipped with the norm $|Z|_N = \max_{0 \le n \le N} \|Z_n\|$, where $\|\cdot\|$ is some norm on $\mathbb{R}^m$. We have the following result, whose proof follows immediately by the form of explicit Runge-Kutta methods.

THEOREM 5 *Let $N > 0$. If the Runge-Kutta method is explicit and has s stages, then $U^{(k)} = Y$ in $z(N, \mathbb{R}^m)$ when $k \ge sN$.*

The result is nice because it guarantees that the discretized dynamic iteration will, on all finite time intervals, reproduce the numerical solution Y of the original problem in a finite number of iterations. However, it is not so nice because it does not provide us with any error estimate for $k < N$, analogous to (4).

Below, we give convergence results for the forward Euler and backward Euler methods. They are proved by mimicking, in the discrete case, the proof of THEOREM 1. As byproducts of the proofs, we get error estimates in the spirit of (4). For this purpose, we need some discrete versions of continuous tools. First, we have two versions of Gronwall's lemma which now involves differences instead of derivatives. The difference operator $\delta : z(\infty, \mathbb{R}^m) \to z(\infty, \mathbb{R}^m)$ is defined as usual by $(\delta Z)_n = Z_{n+1} - Z_n$.

LEMMA 6 (Gronwall I) *Let $Z, \Psi \in z(\infty, \mathbb{R})$, and let $C > -1$. If $(\delta Z)_n \le C Z_n + \Psi_n$ for $n = 0, 1, 2, \ldots$ then*

$$Z_n \le (1+C)^n \left[Z_0 + \sum_{j=1}^{n} (1+C)^{-j} \Psi_{j-1} \right] \quad \text{for } n = 1, 2, 3, \ldots .$$

LEMMA 7 (Gronwall II) *Let $Z, \Psi \in z(\infty, \mathbb{R})$, and let $C < 1$. If $(\delta Z)_n \le C Z_{n+1} + \Psi_{n+1}$ for $n = 0, 1, 2, \ldots$ then*

$$Z_n \le (1-C)^{-n} \left[Z_0 + \sum_{j=1}^{n} (1-C)^{j-1} \Psi_j \right] \quad \text{for } n = 1, 2, 3, \ldots .$$

In addition we have the following combinatorial identity, playing the role as the discrete version of the continuous rule $\int_0^t \tau^r / r! \, d\tau = t^{r+1}/(r+1)!$.

LEMMA 8 *For $r = 0, 1, 2, \ldots$ we have*

$$\sum_{j=1}^{n} \binom{j+r-1}{r} = \binom{n+r}{r+1} .$$

We can now state our results. Let $E^{(k)} = Y - U^{(k)}$.

THEOREM 9 *Let $N \geq 1$. If $\tilde{f}$ is globally Lipschitz continuous with constants K_1 and K_2, and we use the forward Euler method with step size $h > 0$ to compute Y and $U^{(k)}$, then $Y = U^{(k)}$ for $k \geq N$. For $k < N$ we have the error estimate*

$$|E^{(k)}|_N \leq \left(\frac{hK_2}{1+hK_1}\right)^k \binom{N-1}{k} (1+hK_1)^{N-1} |E^{(0)}|_{N-k} \,.$$

THEOREM 10 *Let $N \geq 1$. If $\tilde{f}$ is globally Lipschitz continuous with constants K_1 and K_2, and we use the backward Euler method with step size $h < 1/(K_1+K_2)$ to compute Y and $U^{(k)}$, then $U^{(k)}$ converges to Y in $z(N, \mathbb{R}^m)$ as $k \to \infty$. The asymptotic convergence rate is linear, which follows from the error estimate*

$$|E^{(k)}|_N \leq \left(\frac{hK_2}{1-hK_1}\right)^k \binom{N+k-1}{k} (1-hK_1)^{1-N} |E^{(0)}|_N \,. \tag{6}$$

We will only prove THEOREM 10, since the proof of THEOREM 9 is similar; the major difference being that it relies on LEMMA 6 instead of LEMMA 7.

Proof. First we note that the assumptions on $\tilde{f}$ imply that the Lipschitz constant K of f satisfies $K \leq K_1 + K_2$. Hence the assumption on h implies that $h < 1/K$. This ensures that the solutions Y and $U^{(k)}$ are well defined (see e.g. [4]). The proof is a direct analogue of the proof of THEOREM 1, in that it is based on discrete versions of the bounds (1), (2), (3), and (4). To get the analogue of (1), we bound the differences of the sequence of errors $\|E^{(k)}\| \in z(\infty, \mathbb{R})$, defined by $\|E^{(k)}\|_n = \|Y_n - U_n^{(k)}\|$, as

$$\begin{aligned}
(\delta\|E^{(k)}\|)_n &= \|E^{(k)}\|_{n+1} - \|E^{(k)}\|_n \\
&\leq \|h\tilde{f}(Y_{n+1}, Y_{n+1}) - h\tilde{f}(U_{n+1}^{(k)}, U_{n+1}^{(k-1)})\| \\
&\leq hK_1\|Y_{n+1} - U_{n+1}^{(k)}\| + hK_2\|Y_{n+1} - U_{n+1}^{(k-1)}\| \,;
\end{aligned}$$

hence we get the bound

$$(\delta\|E^{(k)}\|)_n \leq hK_1\|E^{(k)}\|_{n+1} + hK_2\|E^{(k-1)}\|_{n+1} \,. \tag{7}$$

In the continuous case we used Gronwall's lemma to go from (1) to (2). Here we will make use of the discrete version given by LEMMA 7, which applied to (7) yields

$$\|E^{(k)}\|_n \leq hK_2 \sum_{j=1}^{n} (1-hK_1)^{j-n-1} \|E^{(k-1)}\|_j \quad \text{for } n = 1, 2, 3, \ldots$$

which is the analogue of (2). Successive application of this bound yields

$$\|E^{(k)}\|_n \leq (hK_2)^k \sum_{j_1=1}^{n} \sum_{j_2=1}^{j_1} \cdots \sum_{j_k=1}^{j_{k-1}} (1-hK_1)^{j_k-n-k} \|E^{(0)}\|_{j_k} \quad \text{for } n = 1, 2, 3, \ldots \,;$$

this is the analogue of (3). Now all that remains is to take the maximum over all N steps, and estimate the sum. In the continuous case we used the fact that $\int_0^t \tau^r/r!\, d\tau =$

$t^{r+1}/(r+1)!$ to calculate the k iterated integrals with integrand 1. In the discrete case, we use LEMMA 8 to evaluate the k iterated sums with summand 1. This leads to the final estimate (6). A calculation shows that the binomial coefficient appearing in the estimate is bounded from above by $(1+k)^{N-1}$. Hence the asymptotic linear convergence rate is given by

$$\lim_{k\to\infty}\left[C_1 C_2^k \begin{pmatrix} N+k-1 \\ k \end{pmatrix}\right]^{1/k} \le \lim_{k\to\infty}\left[C_1 C_2^k (1+k)^{N-1}\right]^{1/k} = C_2 = \frac{hK_2}{1-hK_1} ,$$

where $C_2 < 1$ by assumption. Hence the estimate depicts, for fixed h, linear convergence of the iteration. The result is proved. ■

Nevanlinna [6, ch. 4] derives convergence results for linear problems discretized by linear multistep methods, by analyzing the corresponding ζ-transformed problem (see also [5]). In particular, he presents the estimate

$$|k_h^{\nu *}(j)| \le (Ch)^\nu \begin{pmatrix} \nu-1+j \\ \nu-1 \end{pmatrix} e^{-\gamma hj} , \; j \ge 0 ,$$

where ν is the iteration index. We see the similarity to our error estimate (6) for non-linear problems, obtained by the more direct approach of using a discrete version of Gronwall's lemma. Since $k_h^{\nu *} = k_h * \cdots * k_h$ (ν times), and since $\mathscr{K}_h^\nu \varphi = k_h^{\nu *} * \varphi$, we obtain for $\gamma \le 0$ the estimate

$$|E^{(\nu)}|_N = |\mathscr{K}_h^\nu E^{(0)}|_N \le (N+1)(Ch)^\nu \begin{pmatrix} \nu-1+N \\ \nu-1 \end{pmatrix} e^{-\gamma T} |E^{(0)}|_N .$$

Hence, for linear problems the two estimates bound the error growth in essentially the same way. We can also note that the term $(1-hK_1)^{-N}$ in (6) approaches e^{TK_1} as $N \to \infty$.

The only place where we have made use of properties special to Euler's method, is in the calculations leading to the bound (7). We anticipate that if similar bounds are provided, our method of proof can be extended to establish convergence results also for more complicated Runge-Kutta methods.

References

[1] A. Bellen and M. Zennaro, *The use of Runge-Kutta formulae in waveform relaxation methods*, Appl. Numer. Math., 11 (1993), pp. 95–114.

[2] M. Bjørhus, *Dynamic iteration for time-dependent partial differential equations: A first approach*, Tech. Report Numerics no. 1, Departments of Mathematics, The University of Trondheim, Norway, 1992.

[3] ——, *On dynamic iteration for delay differential equations*, BIT, 34 (1994), pp. 325–336.

[4] E. Hairer, S. P. Nørsett, and G. Wanner, *Solving Ordinary Differential Equations I*, no. 8 in Springer Series in Computational Mathematics, Springer-Verlag, Berlin Heidelberg, 1987.

[5] U. Miekkala and O. Nevanlinna, *Sets of convergence and stability regions*, BIT, 27 (1987), pp. 554–584.

[6] O. Nevanlinna, *Remarks on Picard-Lindelöf iteration*, BIT, 29 (1989), pp. 328–346, 535–562.

[7] M. Zennaro, *Metodi waveform relaxation per equazioni differenziali ordinarie*. Handwritten lecture notes, University of L'Aquila, Italy, 1992.