# Recomposition of smooth functions in hyposmooth domains

Morten Bjørhus

**Abstract**

The notion of a hyposmooth decomposition of an open subset of $\mathbb{R}^n$ is defined. The subsets constituting a hyposmooth decomposition does not in general have smooth boundaries. However, by the way they are constructed they allow for investigations of the subdomain interfaces by localizing, using only smooth maps. This leads to a result for PDEs, concerning the recomposition of smooth solutions on the subdomains to a global smooth solution. In practical terms, the results ensures that the smooth solutions computed by a domain decomposition method correspond to a solution of the global problem.

## 1 Introduction

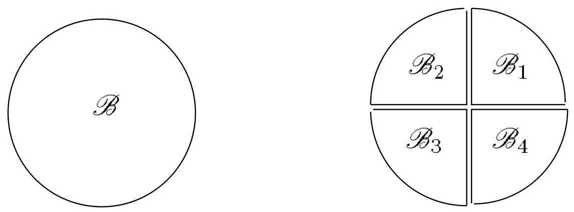
When solving systems of partial differential equations (PDEs) numerically, a natural way to attack the problem is often to decompose the domain into subdomains. This not only accommodates the use of parallel computers for the calculations, but is also in some cases the only way to treat non-smooth domains. These domain decomposition methods are extensively used when solving elliptic systems, and there is also done some work for evolution equations; cf. [7]. In the general case, however, it is not always clear in which sense the solutions of the subproblems correspond to a solution of the global problem. In this paper we discuss this issue in the context of smooth solutions.

Consider the following domain decomposition approach for solving a system of first order PDEs

$$P(x, D)u = f$$

in the unit ball in $\mathbb{R}^2$, possibly subject to boundary conditions on parts of the boundary. One splits the open ball $\mathscr{B}$ into four subdomains $\mathscr{B}_i$, $i = 1, \ldots, 4$; $\mathscr{B}_i$ being the part of $\mathscr{B}$ lying in the interior of the $i$th quadrant. One then solves the PDE on each subdomain, imposing the extra condition that the solutions in two adjacent subdomains should attain the same values on their common boundary.

A natural question to ask is whether the four computed solutions $v_i$ coincide with the restriction to the $\mathscr{B}_i$ of a solution $u$ to the original problem; $u$ having the same degree of smoothness as the $v_i$. This is what we refer to as the recomposition problem. More




precisely, does $v_i \in C^k(\overline{\mathscr{B}_i})$, $Pv_i = f|_{\mathscr{B}_i}$, and $v_i = v_{i'}$ on $\overline{\mathscr{B}_i} \cap \overline{\mathscr{B}_{i'}}$ imply the existence of a $u \in C^k(\overline{\mathscr{B}})$ such that $u|_{\mathscr{B}_i} = v_i$ and $Pu = f$? It turns out to be true if the coefficients of $P$ are elements of $C^{k-1}(\mathscr{B})$, if $f \in C^{k-1}(\mathscr{B})$, and if the two coordinate axes are non-characteristic with respect to $P$; see THEOREM 5.1 below.

The simple way to prove this is to first consider the situation on the two vertical interfaces

$$\overline{\mathscr{B}_1} \cap \overline{\mathscr{B}_2} \cap \mathscr{B} \setminus \{0\} \quad \text{and} \quad \overline{\mathscr{B}_3} \cap \overline{\mathscr{B}_4} \cap \mathscr{B} \setminus \{0\} ,$$

and then to consider the situation on the remaining horizontal surface

$$(\overline{\mathscr{B}_1} \cup \overline{\mathscr{B}_2}) \cap (\overline{\mathscr{B}_3} \cup \overline{\mathscr{B}_4}) \cap \mathscr{B} .$$

In this way we avoid explicit treatment of the corners present at the origin, as we use only the fact that the $\mathscr{B}_i$ can be generated by cutting $\mathscr{B}$ with the two coordinate axes, one after the other; each of which is an infinitely smooth surface.

This example is meant to serve as a motivation for the introduction of hyposmooth decompositions in the next section. This is a class of decompositions of open sets that enables the application of smooth localization in order to check for smoothness across the subdomain interfaces, even if the subdomains themselves are generally not smooth. In Section 3, we make some preliminary investigations of the recomposition problem in the two hemispheres of the unit ball. Then, in Section 4, these local results are used to provide answers to the recomposition problem for hyposmooth decompositions of general open sets. In Section 5 we discuss some generalizations of the results of Section 4.

# 2 Hyposmooth decompositions

Let us begin by fixing some of the notation used. Let $\mathscr{B}$ denote the open unit ball in $\mathbb{R}^n$, i.e.,

$$\mathscr{B} = \left\{ x = (x_1, \ldots, x_n) \;\middle|\; |x| = \sqrt{x_1^2 + \cdots + x_n^2} < 1 \right\} .$$

Let $\mathscr{B}^+ = \mathscr{B} \cap \{x_1 > 0\}$, $\mathscr{B}^0 = \mathscr{B} \cap \{x_1 = 0\}$, and $\mathscr{B}^- = \mathscr{B} \cap \{x_1 < 0\}$. Let $D = (D_1, \ldots, D_n)$ where $D_j = \partial/\partial x_j$, and let as usual $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index. In this notation, $D\varphi$ denotes the gradient of a mapping $\varphi : \mathbb{R}^n \to \mathbb{R}$. For a subset $E \subset \mathbb{R}^n$, $\overline{E}$, $E^\circ$, $E^c$, and $\partial E$ denotes the closure, interior, complement, and boundary of $E$, respectively. Sometimes the alternative notation $cl(E)$ will be used for $\overline{E}$.

We give the definition of a $C^k$ domain. Remember that a $k$-diffeomorphism $\Psi$ is a bijection such that both $\Psi$ and its inverse is of class $C^k$.

**DEFINITION 2.1** *Let $k \geq 0$. An open subset $A \subset \mathbb{R}^n$ is said to be a $C^k$ domain if for each $p \in \partial A$ there exists a neighborhood $U_p$ and a $k$-diffeomorphism $\Psi_p$ such that*

$$\Psi_p(U_p \cap A) = \mathscr{B}^+ \quad , \quad \Psi_p(U_p \cap \partial A) = \mathscr{B}^0 \quad , \quad \Psi_p(U_p \cap \overline{A}^c) = \mathscr{B}^- \quad .$$

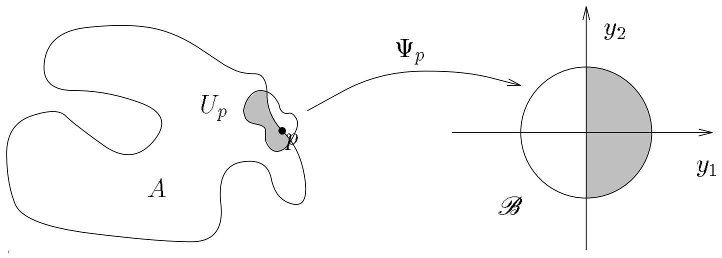



Figure 1: The situation in Definition 2.1

Clearly, the class of $C^k$ domains is in general too restricted to include the types of domains we are likely to encounter in practice. In the case of a domain that is decomposed into subdomains, however, we propose a method for, roughly speaking, treating non-smooth subdomains as if they were $C^k$ domains, by making sure that the decomposition can be constructed according to the following definition.

**DEFINITION 2.2** *Let $k \geq 0$. Let $G \subset \mathbb{R}^n$ be open. We say that a family of sets $\{G_i\}_{i \in I}$ is a hyposmooth decomposition of $G$ of order $k$ if there exists (i) a finite number of $C^k$ domains $A_1, \ldots, A_R$ and (ii) a bijection between the $G_i$ and the connected components of the open set*

$$(\cdots((G \setminus \partial A_1) \setminus \partial A_2) \cdots) \setminus \partial A_R \, . \tag{1}$$

*Similarly, we say that the sets $G_i$, $i \in I$ are hyposmooth domains of order $k$ if there exists an open set $G$ such that the family $\{G_i\}$ is a hyposmooth decomposition of $G$ of order $k$. If the degree of smoothness is not important, we simply use the terms hyposmooth decomposition and hyposmooth domains.*

Hyposmooth decompositions enjoy the following properties.

**PROPOSITION 2.3** *Let $k \geq 0$. Let $G \subset \mathbb{R}^n$ be open, and let $\{G_i\}_{i \in I}$ be a hyposmooth decomposition of $G$ of order $k$. Then*

**(i)** *The $G_i$ are open and disjoint.*

**(ii)** *The $G_i$ are countably many, i.e., $I \subset \mathbb{N}$.*

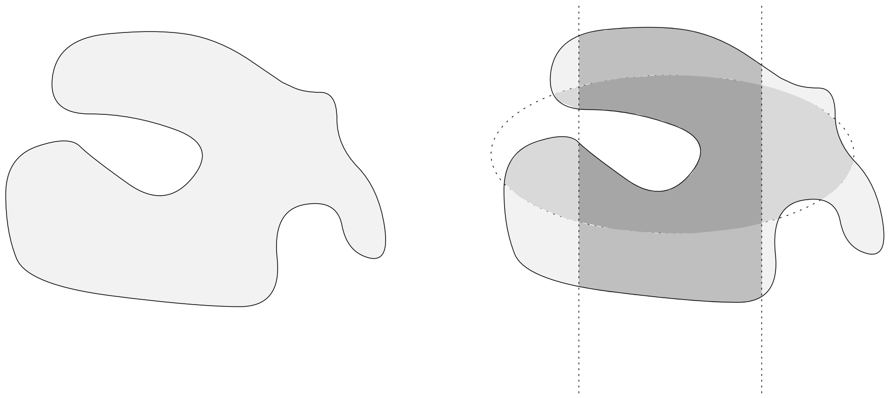


Figure 2: An open set is decomposed into 11 subdomains using an ellipse and a slab. Hence they are hyposmooth domains of order $k$ for any $k \geq 0$.

**(iii)**

$$\bigcup_{i\in I} \overline{G_i} \subset \overline{\bigcup_{i\in I} G_i} = \overline{G} \ .$$

**(iv)**

$$\bigcup_{i\in I} \partial G_i \subset \partial(\bigcup_{i\in I} G_i) = \partial G \cup (\partial A_1 \cap G) \cup \cdots \cup (\partial A_R \cap G) \ .$$

*Proof.* (i) Since $\mathbb{R}^n$ is locally connected, it follows from Proposition V-13.13 in Choquet [4] that since the set (1) is open, its connected components $G_i$ are open. The connected components of an arbitrary set are always disjoint.

(ii) The countability of the $G_i$ follows from the countability of $\mathbb{Q}^n$, since we from each of the open sets $G_i$ can find an element of $\mathbb{Q}^n$ representing it. Since the $G_i$ are disjoint the representatives are unique. They constitute a subset of the countable set $\mathbb{Q}^n$, and are hence countably many.

(iii) The inclusion is true for any family of sets. We show that the equality is true. We have

$$\overline{\bigcup G_i} = \overline{G \cap \left(\bigcap \partial A_j^c\right)} = \overline{\left[G^c \cup \left(\bigcup \partial A_j\right)\right]^c} = \left\{\left[G^c \cup \left(\bigcup \partial A_j\right)\right]^\circ\right\}^c \ .$$

We write $G^c \cup (\bigcup \partial A_j) = (G^c)^\circ \cup \partial G \cup [(\bigcup \partial A_j) \cap G]$. Assume $x \in (\bigcup \partial A_j) \cap G$. $\bigcup \partial A_j$, being a finite union of non-dense sets, is non-dense. Hence all neighborhoods of $x$ must intersect $\bigcup G_i$. If $x \in \partial G$ any neighborhood of $x$ intersects $G$, therefore it must intersect $\bigcup G_i$ since $\bigcup \partial A_j$ is non-dense. But $\bigcup G_i$ is the complement of $G^c \cup (\bigcup \partial A_j)$, so it follows that

$$\left[G^c \cup \left(\bigcup \partial A_j\right)\right]^\circ \subset (G^c)^\circ.$$

Since we also have the opposite inclusion, we can conclude that the two sets are equal. Hence

$$\overline{\bigcup G_i} = \left\{\left[G^c \cup \left(\bigcup \partial A_j\right)\right]^\circ\right\}^c = \{(G^c)^\circ\}^c = \overline{G} \,.$$

(iv) We show the equality first. We have

$$\partial(\bigcup G_i) = cl(\bigcup G_i) \cap (\bigcup G_i)^c = \overline{G} \cap (G^c \cup \partial A_1 \cup \cdots \cup \partial A_R) \,,$$

and the equality follows since $\overline{G} \cap G^c = \partial G$. For the inclusion, we first note that

$$G = \left(\bigcup G_i\right) \cup (\partial A_1 \cap G) \cup \cdots \cup (\partial A_R \cap G) \,.$$

Now, $G_i \subset G$ implies $\partial G_i \subset \overline{G_i} \subset \overline{G}$, and $\partial G_i \cap (\bigcup G_j) = \emptyset$ since the $G_i$ are separated. Hence $\partial G_i = (\partial G_i \cap G) \cup (\partial G_i \cap \partial G)$, which implies

$$\partial G_i \subset \partial G \cup (\partial A_1 \cap G) \cup \cdots \cup (\partial A_R \cap G)$$

and the result is proved. ■

We give some examples of hyposmooth domains in Figure 3. The leftmost figure in the top row corresponds to a hyposmooth decomposition of arbitrary order, by letting $G$ be the entire set, defined as the union of the open annuli $1 - a^{1-2i} < |x| < 1 - a^{-2i}$ for $i = 1, 2, \ldots$ ($a > 1$), and by choosing a single $C^k$ domain whose border does not intersect $G$. The two remaining cases in the top row are hyposmooth decompositions of $\mathscr{B}$ of arbitrary order and of order 0, respectively. All the cases in the bottom row corresponds to hyposmooth decompositions of the open unit square. The left figure corresponds to $R = 3$, using the $C^\infty$ domains $A_1 = \{y > x\}$ and $A_2 = \{y > -x\}$, and the $C^0$ domain $A_3 = \{y > x \sin(1/2x)\}$. Hence it is of class 0. The remaining two cases are hyposmooth domains of arbitrary order.

We have the following definition.

**DEFINITION 2.4** *We say that a family of sets $\{G_i\}_{i\in I}$, where $G_i \subset \mathbb{R}^n$, is locally finite, if for each $x \in \mathbb{R}^n$ there exists a neighborhood of $x$ that intersects only a finite number of the $G_i$.*

**Remark 2.5** Usually, one defines a family $\{G_i\}$ to be locally finite if each compact subset of $\mathbb{R}^n$ intersects only a finite number of the $G_i$, see e.g. Adams [1], ch. IV. However, it is easily verified that the two definitions are equivalent.

Locally finite hyposmooth decompositions allows for a strengthening of properties (iii) and (iv) in PROPOSITION 2.3 as follows.

**PROPOSITION 2.6** *Let $k \geq 0$. Let $G \subset \mathbb{R}^n$ be open, and let $\{G_i\}_{i\in I}$ be a locally finite hyposmooth decomposition of $G$ of order $k$. Then*

**(i)**

$$\bigcup_{i\in I} \overline{G_i} = \overline{\bigcup_{i\in I} G_i} = \overline{G} \,. \tag{2}$$

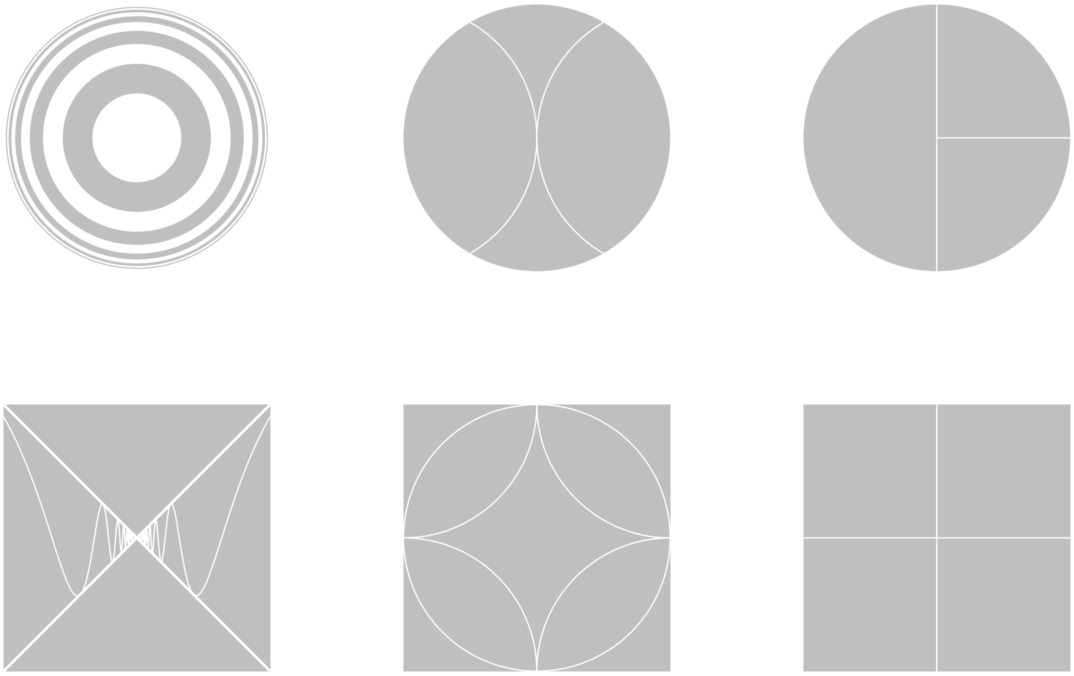


Figure 3: Some examples of hyposmooth domains.

**(ii)**

$$\bigcup_{i\in I} \partial G_i = \partial(\bigcup_{i\in I} G_i) = \partial G \cup (\partial A_1 \cap G) \cup \cdots \cup (\partial A_R \cap G) \,. \tag{3}$$

*Proof.* (i) By PROPOSITION 2.3.(iii), we only have to show that $\bigcup \overline{G_i} \supset \overline{\bigcup G_i}$. We first show that $\bigcup \overline{G_i}$ is closed. Let $x \in cl(\bigcup \overline{G_i})$. Then, since the $G_i$ constitute a locally finite family, we must have $x \in cl(\bigcup_{j\in J} \overline{G_j})$, where $J \subset I$ is finite. Hence

$$x \in \bigcup_{j\in J} \overline{\overline{G_j}} = \bigcup_{j\in J} \overline{G_j} \subset \bigcup_{i\in I} \overline{G_i} \,.$$

So $\bigcup \overline{G_i}$ is closed. But then since $\bigcup G_i \subset \bigcup \overline{G_i}$, we apply the closure operator to get $\overline{\bigcup G_i} \subset \bigcup \overline{G_i}$.

(ii) By PROPOSITION 2.3.(iv), we only have to show that $\bigcup \partial G_i \supset \partial(\bigcup G_i)$. This follows from (2), since then

$$\bigcup \partial G_i = \bigcup \left(\overline{G_i} \setminus G_i\right) \supset \bigcup \left(\overline{G_i} \setminus \bigcup G_j\right) = \left(\bigcup \overline{G_i}\right) \setminus \left(\bigcup G_j\right) = \partial \left(\bigcup G_i\right) \,.$$

∎

If we consider the hyposmooth decompositions in Figure 3, all are locally finite except for the two on the left. For if we choose a point on the unit circle in the upper left case, every neighborhood of this point will intersect infinitely many of the annuli. Similarly in the lower left case, where any neighborhood of zero will intersect infinitely many subdomains.

In the upper case we see that $\bigcup \overline{G_i}$ does not intersect the unit circle, while $\overline{\bigcup G_i}$ does. Hence (2) is not satisfied. It is satisfied, however, in the lower case, and this shows that the condition of local finiteness PROPOSITION 2.6 is only a sufficient one.

**DEFINITION 2.7** *Let $k \geq 0$. Let $G \subset \mathbb{R}^n$ be open, and let $\{G_i\}_{i\in I}$ be a hyposmooth decomposition of $G$ of order $k$. If (2) is satisfied, we say that $\{G_i\}$ has the closure property.*

All but the upper left example in Figure 3 have the closure property. From the proof of PROPOSITION 2.6.(ii), it is clear that (3) is true whenever a hyposmooth decomposition has the closure property.

# 3 Some smoothness results

In this section we will first show a lemma concerning the existence of smooth extensions from an open set $A \subset \mathbb{R}^n$ to $\mathbb{R}^n$ of functions in a certain subspace of $C^k(A)$. This lemma will be used in the proof of another result, giving sufficient conditions for the recomposition of two functions defined in each of the hemispheres $\mathscr{B}^{\pm}$.

## 3.1 Smooth extensions of functions in $C^k(\overline{A})$

We start with a definition.

**DEFINITION 3.1** *Let $A \subset \mathbb{R}^n$ be open. For $k \geq 0$, $C^k(\overline{A})$ is the subspace of $C^k(A)$ consisting of those functions $u$ satisfying the property that for each $|\alpha| \leq k$, $D^\alpha u$ can be extended to a continuous function on $\overline{A}$. This extension is unique and will be denoted by $\widetilde{D^\alpha} u$.*

We have the following result:

**LEMMA 3.2** *Let $k \geq 0$. Let $A \subset \mathbb{R}^n$ be open, and let $f \in C^k(\overline{A})$. If $A$ is relatively compact, then there exists a function $F \in C^k(\mathbb{R}^n)$ with the property that*

$$D^\alpha F|_{\overline{A}} = \widetilde{D^\alpha} f$$

*for $|\alpha| \leq k$.*

*Proof.* The proof is an application of Whitney's extension theorem [9]. This theorem says that the extension $F$ exists if we can show that the function $\tilde{f}$ (the extension of $f$), which is defined on the closed set $\overline{A}$, is *of class $C^k$ in $\overline{A}$ in terms of the functions $\widetilde{D^\alpha} f$, $|\alpha| \leq k$.* This means that we must show that each $\widetilde{D^\alpha} f$ has the formal Taylor expansion

$$\widetilde{D^\alpha} f(x) = \sum_{|\beta| \leq k - |\alpha|} \frac{\widetilde{D^{\alpha+\beta}} f(y)}{\beta!}(x-y)^\beta + R_\alpha(x;y)$$

for $|\alpha| \leq k$, where the remainder $R_\alpha$ has the following property $(\star)$:

$$(\star) \quad \begin{cases} \text{Given } z \in \overline{A} \text{ and any } \varepsilon > 0, \text{ there is a } \delta > 0, \text{ independent of } z, \text{ such that} \\ \qquad\qquad |R_\alpha(x;y)| < \varepsilon |x-y|^{k-|\alpha|} \\ \text{whenever } x, y \in \overline{A} \text{ satisfies } |x-z| < \delta\,, \quad |y-z| < \delta\,. \end{cases}$$

In other words, property $(\star)$ expresses the requirement that the remainder goes to zero like $o(|x-y|^{k-|\alpha|})$ as $x \to y$, uniformly in $\overline{A}$. See also Ziemer [10], p. 131.

Consider the functions $\rho_\alpha(x,y)$, $|\alpha| \leq k$, defined in $\overline{A} \times \overline{A}$ by

$$\rho_\alpha(x,y) = \left( \widetilde{D^\alpha} f(x) - \sum_{|\beta| \leq k-|\alpha|} \frac{\widetilde{D^{\alpha+\beta} f}(y)}{\beta!}(x-y)^\beta \right) |x-y|^{-(k-|\alpha|)} \quad \text{for} \quad x \neq y\,,$$

and let $\rho_\alpha(x,y) = 0$ for $x = y$. Let $|\alpha| \leq k$. We will show that $\rho_\alpha(x,y)$ is continuous in $\overline{A} \times \overline{A}$. Since all the $\widetilde{D^{\alpha+\beta}} f$ are continuous in $\overline{A}$, and $|x-y|^{-(k-|\alpha|)}$ is a continuous function in $\overline{A} \times \overline{A}$ when $x \neq y$, we conclude that $\rho_\alpha(x,y)$ is continuous in all points where $x \neq y$.

On the diagonal of $A \times A$, we know by Taylor's theorem (see e.g. [8]) that $\tilde{f}$ satisfies at least a non-uniform version of $(\star)$; i.e., given $z \in A$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|\rho_\alpha(z,z) - \rho_\alpha(x,y)| = |\rho_\alpha(x,y)| < \varepsilon \ \text{ whenever } \ |x-z| < \delta,\ |y-z| < \delta\,. \tag{4}$$

Hence $\rho_\alpha(x,y)$ is continuous in all points where $x = y$ and $y \in A$. It is even uniformly continuous in these points. To see this, we use Taylor's theorem with remainder (see e.g. [5]) to get, for $x, y \in A$,

$$|\rho_\alpha(x,y)| \cdot |x-y|^{k-|\alpha|} \leq \sum_{|\beta|=k-|\alpha|} \frac{|D^{\alpha+\beta} f(\xi) - D^{\alpha+\beta} f(y)|}{\beta!} |(x-y)^\beta|$$

where $\xi = y + \theta(x-y)$ for some $\theta \in (0,1)$. Let $K = K(n)$ be a constant such that

$$|x_1| + \cdots + |x_n| \leq K(|x_1|^2 + \cdots + |x_n|^2)^{1/2} = K|x| \text{ for all } x \in \mathbb{R}^n\,.$$

By the assumptions on $f$ it follows that the $D^{\alpha+\beta} f$ are uniformly continuous in $A$. Hence, given $\varepsilon > 0$, we can choose $\delta > 0$, independent of $x, y \in A$, such that $|D^{\alpha+\beta} f(\xi) - D^{\alpha+\beta} f(y)| < \varepsilon K^{|\alpha|-k}$ for all $|\beta| \leq k - |\alpha|$ whenever $|x-y| < \delta$. Hence

$$|\rho_\alpha(x,y)| \cdot |x-y|^{k-|\alpha|} \leq \frac{\varepsilon}{K^{k-|\alpha|}} \sum_{|\beta|=k-|\alpha|} \frac{1}{\beta!} |(x-y)^\beta| \leq \frac{\varepsilon}{K^{k-|\alpha|}} \left( \sum_{j=1}^n |x_j - y_j| \right)^{k-|\alpha|}$$

for $|x-y| < \delta$, by the multinomial theorem. Since $\sum_{j=1}^n |x_j - y_j| < K|x-y|$, the last term is bounded by $\varepsilon |x-y|^{k-|\alpha|}$. In conclusion, $\rho_\alpha(x,y)$ is uniformly continuous on the diagonal of $A \times A$; i.e., the $\delta$ in (4) can be chosen independently of $z$.

It remains to show continuity in the points where $x = y$ and $y \in \partial A$. This is a consequence of the uniform continuity on the diagonal of $A \times A$. Let $z \in \partial A$ and let $\varepsilon > 0$. From above, we can find a $z' \in A$ and a $\delta > 0$ such that $|\rho_\alpha(x, y)| < \varepsilon$ whenever $|x - z'| < 2\delta$ and $|y - z'| < 2\delta$, and such that $|z' - z| < \delta$. But then it follows that (4) holds. Hence $\rho_\alpha(x, y)$ is continuous also in the points where $x = y$ and $y \in \partial A$.

Now, $\rho_\alpha(x, y)$ being a continuous function on the compact set $\overline{A} \times \overline{A}$, it is uniformly continuous in $\overline{A} \times \overline{A}$. In particular, it is uniformly continuous on the diagonal of $\overline{A} \times \overline{A}$; i.e., given $z \in \overline{A}$ and $\varepsilon > 0$ there exists a $\delta > 0$, independent of $z$, such that (4) holds for $x, y \in \overline{A}$. But this is exactly property ($\star$), and since it holds for any $|\alpha| \leq k$, the extension $F \in C^k(\mathbb{R}^n)$ exists according to Whitney's extension theorem. ■

## 3.2 Recomposition in two hemispheres

We use the following notation when it comes to differential operators.

**DEFINITION 3.3** *Let $m \geq 0$. Let $A \subset \mathbb{R}^n$ be open, and let $\mathscr{H}$ be a space of functions in $A$. Let $a_\alpha \in \mathscr{H}$ for $|\alpha| \leq m$. If there exists at least one $\alpha$, $|\alpha| = m$, such that $a_\alpha \not\equiv 0$ we say that*

$$P(x, D) = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

*is a differential operator of order $m$ in $A$, with coefficients in $\mathscr{H}$. We call*

$$P^\bullet(x, D) = \sum_{|\alpha| = m} a_\alpha(x) D^\alpha$$

*the principal part of $P$. If $S$ is a subset of the boundary of a $C^1$ domain, and $S \cap A \neq \emptyset$, we say that $S$ is non-characteristic for $P$ if $P^\bullet(p, D\Psi_p^1(p)) \neq 0$ for all $p \in S \cap A$; where $\Psi_p = (\Psi_p^1, \ldots, \Psi_p^n)$ is the mapping defining the $C^1$ domain (see* DEFINITION 2.1*).*

The remainder of this section will be used to prove the following Lemma, and to discuss some possible generalizations.

**LEMMA 3.4** *Let $m \geq 1$ and $k \geq m$. Let $P(x, D)$ be a differential operator of order $m$ in $\mathscr{B}$ with coefficients in $C^{k-m}(\mathscr{B})$. Let $w^\pm \in C^k(\overline{\mathscr{B}^\pm})$ satisfy*

$$\widetilde{D_1^j w^-} = \widetilde{D_1^j w^+} \quad \text{on } \mathscr{B}^0 \text{ for } 0 \leq j \leq m - 1 \,, \tag{5}$$

*and*

$$P(x, D) w^\pm = f|_{\mathscr{B}^\pm} \,, \tag{6}$$

*where $f \in C^{k-m}(\mathscr{B})$. Then, if $\mathscr{B}^0$ is non-characteristic for $P$, there exists a unique function $w \in C^k(\mathscr{B})$ with the property that $w|_{\mathscr{B}^\pm} = w^\pm$.*

**Remark 3.5** To verify that it is not in general sufficient to merely have $w^{\pm} \in C^k(\mathscr{B}^{\pm})$ we give a counterexample. Let $n = 2$, let $P(x, D) = D_1$ (i.e., $m = 1$), and consider the function

$$w(x_1, x_2) = |x_2| \exp(\frac{-x_1^2}{|x_2|}) \in C(\overline{\mathscr{B}}) .$$

Let $w^{\pm} = w|_{\mathscr{B}^{\pm}}$. Then $w^{\pm} \in C(\overline{\mathscr{B}^{\pm}})$, and for the first partial derivatives we have

$$D_1 w^{\pm} = -2x_1 \exp(\frac{-x_1^2}{|x_2|}) \in C(\overline{\mathscr{B}^{\pm}}) ,$$

and

$$D_2 w^{\pm} = \left( \frac{|x_2|}{x_2} + \frac{x_1^2}{x_2} \right) \exp(\frac{-x_1^2}{|x_2|}) \in C(\mathscr{B}^{\pm}) .$$

Hence the conditions of the lemma are satisfied for $k = 1$ with $f := D_1 w$, except for the slightly weaker condition on the smoothness of $D_2 w^{\pm}$. But $w$, which is the only possible continuous extension, is not in $C^1(\mathscr{B})$ since $D_2 w$ does not exist at the origin.

*Proof.* Existence of $w \in C(\mathscr{B})$ follows directly by defining $w$ as $w^{\pm}$ in the two hemispheres $\mathscr{B}^{\pm}$, and defining it to be their common limit on $\mathscr{B}^0$.

To show that $w \in C^k(\mathscr{B})$ we show, for each $j = 0, 1, \ldots, k$, that $D^{(j,\beta)} w$, $|\beta| \leq k - j$ exist and are continuous in $\mathscr{B}$. For each $j$ we first take care of the purely normal derivative with respect to $\mathscr{B}^0$ (i.e., when $|\beta| = 0$). Then we look at the derivatives of this function in the purely tangential directions.

First, let $j = 0$. By LEMMA 3.2 we can conclude that $\widetilde{D^{(0,\beta)} w^{\pm}} = D^{(0,\beta)} \widetilde{w^{\pm}}$ on $\mathscr{B}^0$ for $|\beta| \leq k$. Hence $D^{(0,\beta)} w$ exists and is continuous in $\mathscr{B}$ for $|\beta| \leq k$.

Next, let $1 \leq j \leq m - 1$. By induction on $j$ (starting with $j = 1$), LEMMA 3.2 lets us conclude that $\widetilde{D^{(j,0)} w^{\pm}}$ on $\mathscr{B}^0$ equals the one-sided derivatives of $D^{(j-1,0)} w$ with respect to $x_1$. By assumption these are equal, hence $D^{(j,0)} w$ exists and is continuous. Now, arguing as we did in the case $j = 0$, we conclude that $D^{(j,\beta)} w$ exists and is continuous for $|\beta| \leq k - j$.

Now for the remaining case $m \leq j \leq k$. By induction on $j$, (starting with $j = m$), we can by (6) and the assumption of non-characteristicity solve for $\widetilde{D^{(j,0)} w^{\pm}}$ on $\mathscr{B}^0$ in terms of functions that are continuous in $\mathscr{B}$. Hence $\widetilde{D^{(j,0)} w^-} = \widetilde{D^{(j,0)} w^+}$ on $\mathscr{B}^0$. Again by LEMMA 3.2, we can conclude that these are the one-sided derivatives of $D^{(j-1,0)}$. Hence $D^{(j,0)} w$ exists and is continuous in $\mathscr{B}$. Arguing as before, we get that $D^{(j,\beta)} w$ exists and is continuous for $|\beta| \leq k - j$.

Summing up, we have found a function $w$ with $D^{\alpha} w \in C(\mathscr{B})$ for $|\alpha| \leq k$, which is equivalent to $w \in C^k(\mathscr{B})$. By construction, we have $w|_{\mathscr{B}^{\pm}} = w^{\pm}$. That $w$ is unique follows by the density of $\mathscr{B}^+ \cup \mathscr{B}^-$ in $\mathscr{B}$. ■

We can generalize the interface conditions (5) somewhat, and still get the same result.

**COROLLARY 3.6** *Let $m \geq 1$ and $k \geq m$. Let $P(x, D)$ be a differential operator of order $m$ in $\mathscr{B}$ with coefficients in $C^{k-m}(\mathscr{B})$. For each $j = 0, \ldots, m - 1$, let $B_j(x, D)$ be a* *differential operator of order $j$ in $\mathscr{B}$ with coefficients in $C^0(\mathscr{B})$. Let $w^\pm \in C^k(\overline{\mathscr{B}}^\pm)$ satisfy*

$$\widetilde{B_j w^-} = \widetilde{B_j w^+} \quad on \quad \mathscr{B}^0 \quad for \quad 0 \le j \le m-1 \;, \tag{7}$$

*and*

$$P(x,D)w^\pm = f|_{\mathscr{B}^\pm} \;,$$

*where $f \in C^{k-m}(\mathscr{B})$. Then, if $\mathscr{B}^0$ is non-characteristic for $P$ and $B_j$, $j = 0, \ldots, m-1$, there exists a unique function $w \in C^k(\mathscr{B})$ with the property that $w|_{\mathscr{B}^\pm} = w^\pm$.*

*Proof.* The proof is the same as above, except for a small addition to the induction argument in the case $1 \le j \le m-1$. Assume that $D^\alpha w$ exists and is continuous for $|\alpha| \le k$, $\alpha_1 < j$. Since

$$B_j(x,D) = b^j_{(j,0,\ldots,0)}(x) D_1^j + \sum_{|\alpha|\le j,\, \alpha_1 < j} b^j_\alpha(x) D^\alpha \;,$$

the corresponding condition in (7) is equivalent to

$$b^j_{(j,0,\ldots,0)} \widetilde{D_1^j w^-} = b^j_{(j,0,\ldots,0)} \widetilde{D_1^j w^+} \quad \text{on} \quad \mathscr{B}^0 \;.$$

$\mathscr{B}^0$ is non-characteristic for $B_j$ if and only if $b^j_{(j,0,\ldots,0)} \ne 0$ on $\mathscr{B}^0$, hence we have

$$\widetilde{D_1^j w^-} = \widetilde{D_1^j w^+} \quad \text{on} \quad \mathscr{B}^0 \;,$$

and we can complete the induction step as above to conclude that $D^\alpha w$ exists and is continuous for $|\alpha| \le k$, $\alpha_1 < j+1$. Since (7) for $j = 0$ implies that $D^\alpha w$ exists and is continuous for $|\alpha| \le k$, $\alpha_1 = 0$, our modified induction argument is proved. ■

It is obvious that we can generalize the interface conditions further, as long as we can reduce them to the case above. Hence we have the following result.

**COROLLARY 3.7** *Let $m \ge 1$ and $k \ge m$. Let $P(x,D)$ be a differential operator of order $m$ in $\mathscr{B}$ with coefficients in $C^{k-m}(\mathscr{B})$. For each $j = 0, \ldots, m-1$, let $B_j(x,D)$ be a differential operator of order $j$ in $\mathscr{B}$ with coefficients in $C^0(\mathscr{B})$. Let $w^\pm \in C^k(\overline{\mathscr{B}}^\pm)$ satisfy*

$$\sum_{j=0}^{m-1} a_{ij} \widetilde{B_j w^-} = \sum_{j=0}^{m-1} a_{ij} \widetilde{B_j w^+} \quad on \quad \mathscr{B}^0 \quad for \quad 0 \le i \le m-1 \;, \tag{8}$$

*and*

$$P(x,D)w^\pm = f|_{\mathscr{B}^\pm} \;,$$

*where $f \in C^{k-m}(\mathscr{B})$. Then, if $\mathscr{B}^0$ is non-characteristic for $P$ and $B_j$, $j = 0, \ldots, m-1$, and if the $m \times m$ matrix $(a_{ij})$ is non-singular on $\mathscr{B}^0$, there exists a unique function $w \in C^k(\mathscr{B})$ with the property that $w|_{\mathscr{B}^\pm} = w^\pm$.*

*Proof.* Since $(a_{ij})$ is non-singular on $\mathscr{B}^0$, we can solve for the $m$ quantities $\widetilde{B_j w^-} - \widetilde{B_j w^+}$, which must be zero. Hence we are in the position to use COROLLARY 3.6. ■

## 4 Recomposition in hyposmooth domains

We begin with the following definition, generalizing the definition of $C^k(\overline{A})$ to the case where we only require the existence of a continuous extension to a subset of the boundary $\partial A$.

**DEFINITION 4.1** *Let $A \subset \mathbb{R}^n$ be open, and let $b \subset \partial A$. For $k \geq 0$, $C^k(A \cup b)$ is the subspace of $C^k(A)$ consisting of those functions $u$ satisfying the property that for each $|\alpha| \leq k$, $D^\alpha u$ can be extended to a continuous function on $A \cup b$. This extension is unique and will be denoted by $\widetilde{D^\alpha u}$.*

We will need the concept of a non-characteristic hyposmooth decomposition of a set.

**DEFINITION 4.2** *Let $k \geq 1$. Let $G \subset \mathbb{R}^n$ be open, and let $\{G_i\}_{i\in I}$ be a hyposmooth decomposition of order $k$ of $G$. Let $m \geq 0$, and let $P(x, D)$ be a differential operator of order $m$ in $G$. We say that the hyposmooth decomposition is non-characteristic with respect to $P$ if the surfaces $\partial A_j \cap G$ are all non-characteristic for $P$.*

In a Cauchy problem corresponding to a differential operator of order $m$, one usually specifies data on a non-characteristic surface. Only the normal derivatives of order $\leq m-1$ are needed, since the higher order derivatives can be computed by differentiating the PDE itself. We will be in a similar situation concerning the interface conditions, and must therefore define properly what we mean by higher order normal derivatives of a function $f \in C^k(A \cup b)$. We use the same definition as Nečas [6]. Let $A \subset \mathbb{R}^n$ be open, and let $b \subset \partial A$. Let $\nu : b \to \mathbb{R}^n$. If $f \in C^k(A \cup b)$, $k \geq 0$, we define the directional derivatives of $f$ on $b$ with respect to $\nu$ as

$$D^j_\nu f(x) = \sum_{|\alpha|=j} \begin{pmatrix} j \\ \alpha \end{pmatrix} \widetilde{D^\alpha f}(x)\nu(x)^\alpha \ ,$$

where $\binom{j}{\alpha} = j!/\alpha_1! \cdots \alpha_n!$ is the multinomial coefficient, and $\nu^\alpha = \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n}$. In this paper, we will only need this construct on sets $b$ that possess normal vectors everywhere. Hence $\nu(x)$ will always denote a unit normal at $x$, and $D^j_\nu f$ are the corresponding normal derivatives.

We will now present sufficient conditions ensuring that if we have solved a PDE on a hyposmooth decomposition, then we have in fact also solved it on the original domain. To obtain this, we have to introduce interface conditions on the smooth parts of the interfaces, that is, on the sets

$$\Gamma_{ii'} = \partial G_i \cap \partial G_{i'} \cap G \setminus \left( \bigcup_{k \neq i,i'} \partial G_k \right) \ .$$

In other words, $\Gamma_{ii'}$ is the subset of $\partial G_i \cap \partial G_{i'}$ which lies in the open set $G$ and which does not intersect any other $\partial G_k$.

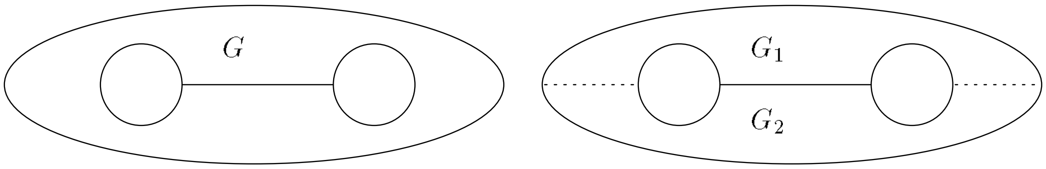



Figure 4: The part of $\partial G$ connecting the two holes is not regular.

**Remark 4.3** We note that each point in $\Gamma_{ii'}$, $i \neq i'$, belongs to essentially only one of the $\partial A_j$. By this we mean that if $p \in \Gamma_{ii'} \in \partial A_j \cap \partial A_{j'}$, then $\partial A_j$ and $\partial A_{j'}$ must coincide in a neighborhood of $p$.

**DEFINITION 4.4** *Let $\{G_i\}$ be a hyposmooth decomposition of $G$. A subset $\partial G^\star \subset \partial G$ is said to be regular (with respect to $\{G_i\}$) if*

$$\partial G^\star \subset \partial G \setminus \left( \bigcup_{i \neq i'} (\partial G_i \cap \partial G_{i'}) \setminus \bigcup_{i,i'} \overline{\Gamma_{ii'}} \right) .$$

In other words, $\partial G^\star \subset \partial G$ is regular if each point of $\partial G^\star$ is either in none of the $\partial G_i \cap \partial G_{i'}$, or in $\partial G_i \cap \partial G_{i'}$ and simultaneously in the closure of $\Gamma_{ii'}$. We have the following result.

**THEOREM 4.5** *Let $m \geq 1$, and $k \geq m$. Let $G \subset \mathbb{R}^n$ be open, let $P(x, D)$ be a differential operator of order $m$ in $G$ with coefficients in $C^{k-m}(G)$, and let and let $\{G_i\}_{i \in I}$ be a locally finite hyposmooth decomposition of $G$ of order $k$, non-characteristic with respect to $P$.*

*Let $\partial G^\star \subset \partial G$ be regular, let $G^\star = G \cup \partial G^\star$, and let $\partial G_i^\star = \partial G_i \cap G^\star$. Assume $v_i \in C^k(G_i \cup \partial G_i^\star)$ satisfies*

$$D_\nu^j v_i = D_\nu^j v_{i'} \quad \text{on} \quad \Gamma_{ii'} \quad \text{for} \quad 0 \leq j \leq m-1 , \tag{9}$$

*and*

$$P(x, D) v_i = f|_{G_i} ,$$

*where $f \in C^{k-m}(G)$. Then there exists a unique function $u \in C^k(G^\star)$ with the properties $u|_{G_i} = v_i$ and $P(x, D)u = f$.*

**Remark 4.6** Note that an important special case is $\partial G^\star = \emptyset$, yielding the existence of $u$ in $C^k(G)$. The reason why we must restrict $\partial G^\star$ to be regular is apparent from the case in Figure 4. There is no interface condition on the part of $\partial G$ connecting the two holes. Hence we can not conclude continuity in the set $G \cup \partial G$ even if we have functions in $G_i \cup \partial G_i$.

The proof of THEOREM 4.5 successively takes care of the interfaces generated by the domains $A_1, \ldots, A_R$; starting with $A_R$. We illustrate the idea by a simple example where $\partial G^\star = \emptyset$ and $R = 2$. Let $G$ be the interior of the circle drawn in Figure 5. Decompose $G$ into the four subdomains $G_i$, $i = 1, 2, 3, 4$ by the technique of DEFINITION 2.2, using the

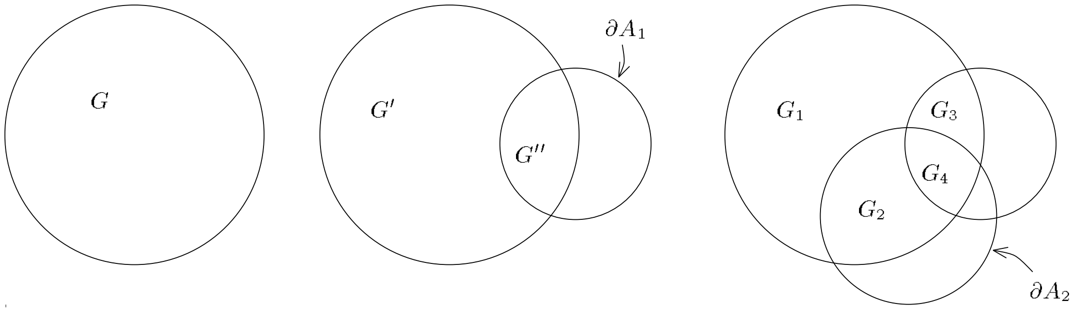



Figure 5: A simple example

two balls $A_1$ and $A_2$. We denote the two subdomains we get by decomposing $G$ using only $A_1$ by $G'$ and $G''$. Note that $\{G_1, G_2\}$ is a hyposmooth decomposition of $G'$, and similarly for $\{G_3, G_4\}$ and $G''$.

Consider first the two subdomains $G_1$ and $G_2$. The whole idea of the proof of THEOREM 4.5 is now to show that the assumptions we impose imply the existence of a function $v' \in C^k(G')$ having the property that $v'|_{G_i} = v_i$, $i = 1, 2$. Assuming this is the case, we can argue in the same manner that there exists a function $v'' \in C^k(G'')$ with the property $v''|_{G_i} = v_i$, $i = 3, 4$. As a result, we have reduced our problem from the case $R = 2$ to the case $R = 1$; we are left with the decomposition $\{G', G''\}$ of $G$, and the corresponding functions $v'$, $v''$. Since this situation is completely analogous to what we started out with, we can argue that there exists a function $v \in C^k(G)$ having the property $v|_{G'} = v'$ and $v|_{G''} = v''$, or equivalently $v|_{G_i} = v_i$, $i = 1, 2, 3, 4$. At this point the result would be proved by letting $u = v$.

Before proving THEOREM 4.5, we make some preparations. The following definition introduces a suitable enumeration of the subsets we will need to refer to below.

**DEFINITION 4.7** *Let $G^{(0)}_\alpha$, $\alpha \in I_0 \subset \mathbb{N}$ be the connected components of $G$. For $j = 1, \ldots, R$ we define the sets $G^{(j)}_{(\alpha,\kappa)}$, $K^{(j)}_\alpha$, and $I_j$ recursively as follows:*

- *For each $\alpha \in I_{j-1}$, let $G^{(j)}_{(\alpha,\kappa)}$, $\kappa \in K^{(j)}_\alpha \subset \mathbb{N}$ be the connected components of $G^{(j-1)}_\alpha \setminus \partial A_j$, i.e.,*

$$G^{(j-1)}_\alpha \setminus \partial A_j =: \bigcup_{\kappa \in K^{(j)}_\alpha}^{connected} G^{(j)}_{(\alpha,\kappa)} \, . \tag{10}$$

- *Let $I_j \subset \mathbb{N}^{j+1}$ be the set of multi-indices*

$$I_j = \left\{ (\alpha, \kappa) \mid \alpha \in I_{j-1},\, \kappa \in K^{(j)}_\alpha \right\} \, . \tag{11}$$

To illustrate, we enumerate the sets used in the simple example above:

| $j=0$ | $j=1$ | $j=2$ |
|---|---|---|
| $G_1^{(0)} = G$ | $G^{(1)}_{(1,1)} = G'$ | $G^{(2)}_{(1,1,1)} = G_1,\ G^{(2)}_{(1,1,2)} = G_2$ |
| | $G^{(1)}_{(1,2)} = G''$ | $G^{(2)}_{(1,2,1)} = G_3,\ G^{(2)}_{(1,2,2)} = G_4$ |
| $I_0 = \{1\}$ | $I_1 = \{(1,1),(1,2)\}$ | $I_2 = \{(1,1,1),(1,1,2),(1,2,1),(1,2,2)\}$ |

In the general case we get, by repeated use of (10) and (11),

$$(\cdots((G \setminus \partial A_1) \setminus \partial A_2)\cdots) \setminus \partial A_R = (\cdots((\bigcup_{\alpha\in I_1} G^{(1)}_\alpha) \setminus \partial A_2)\cdots) \setminus \partial A_R = \ \cdots\ = \bigcup_{\alpha \in I_R} G^{(R)}_\alpha \ ,$$

where all the unions are over disjoint and connected sets. Hence the subdomains $G_i$ constituting the hyposmooth decomposition of $G$ are exactly the subsets $G^{(R)}_\alpha$, $\alpha \in I_R$. We also note that for each $j = 0, \ldots, R-1$, the family $\{G^{(j+1)}_{(\alpha,\kappa)}\}_\kappa$ is a hyposmooth decomposition of $G^{(j)}_\alpha$ of order $k$ for each $\alpha \in I_j$, using the single $C^k$ domain $A_{j+1}$.

As indicated in the example above, we can prove THEOREM 4.5 if we can prove it in the case $R = 1$; that is, in the case where a single $C^k$ domain is used to generate the subdomains.

**LEMMA 4.8** *Assume* THEOREM 4.5 *is true for $R = 1$. Then it is true for any $R$.*

*Proof.* We prove the lemma by induction. Assume THEOREM 4.5 is true for $R = r$. We will show that this implies that it is true for $R = r + 1$. Let $\alpha \in I_r$, and consider the family $\{G^{(r+1)}_{(\alpha,\kappa)}\}_{\kappa\in K^{(r+1)}_\alpha}$, which is a locally finite hyposmooth decomposition of $G^{(r)}_\alpha$, using $\partial A_{r+1}$. Hence we can use THEOREM 4.5 for $R = 1$ to conclude that there is a function $v^{(r)}_\alpha \in C^k(G^{(r)}_\alpha \cup \partial G^\star)$ such that $Pv^{(r)}_\alpha = f|_{G^{(r)}_\alpha}$. In this way we have 'removed' the interfaces generated by $\partial A_{r+1}$. Renaming the sets $G^{(r)}_\alpha$, $\alpha \in I_r$ as $G_i$, $i \in I$, we are almost in the position where we can use THEOREM 4.5 for $R = r$. The only thing remaining is to verify condition (9), since some points that was not in any of the $\Gamma_{ii'}$ before the renaming, will be afterwards (in our simple example, the point $\partial A_1 \cap \partial A_2 \cap G$ is of this type). However, by Remark 4.3 it follows that these points are non-dense in the new interfaces $\Gamma_{ii'}$; hence condition (9) holds by extending it continuously to these points. Applying THEOREM 4.5 for $R = r$ yields the result. ■

We are now in the position to prove THEOREM 4.5.

*Proof.* By LEMMA 4.8, we only have to prove the theorem for $R = 1$. Let the function $w : G \to \mathbb{R}$ be defined by $w = \tilde{v}_i$ in $(G_i \cup \partial G_i^\star) \cap G$. It is defined in the whole of $G$ since

$$\bigcup_{i\in I} (G_i \cup \partial G_i^\star) = \bigcup_{i\in I} (cl(G_i) \cap G^\star) = G^\star \cap \left(\bigcup_{i\in I} cl(G_i)\right) = G^\star \cap cl\left(\bigcup_{i\in I} G_i\right) = G^\star \ , \qquad (12)$$

where the last two equalities follows from PROPOSITION 2.6. Since $\tilde{v}_i$ and $\tilde{v}_{i'}$ attain the same values on the intersection

$$[(G_i \cup \partial G_i^\star) \cap G] \cap [(G_{i'} \cup \partial G_{i'}^\star) \cap G] = \partial G_i \cap \partial G_{i'} \cap G = \Gamma_{ii'}$$

by (9), $w$ is well-defined, and we have $w \in C(G)$. Moreover, by (12) we have $w \in C(G^\star)$, since an extension of $w$ is provided by letting $\tilde{w} = \tilde{v}_i$ in $G_i \cup \partial G_i^\star$ .

Assume for the moment that the following is true:

$$(\star) \begin{cases} \text{For each } p \in \partial A_1 \cap G \text{ there exists an open} \\ \text{set } \mathscr{O}_p \subset G \text{ such that } w|_{\mathscr{O}_p} \in C^k(\mathscr{O}_p) \, . \end{cases}$$

In this case we have, since $w|_{G_i} = v_i \in C^k(G_i)$,

$$w \in C^k\left(\left\{\bigcup_i G_i\right\} \cup \left\{\bigcup_p \mathscr{O}_p\right\}\right) = C^k(G) \, . \tag{13}$$

The last equality follows from the fact that

$$\left(\bigcup_i G_i\right) \cup \left(\bigcup_p \mathscr{O}_p\right) \subset G$$

since $G_i$ and $\mathscr{O}_p$ are subsets of $G$, and from the converse inclusion

$$G = (G \setminus \partial A_1) \cup (G \cap \partial A_1) = \left(\bigcup_i G_i\right) \cup (G \cap \partial A_1) \subset \left(\bigcup_i G_i\right) \cup \left(\bigcup_p \mathscr{O}_p\right) \, .$$

By (12) and the assumptions made on the $v_i$, it follows that for each $|\alpha| \leq k$, $D^\alpha w$ has an extension

$$\widetilde{D^\alpha w} \in C\left(\bigcup_i (G_i \cup \partial G_i^\star)\right) = C(G^\star) \, . \tag{14}$$

By DEFINITION 4.1, it follows from (13) and (14) that $w \in C^k(G^\star)$. In addition, since the continuous functions $Pw$ and $f$ are equal on a dense subset of $G$, they are equal in all of $G$. Hence, by letting $u = w$ the result follows.

Now it remains to verify that property $(\star)$ is satisfied. Let $p \in \partial A_1 \cap G$, and let $U_p$ be the neighborhood of $p$ from DEFINITION 2.1. We know that the $k$-diffeomorphism $\Psi_p$ maps $U_p$ onto $\mathscr{B}$. Hence we can find an open set $\mathscr{O}_p$ such that $\overline{\mathscr{O}}_p \subset U_p \cap G$ and such that $\Psi_p$ maps $\mathscr{O}_p$ onto $r\mathscr{B}$ for some $0 < r \leq 1$; without loss of generality, we can assume $r = 1$. Since $A_1$ and $\overline{A_1}^c$ are not connected, $\mathscr{O}_p \cap A_1 \subset G_i$ and $\mathscr{O}_p \cap \overline{A}_1^c \subset G_{i'}$ for some $i, i' \in I$, $i \neq i'$.

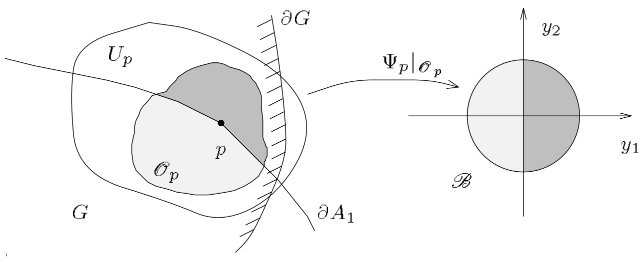
To simplify the notation, let $\mathscr{O}^+ = \mathscr{O}_p \cap A_1$ and $\mathscr{O}^- = \mathscr{O}_p \cap \overline{A}_1^c$, and let $w^+ = w|_{\mathscr{O}^+}$ and $w^- = w|_{\mathscr{O}^-}$. Let $\mathscr{O}^0 = \mathscr{O}_p \cap \partial A_1$. By the assumptions made on the $v_i$, we have

$$w^\pm \in C^k(\overline{\mathscr{O}}^\pm) \, .$$

Let $w_* = w|_{\mathscr{O}_p} \circ \Psi_p^{-1}$, and $w_*^\pm = w^\pm \circ \Psi_p^{-1}$. Then we have

$$w_*^\pm \in C^k(\overline{\mathscr{B}}^\pm)$$

by the $C^k$-smoothness of $\Psi_p^{-1}$. Similarly, let $f_* = f \circ \Psi_p^{-1}$; then $f_* \in C^{k-m}(\mathscr{B})$.




For each $j = 0, \ldots, m-1$, the corresponding interface condition in (9) can locally be written as

$$\widetilde{B_j w^+} = \widetilde{B_j w^-} \quad \text{on} \quad \mathscr{O}^0 \ ,$$

where

$$B_j(x, D) = \sum_{|\alpha|=j} \begin{pmatrix} j \\ \alpha \end{pmatrix} \frac{(D\Psi_p^1(x))^\alpha}{|D\Psi_p^1(x)|^j} D^\alpha \tag{15}$$

is a differential operator of order $j$ in $\mathscr{O}_p$ with coefficients in $C^{k-1}(\mathscr{O}_p)$. It transforms to

$$\widetilde{B_{j*} w_*^+} = \widetilde{B_{j*} w_*^-} \quad \text{on} \quad \mathscr{B}^0 \ ,$$

where $B_{j*}(x, D)$ is a differential operator of order $j$ in $\mathscr{B}$ with coefficients in $C^{\min\{k-1, k-j\}}(\mathscr{B})$, hence in $C^0(\mathscr{B})$. Since

$$B_j^\bullet(x, D\Psi_p^1(x)) = \sum_{|\alpha|=j} \begin{pmatrix} j \\ \alpha \end{pmatrix} \frac{(D\Psi_p^1(x))^{2\alpha}}{|D\Psi_p^1(x)|^j} > 0 \quad \text{on} \quad \mathscr{O}^0 \ ,$$

$\mathscr{O}^0$ is non-characteristic for $B_j$. Using the relation

$$B_{j*}^\bullet(\Psi_p(x), (1, 0, \ldots, 0)) = B_j^\bullet(x, D\Psi_p^1(x)) \quad \text{on} \quad \mathscr{O}^0 \tag{16}$$

(see e.g. Section 1.7 in Chazarain, Piriou [3]), it follows that $\mathscr{B}^0$ is non-characteristic for $B_{j*}$. Similarly, let $P_*$ denote the $\Psi_p$-transformed version of $P$, a differential operator of order $m$ in $\mathscr{B}$ with coefficients in $C^{k-m}(\mathscr{B})$. Clearly, $P_* w_*^\pm = f_*|_{\mathscr{B}^\pm}$. Moreover, since $\mathscr{O}^0$ is non-characteristic for $P$, $\mathscr{B}^0$ is non-characteristic for $P_*$.

We are now in the position to apply COROLLARY 3.6 and conclude that $w_* \in C^k(\mathscr{B})$. Transforming back, we have $w|_{\mathscr{O}_p} \in C^k(\mathscr{O}_p)$, and property $(\star)$ is verified. ■

**Remark 4.9** In the proof, the assumption about the hyposmooth decomposition being locally finite was used only to establish (12), hence for $R = 1$ the theorem holds also under the weaker assumption that the decomposition has the closure property. Presently, it is not clear to us how this weakening of the assumption affects the general case $R > 1$.

It follows immediately that we have the following extension of the theorem.

**COROLLARY 4.10** *The conclusion of* THEOREM 4.5 *still holds if condition* (9) *is replaced by*

$$\sum_{j=0}^{m-1} a_{kj} D_\nu^j v_i = \sum_{j=0}^{m-1} a_{kj} D_\nu^j v_{i'} \quad on \;\; \Gamma_{ii'} \;\; for \;\; 0 \le k \le m-1 \; , \tag{17}$$

*where the matrix* $(a_{kj})$ *is non-singular on* $\bigcup_{j=1}^R (\partial A_j \cap G)$.

*Proof.* As before, the system of interface conditions (17) is reduced to the interface conditions (9) by premultiplying with $(a_{kj})^{-1}$. ■

The only property of the normal derivatives that was essential in the proof of the theorem is that they locally can be embedded in the differential operators (15), and that the interfaces are non-characteristic for these. This motivates the introduction of a more general type of interface conditions.

**DEFINITION 4.11** *Let* $k \ge 0$, *and let* $\{G_i\}$ *be a hyposmooth decomposition of* $G$. *If a mapping*

$$\mathscr{M} : C^k(G_i \cup (\bigcup_{i' \ne i} \Gamma_{ii'})) \to C^0(\bigcup_{i' \ne i} \Gamma_{ii'}) \; , \; i \in I$$

*satisfies, for each* $j = 1, \ldots, R$,

$$(\mathscr{M} v_i)|_{\partial A_j} = \widetilde{B^{(j)} v_i}|_{\partial A_j} \; ,$$

*where* $B^{(j)}$ *is a differential operator of order* $k$ *in an open neighborhood* $\mathscr{A}_j$ *of* $\partial A_j \cap G$ *with coefficients in* $C^0(\mathscr{A}_j)$, *we say that* $\mathscr{M}$ *is a compatible interface operator of order* $k$. *We say that the hyposmooth decomposition (of order* $\ge 1$*) is non-characteristic with respect to* $\mathscr{M}$ *if, for each* $j = 1, \ldots, R$, *the surface* $\partial A_j \cap G$ *is non-characteristic for* $B^{(j)}$.

**Remark 4.12** It can be verified that $D_\nu^k$ is a compatible interface operator of order $k$, by constructing the $B^{(j)}$ from the $B_j$ introduced in the proof above.

We get the following more general statement of the theorem, in the same spirit as COROLLARY 3.7.

**COROLLARY 4.13** *The conclusion of* THEOREM 4.5 *still holds if condition* (9) *is replaced by*

$$\sum_{j=0}^{m-1} a_{kj} \mathscr{M}_j v_i = \sum_{j=0}^{m-1} a_{kj} \mathscr{M}_j v_{i'} \quad on \;\; \Gamma_{ii'} \;\; for \;\; 0 \le k \le m-1 \; ,$$

*where the* $m \times m$ *matrix* $(a_{kj})$ *is non-singular on* $\bigcup_{j=1}^R (\partial A_j \cap G)$, *and for each* $j = 0, \ldots, m-1$, $\mathscr{M}_j$ *is a compatible interface operator of order* $j$, *and the hyposmooth decomposition is non-characteristic with respect to* $\mathscr{M}_j$.

# 5 Various generalizations

In this section, we will briefly discuss possible generalizations of the results above. The first thing to notice, is that all results in the two previous sections generalize directly to the case of systems of PDEs. The only formal change is that the requirement $P^\bullet(p, D\Psi_p^1(p)) \neq 0$ for non-characteristicity is replaced by the requirement

$$\det P^\bullet(p, D\Psi_p^1(p)) \neq 0 \ ;$$

cf. DEFINITION 3.3. The results generalizing THEOREM 4.5 and COROLLARY 4.13 are as follows.

**THEOREM 5.1** *Let $m, r \geq 1$, and $k \geq m$. Let $G \subset \mathbb{R}^n$ be open, let $P(x, D)$ be a differential operator of order $m$ in $G$ with coefficients in $C^{k-m}(G)^{r\times r}$, and let and let $\{G_i\}_{i\in I}$ be a locally finite hyposmooth decomposition of $G$ of order $k$, non-characteristic with respect to $P$.*

*Let $\partial G^\star \subset \partial G$ be regular, let $G^\star = G \cup \partial G^\star$, and let $\partial G_i^\star = \partial G_i \cap G^\star$. Assume $v_i \in C^k(G_i \cup \partial G_i^\star)^r$ satisfies*

$$D_\nu^j v_i = D_\nu^j v_{i'} \ \ \text{on} \ \ \Gamma_{ii'} \ \ \text{for} \ \ 0 \leq j \leq m-1 \ , \tag{18}$$

*and*

$$P(x, D)v_i = f|_{G_i} \ ,$$

*where $f \in C^{k-m}(G)^r$. Then there exists a unique function $u \in C^k(G^\star)^r$ with the properties $u|_{G_i} = v_i$ and $P(x, D)u = f$.*

In the systems case, if the operators $B^{(j)}$ in DEFINITION 4.11 have coefficients in $C^0(\mathscr{A}_j)^{r\times r}$, we say that the corresponding compatible interface operator has $r$ components.

**COROLLARY 5.2** *The conclusion of* THEOREM 5.1 *still holds if condition* (18) *is replaced by*

$$\sum_{j=0}^{m-1} a_{kj}\mathscr{M}_j v_i = \sum_{j=0}^{m-1} a_{kj}\mathscr{M}_j v_{i'} \ \ \text{on} \ \ \Gamma_{ii'} \ \ \text{for} \ \ 0 \leq k \leq m-1 \ ,$$

*where the $a_{kj}$ are $r\times r$ matrices, the $mr\times mr$ matrix $(a_{kj})$ is non-singular on $\bigcup_{j=1}^R(\partial A_j \cap G)$, and for each $j = 0, \ldots, m-1$, $\mathscr{M}_j$ is a compatible interface operator of order $j$ with $r$ components, and the hyposmooth decomposition is non-characteristic with respect to $\mathscr{M}_j$.*

The next thing to notice, is that THEOREM 5.1 will hold for any decomposition of $G$ that can be made into a hyposmooth decomposition by subtracting the boundaries of finitely many $C^k$ domains. As an example, consider the recomposition problem for the upper right decomposition of Figure 3, which is a hyposmooth decomposition of order $k$ only when $k = 0$. Let $A_3$ be a $C^k$ domain whose boundary includes the horizontal interface already present. The interface conditions will hold automatically on the new interface introduced by subtracting $\partial A_3$, and the resulting decomposition of $\mathscr{B}$ is hyposmooth of order $k$. Hence we can apply THEOREM 5.1 to this case; in particular, this provides an answer for the original decomposition.

We can also relax the requirement of non-characteristicity somewhat in the assumptions of THEOREM 5.1. For the proof to go through, it is sufficient that the following property is satisfied:

**ASSUMPTION 5.3** *The interfaces $\Gamma_{ii'}$ corresponding to each of the $R$ hyposmooth decompositions*

$$G \setminus \partial A_1,\ G \setminus \partial A_1 \setminus \partial A_2,\ \ldots, G \setminus \partial A_1 \setminus \partial A_2 \setminus \cdots \setminus \partial A_R$$

*are all non-characteristic for $P$.*

Note that the ordering of the $A_j$ plays a role here. As an example, consider the lower middle decomposition in Figure 3. It is a hyposmooth decomposition of any order. The boundaries of the five balls generating the decomposition are all characteristic with respect to $P = D_1$ at some points, so apparently THEOREM 5.1 can not be applied to this situation. However, the parts of the boundaries lying in the interior of the square are non-characteristic; hence we can apply the theorem as before.

It is even sufficient that the decomposition can be made into a hyposmooth decomposition having this property, by subtracting the boundaries of a finite number of $C^k$ domains. As an example, consider the upper middle decomposition in Figure 3. The boundaries of the two balls $A_1$ and $A_2$ generating this decomposition are characteristic with respect to $P = D_2$ at the center of $\mathscr{B}$. Subtract the boundary of the upper half-space, which we denote by $A_3$. As above, the interface conditions are automatically satisfied on the new interfaces. Now, by renumbering the three $C^k$ domains in such a way that $A_1$ becomes the upper half-space, we see that ASSUMPTION 5.3 is satisfied. Hence THEOREM 5.1 can be applied, and we see that this also answers the recomposition problem for the case we started out with.

We summarize this discussion in the following result.

**COROLLARY 5.4** *The conclusion of* THEOREM 5.1 *holds for any family $\{G_i\}$ if the family of sets we get by subtracting the boundaries of finitely many $C^k$ domains from the family $\{G_i\}$ is a locally finite hyposmooth decomposition of $G$ of order $k$, and if, after a possible renumbering of all the $C^k$ domains generating the new decomposition,* ASSUMPTION 5.3 *is satisfied.*

A direct application of the results presented in this paper can be found in [2], where the author discusses the question of equivalence between a decomposed problem and the corresponding original problem.

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