Waveform relaxation as a dynamical system

Morten Bjørhus

and

Andrew Stuart*

Abstract

In this paper the properties of waveform relaxation are studied when applied to the dynamical system generated by an autonomous ordinary differential equation. In particular, the effect of the waveform relaxation on the invariant sets of the flow is analysed. Windowed waveform relaxation is studied, whereby the iterative technique is applied on successive time intervals of length T and a fixed, finite, number of iterations taken on each window. This process does not generate a dynamical system on $\mathbb{R}^+$ since two different applications of the waveform algorithm over different time intervals do not, in general, commute. In order to generate a dynamical system it is necessary to consider the time T map generated by the relaxation process. This is done, and C^1-closeness of the resulting map to the time T map of the underlying ordinary differential equation is established. Using this, various results from the theory of dynamical systems are applied, and the results discussed.

1 Introduction

Consider the *waveform relaxation* (or dynamic iteration) scheme

$$\begin{cases} \text{For each } \ \ell = 1,2,3,\ldots \ \text{ find } \ u^{(\ell)} : \mathbb{R}^+ \to \mathbb{R}^m \ \text{ satisfying} \\ \\ \dot{u}^{(\ell)} = \tilde{f}(u^{(\ell)}, u^{(\ell-1)}) \ , \quad u^{(\ell)}(0) = U \ , \\ \qquad\qquad u^{(0)}(t) \equiv U \ , \end{cases} \tag{1}$$

used to approach the solution $u : \mathbb{R}^+ \to \mathbb{R}^m$ of the initial value problem

$$\dot{u} = f(u) \quad , \quad u(0) = U \ . \tag{2}$$

Here $\tilde{f}(v,w)$, the *splitting* of f, must satisfy $\tilde{f}(u,u) = f(u)$ for all $u \in \mathbb{R}^m$ for consistency. The special splitting $\tilde{f}(v,w) := f(w)$ yields the classical Picard iteration.

*Scientific Computing/Computational Mathematics program, Stanford University

Several papers have been written on the mathematical and numerical properties of this and related methods during the last decade; from the first article by Lelarasmee et al. [7], motivated by applications in electrical engineering, to the much referenced article by Miekkala and Nevanlinna [8], and later works by, for example, Nevanlinna [9], Bellen and Zennaro [1], and Bjørhus [2]. Much of the interest in the technique is motivated by the possibility of using it to exploit parallelism in software for initial value problems.

Under appropriate global Lipschitz conditions on $\tilde{f}$, the convergence of (1) to the solution of (2) is superlinear on any finite time interval. However the convergence may be slow initially and so, instead of performing the waveform relaxation on the entire time interval of interest, one often confines the iteration to windows. That is, we choose $T > 0$ and iterate successively on intervals of length T, starting with the interval $[0, T]$. In a practical implementation it is necessary to stop the iteration in a given window after a finite number of iterations, and then move on to the next window. There are at least two possible strategies one can employ to determine the number of iterations to take. The first is to iterate a fixed number of times in each window, and the other is to stop the iteration when the difference between two successive iterates is bounded from above by some given tolerance. In the present paper we study the first strategy.

Define the nth window $I_n = (nT, nT+T)$ and let k denote the fixed number of iterations taken in each time window. Let $u_n^{(\ell)} : \overline{I_n} \to \mathbb{R}^m$ denote iterate number ℓ on the nth window. With this windowing strategy, our waveform relaxation scheme becomes, given $u_{-1}^{(k)}(0) = U$, the following:

$$\begin{cases} \text{For } \; n = 0, 1, 2, \ldots \; \text{ find } \; u_n^{(k)} \; \text{ by solving} \\ \\ \dot{u}_n^{(\ell)} = \tilde{f}(u_n^{(\ell)}, u_n^{(\ell-1)}) \; \text{ in } \; I_n \, , \;\; u_n^{(\ell)}(nT) = u_{n-1}^{(k)}(nT) \, , \\ \qquad\qquad u_n^{(0)}(t) \equiv u_{n-1}^{(k)}(nT) \; \text{ in } \; I_n \\ \\ \text{for } \; \ell = 1, 2, \ldots, k \, . \end{cases} \tag{3}$$

Let $\mathscr{S}(t)$, $t \geq 0$ be the (generally nonlinear) semigroup of mappings from $\mathbb{R}^m$ to $\mathbb{R}^m$ corresponding to the solution of (2), so that

$$\mathscr{S}(t)U = u(t) \;\text{ for }\; 0 \leq t < \infty \, .$$

If we define a similar family of mappings $\mathscr{S}_k(t)$ corresponding to our approximation (3), so that

$$\mathscr{S}_k(t)U = u_{[t/T]}^{(k)}(t) \;\text{ for }\; 0 \leq t < \infty \, , \tag{4}$$

where $[\cdot]$ denotes the integer part, it is easily seen that this family is in general not a semigroup. For example, take the Picard splitting with $k = 1$. We then have

$$\mathscr{S}_1(t)U = U + \int_0^t f(U)\, d\tau \;\text{ for }\; 0 \leq t < T \, , \tag{5}$$

which is equivalent to approximating $u(t)$ using Euler's method. From (5) and the definition (4) of $\mathscr{S}_1(t)$ we have

$$\begin{aligned}\mathscr{S}_1(3T/2)U &= \mathscr{S}_1(T/2)\mathscr{S}_1(T)U \\ &= (U+Tf(U)) + \int_0^{T/2} f(U+Tf(U))\,d\tau \\ &= U+Tf(U)+\frac{T}{2}f(U+Tf(U))\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\mathscr{S}_1(T)\mathscr{S}_1(T/2)U &= \mathscr{S}_1(T)\left(U+\frac{T}{2}f(U)\right) \\ &= U+\frac{T}{2}f(U)+\int_0^{T} f(U+\frac{T}{2}f(U))\,d\tau \\ &= U+\frac{T}{2}f(U)+Tf(U+\frac{T}{2}f(U))\,.\end{aligned}$$

Thus $\mathscr{S}_1(T)\mathscr{S}_1(T/2) \neq \mathscr{S}_1(3T/2)$. This reflects the well-known fact that the mappings associated with Euler's method for two different steps do not commute. Hence the family $\{\mathscr{S}_k(t)\}$ can not in general be a semigroup.

Our objective here is to examine the effect of the approximation (3) on the dynamical system generated by (2). Since the mapping $\mathscr{S}_k(t)$ does not define a dynamical system we are immediately faced with a conceptual problem. With this in mind we choose to formulate the problem in a different way, considering only the values of u and $u_n^{(k)}$ at the points nT, $n = 0, 1, 2, \ldots$ Thus, we only deal with the two sequences

$$\{U, u(T), u(2T), \ldots\} \quad\text{and}\quad \{U, u_0^{(k)}(T), u_1^{(k)}(2T), \ldots\}\,. \tag{6}$$

Let S^n, $n \geq 0$ be the discrete semigroup of mappings from $\mathbb{R}^m$ to $\mathbb{R}^m$ defined by $S^n = \mathscr{S}(nT)$; that is

$$S^nU = u(nT) \quad\text{for}\quad n = 0, 1, 2, \ldots\ .$$

The semigroup property implies that the rule $S^{n+m} = S^nS^m = S^mS^n$ holds for all $n, m \geq 0$, and that $S^0 = I$. Similarly, for our waveform relaxation scheme (3), we define the family of mappings S_k^n as the mappings taking U to the approximation to the solution at time nT; thus

$$S_k^nU = u_{n-1}^{(k)}(nT) \quad\text{for}\quad n = 0, 1, 2, \ldots\ .$$

Now S_k^n, $n \geq 0$ is a semigroup, since it may be verified that $S_k^0 = I$ and $S_k^{n+m} = S_k^nS_k^m = S_k^mS_k^n$ for $n, m \geq 0$.

Throughout the paper we use $\|\cdot\|$ to denote the euclidean norm on $\mathbb{R}^m$ (in LEMMA 2.3 on $\mathbb{R}^{2m}$) and also the induced operator norm on matrices in $L(\mathbb{R}^m, \mathbb{R}^m)$. By $B(x, r)$ we denote the closed ball in $\mathbb{R}^m$ centered in x with radius r. For $g : \mathbb{R}^m \to \mathbb{R}^m$ we let $Dg(x) \in L(\mathbb{R}^m, \mathbb{R}^m)$ denote the Frechet derivative of g at x. Similarly, for $g : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ we let D_ig denote the derivative with respect to component i, $i = 1, 2$. We make the following simplifying assumption throughout the paper:

ASSUMPTION 1.1 *The vector fields $f \in C^1(\mathbb{R}^m, \mathbb{R}^m)$ and $\tilde{f} \in C^1(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$ are globally Lipschitz continuous, that is, there exist constants K, K_1, and $K_2 > 0$ such that, for all $x, y, v, w \in \mathbb{R}^m$,*

$$\begin{aligned} \|f(v) - f(w)\| &\leq K\|v - w\| && (7) \\ \|\tilde{f}(x, v) - \tilde{f}(y, w)\| &\leq K_1\|x - y\| + K_2\|v - w\|; && (8) \end{aligned}$$

furthermore, the first derivatives of f and $\tilde{f}$ are globally Lipschitz continuous, that is, there exist constants B, B_I and B_{II} such that, for all $x, y, z, x', y', z' \in \mathbb{R}^m$,

$$\begin{aligned} \|Df(x)y - Df(x')y'\| &\leq B\left(\|x - x'\| + \|y - y'\|\right) , \\ \|D_1\tilde{f}(x, y)z - D_1\tilde{f}(x', y')z'\| &\leq B_I\left(\|x - x'\| + \|y - y'\| + \|z - z'\|\right) , \\ \|D_2\tilde{f}(x, y)z - D_2\tilde{f}(x', y')z'\| &\leq B_{II}\left(\|x - x'\| + \|y - y'\| + \|z - z'\|\right) ; \end{aligned}$$

finally, $\tilde{f}$ is a consistent splitting of f, that is,

$$\tilde{f}(u, u) = f(u) \quad \forall\, u \in \mathbb{R}^m . \tag{9}$$

Of course, imposing such global Lipschitz bounds is an unnecessarily strong assumption which can be removed with a slightly more subtle analysis. In several places we will comment on extensions to the locally Lipschitz case.

In Section 2 we prove some basic approximation results, showing C^1-closeness of the maps S^n and S_k^n and using this to study the difference between $u(t)$ and $u_n^{(k)}(t)$. In Sections 3 and 4 these results are used to study the effect of the waveform relaxation on equilibria, phase portraits, periodic solutions and strange attractors. Finally note that we do not explicitly consider the effect of numerical approximation on the relaxation (3). However, the effect of this additional approximation could be incorporated in the analysis without great difficulty by using the techniques in [10] to establish C^1-closeness of the map generated by numerical approximation of (3) to the map S_k^1.

2 Some approximation results

In order to use certain results from dynamical systems theory in the remainder of the paper, it is useful to show that our discrete semigroups satisfy, under suitable assumptions, the two following conditions:

(i)

$$\|S_k^1 U - S^1 U\| \to 0 \quad \text{as} \quad k \to \infty,$$

uniformly for U in bounded sets.

(ii)

$$\|DS_k^1(U) - DS^1(U)\| \to 0 \quad \text{as} \quad k \to \infty,$$

uniformly for U in bounded sets.

Property **(i)** (resp. **(ii)**) is referred to as C^0-closeness (resp. C^1-closeness) of the mappings S_k^1 and S^1.

As an immediate consequence of the C^0-closeness, we can, if f is globally Lipschitz continuous with constant K, bound the error $\mathscr{E}_k^{n+1} := S_k^{n+1}U - S^{n+1}U$ by

$$\begin{aligned}\|\mathscr{E}_k^{n+1}\| &= \|S_k^{n+1}U - S^{n+1}U\| = \|S_k^1 S_k^n U - S^1 S^n U\| \\ &\leq \|S_k^1 S_k^n U - S^1 S_k^n U\| + \|S^1 S_k^n U - S^1 S^n U\|\end{aligned}$$

so that, by continuity of the semigroup $\mathscr{S}(t)$,

$$\|\mathscr{E}_k^{n+1}\| \leq \|S_k^1 S_k^n U - S^1 S_k^n U\| + e^{KT}\|\mathscr{E}_k^n\| \; . \tag{10}$$

Successive application of this estimate, together with **(i)**, will yield an estimate of $\|\mathscr{E}_k^N\|$. Hence we can analyze convergence properties of our iteration. We will give the precise result later in this section. The C^1-closeness will allow us to consider the effect for the approximation (3) on more complicated invariant sets of (2).

We will need the following lemma, relating the iterates appearing in (1) to the solution of (2).

LEMMA 2.1 *Let* (7), (8) *and* (9) *of* ASSUMPTION 1.1 *hold. Let u solve* (2) *with $u(0) = U$ and let $v^{(k)}$ solve* (1) *with $v^{(k)}(0) = V$, $v^{(0)}(t) \equiv V$. Then, if*

$$C_1(t) = \frac{1}{K} e^{(K_1+K)t} \quad , \quad \varepsilon(k,t) = \frac{(K_2 t)^k}{k!} \; , \tag{11}$$

it follows that

$$\|v^{(k)}(t) - u(t)\| \leq C_1(t)\varepsilon(k,t)\|f(V)\| + e^{Kt}\|U - V\| \quad \forall\, t \geq 0 \; .$$

Proof. Let v be the solution of (2) with $v(0) = V$. We have

$$\|v^{(k)}(t) - u(t)\| \leq \|v^{(k)}(t) - v(t)\| + \|v(t) - u(t)\| \; .$$

The last quantity on the right satisfies the standard estimate

$$\|v(t) - u(t)\| \leq e^{Kt}\|V - U\| \; .$$

It remains to estimate the quantity $\|v^{(k)}(t) - v(t)\|$. First we note that we have (on the open intervals where $v^{(k)}(t) \neq v(t)$) the differential inequality

$$\frac{d}{dt}\|v^{(k)}(t) - v(t)\| \leq K_1\|v^{(k)}(t) - v(t)\| + K_2\|v^{(k-1)}(t) - v(t)\| \; .$$

An application of Gronwall's lemma yields a bound for $\|v^{(k)}(t) - v(t)\|$ in terms of K_1, K_2, and $\|v^{(k-1)}(t) - v(t)\|$. Successive application of this bound yields an iterated integral which can be estimated crudely to give

$$\|v^{(k)}(t) - v(t)\| \leq \varepsilon(k,t) e^{K_1 t} \max_{0\leq s\leq t} \|v^{(0)}(s) - v(s)\| \; ;$$

see, for example, [2]. The last norm can be estimated, again by using Gronwall's lemma in a standard fashion, as

$$\max_{0\le s\le t}\|v^{(0)}(s)-v(s)\| = \max_{0\le s\le t}\|V-v(s)\| \le \frac{e^{Kt}}{K}\|f(V)\| ,$$

and we thus obtain the estimate

$$\|v^{(k)}(t)-v(t)\| \le C_1(t)\varepsilon(k,t)\|f(V)\| .$$

This completes the proof. ∎

Remark 2.2 Although we have assumed globally Lipschitz f and $\tilde{f}$, it is worth remarking that superlinear convergence can be established in the locally Lipschitz case in the following two situations: (a) if T is sufficiently small then the map from the function $u^{(\ell)}(t)$ to the function $u^{(\ell+1)}(t)$ takes a bounded set into itself and hence the globally Lipschitz result may be adapted to apply within this bounded set; (b) if the sequence of functions $\{u^{(\ell)}(t)\}_{\ell=0}^{\infty}$ is uniformly bounded in the supremum norm on $[0,T]$ then the globally Lipschitz result may be adapted to apply, on a set satisfying the appropriate bound, to deduce superlinear convergence of the iteration.

In the following, let $C_1'(t)$ and $\varepsilon'(k,T)$ be given by $C_1(t)$ and $\varepsilon(k,T)$ (see (11)) with $K \mapsto \sqrt{2}(B+K), K_i \mapsto \sqrt{2}(B_I+B_{II}+K_i)$ for $i=1,2$. Now we are in the position to show the following result.

LEMMA 2.3 *Let* ASSUMPTION 1.1 *hold. Then, for all $T>0$,*

$$\|S_k^1U - S^1U\| + \|DS_k^1(U) - DS^1(U)\| \le \sqrt{2}\cdot C_1'(T)\cdot\varepsilon'(k,T)\,(\|f(U)\|+K) \tag{12}$$

for all $U\in\mathbb{R}^m$.

Proof. In the following D_U denotes differentiation with respect to U. Let $u(t)=u(U,t)$, and define for $\ell=1,2,\ldots$ and for all $\xi\in\mathbb{R}^m$

$$\eta_\xi(t) = \begin{pmatrix} u(t) \\ D_U u(U,t)\xi \end{pmatrix} \ , \ \ \eta_\xi^{(\ell)}(t) = \begin{pmatrix} u^{(\ell)}(t) \\ D_U u^{(\ell)}(U,t)\xi \end{pmatrix} .$$

We can see that η_ξ and $\eta_\xi^{(\ell)}$ satisfy the initial value problems

$$\dot{\eta}_\xi = g(\eta_\xi) \ , \ \ \eta_\xi(0) = \begin{pmatrix} U \\ \xi \end{pmatrix}$$

and

$$\dot{\eta}_\xi^{(\ell)} = \tilde{g}(\eta_\xi^{(\ell)},\eta_\xi^{(\ell-1)}) \ , \ \ \eta_\xi^{(\ell)}(0) = \begin{pmatrix} U \\ \xi \end{pmatrix} \ , \ \ \eta_\xi^{(0)}(t) \equiv \begin{pmatrix} U \\ \xi \end{pmatrix} ;$$

here the functions $g : \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ and $\tilde{g} : \mathbb{R}^{2m} \times \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ are given by (for $x_i, y_i \in \mathbb{R}^m$)

$$g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{pmatrix} f(x_1) \\ Df(x_1)x_2 \end{pmatrix} ,$$

and

$$\tilde{g}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{pmatrix} \tilde{f}(x_1, y_1) \\ D_1\tilde{f}(x_1, y_1)x_2 + D_2\tilde{f}(x_1, y_1)y_2 \end{pmatrix} .$$

Note that $\tilde{g}(\eta, \eta) = g(\eta)$ by ASSUMPTION 1.1. We see that the equations satisfied by η_ξ and $\eta_\xi^{(\ell)}$ are completely analogous to equations (2) and (1), respectively. Recall that $\|\cdot\|$ defines the euclidean norm on $\mathbb{R}^{2m}$. Writing vectors $w \in \mathbb{R}^{2m}$ in the form $w = (u^T, v^T)^T$, with $u, v \in \mathbb{R}^m$, we can define an equivalent norm $|\cdot|$ on $\mathbb{R}^{2m}$ by

$$|w| = \|u\| + \|v\| ,$$

satisfying

$$\|w\| \leq |w| \leq \sqrt{2}\|w\| . \tag{13}$$

Using this it follows that g and $\tilde{g}$ satisfy Lipschitz conditions analogous to (7) and (8) with $K \mapsto \sqrt{2}(B + K), K_i \mapsto \sqrt{2}(B_I + B_{II} + K_i)$ for $i = 1, 2$. Thus we can we can verify the assumptions of LEMMA 2.1 (for the case $V = U$), with g, $\tilde{g}$, $2m$ instead of f, $\tilde{f}$, m. We find that

$$\|u^{(k)}(T) - u(T)\| + \|D_U u^{(k)}(U, T)\xi - D_U u(U, T)\xi\| \leq \sqrt{2} \cdot \|\eta_\xi^{(k)}(T) - \eta_\xi(T)\|.$$

By LEMMA 2.1 we know that this last norm is bounded by $C_1'(T) \cdot \varepsilon'(k, T) \cdot \|g(\eta_\xi(0))\|$. But, by (13),

$$\sqrt{2} \cdot C_1'(T) \cdot \varepsilon'(k, T) \cdot \|g(\eta_\xi(0))\| \leq \sqrt{2} \cdot C_1'(T) \cdot \varepsilon'(k, T) \cdot (\|f(U)\| + \|Df(U)\xi\|).$$

Since f is globally Lipschitz with constant K so that $\|Df(U)\| \leq K$, this yields the estimate (12) after taking the supremum over $\|\xi\| = 1$. The proof is complete. ■

The estimates we set out to establish now follow from LEMMA 2.3:

PROPOSITION 2.4 *Let* ASSUMPTION 1.1 *hold. Then there exist constants* $C_2 = C_2(T, R)$ *and* $C_3 = C_3(T, R)$ *such that, for all* $U \in B(0, R)$ *and all* $T > 0$,

$$\|S_k^1 U - S^1 U\| \leq C_2 \varepsilon(k, T) \tag{14}$$

$$\|DS_k^1(U) - DS^1(U)\| \leq C_3 \varepsilon'(k, T). \tag{15}$$

Proof. In fact, from LEMMA 2.3, we see that for a given U both quantities can be bounded by the same expression, namely

$$\sqrt{2} \cdot C_1'(T) \cdot \varepsilon'(k, T) \, (\|f(U)\| + K) \ .$$

However, $f(U)$ being a continuous function in $\mathbb{R}^m$ implies that its composition with a norm is continuous. Since a continuous function always attains its maximum on a compact set, we can find a constant $C_f(R)$ such that

$$\sup_{U\in B(0,R)} \|f(U)\| \leq C_f(R) \; .$$

Hence we get the estimate (15) by putting $C_3 = \sqrt{2}\cdot C_1'(T)(C_f(R)+K)$. LEMMA 2.1 (with $V = U$) yields estimate (14) by letting $C_2(T,R) = C_1(T)\cdot C_f(R)$. ■

Now we can give the precise form of the convergence result promised at the beginning of this section.

THEOREM 2.5 *Let* ASSUMPTION 1.1 *hold and let* $\delta > 0$. *Assume that* $\|S^n U\| \leq r$ *for all* $0 \leq n \leq N$ *and that* k *is chosen so that*

$$\left(\frac{e^{NKT}-1}{e^{KT}-1}\right) C_4\varepsilon(k,T) \leq \delta \; ,$$

where $C_4 = C_2(T, r+\delta)$. *Then*

$$\|S_k^n U - S^n U\| \leq \left(\frac{e^{nKT}-1}{e^{KT}-1}\right) C_4\varepsilon(k,T) \quad \forall\, 0 \leq n \leq N \; . \tag{16}$$

Proof. Assume for the purposes of induction that (16) holds for $n = m \leq N-1$, noting that this is true for $n = 0$. By (16) it follows that

$$\|S_k^m U - S^m U\| \leq \left(\frac{e^{NKT}-1}{e^{KT}-1}\right) C_4\varepsilon(k,T) \leq \delta$$

so that $S_k^m U \in B(0, r+\delta)$. Thus, by (10), and (14),

$$\begin{aligned}
\|\mathcal{E}_k^{m+1}\| &\leq C_4\varepsilon(k,T) + e^{KT}\|\mathcal{E}_k^m\| \\
&\leq C_4\varepsilon(k,T) + e^{KT}\left(\frac{e^{mKT}-1}{e^{KT}-1}\right) C_4\varepsilon(k,T) \\
&= \left(\frac{e^{(m+1)KT}-1}{e^{KT}-1}\right) C_4\varepsilon(k,T) \; .
\end{aligned}$$

The result follows by induction. ■

COROLLARY 2.6 *Let the assumptions of* THEOREM 2.5 *hold, and let* $t = nT + \tau$ *where* $0 \leq n \leq N$ *and* $\tau \in [0,T)$. *Then there exists* $C_5 = C_5(T, r+\delta)$ *such that*

$$\|u_n^{(k)}(t) - u(t)\| \leq C_5\varepsilon(k,T) \; .$$

Proof. Applying LEMMA 2.1 we have

$$\begin{aligned}\|u_n^{(k)}(t) - u(t)\| &\leq \|\mathscr{S}_k(t)U - \mathscr{S}(t)U\| \\ &= \|\mathscr{S}_k(\tau)S_k^n U - \mathscr{S}(\tau)S^n U\| \\ &\leq C_1(\tau)\varepsilon(k,\tau)\|f(S_k^n U)\| + e^{K\tau}\|S_k^n U - S^n U\| \ .\end{aligned}$$

Noting that $S_k^n U \in B(0, r+\delta)$ and that $C_1(\tau) \leq C_1(T)$, $\varepsilon(k,\tau) \leq \varepsilon(k,T)$, applying THEOREM 2.5 gives the desired result. ■

Remark 2.7 Note that, to prove THEOREM 2.5 and COROLLARY 2.6 we actually only use (7), (8) and (9) of ASSUMPTION 1.1 since only (14), and not (15), is used in the proof. Point (15) will be used in Sections 3 and 4.

3 Neighbourhood of an equilibrium point

In this section we compare the behaviour of (2) and (3) in the neighbourhood of an equilibrium point of (2). We use the C^1-closeness established in Section 2 to apply some basic results from the theory of dynamical systems.

We make the following definitions:

$$\begin{aligned}\mathcal{E} &= \{v \in \mathbb{R}^m \mid \mathscr{S}(t)v \equiv v \ \forall\, t \geq 0\} \ , \\ \mathcal{E}_k &= \{v \in \mathbb{R}^m \mid \mathscr{S}_k(t)v \equiv v \ \forall\, t \geq 0\} \ .\end{aligned} \tag{17}$$

Thus $\mathcal{E}$ and $\mathcal{E}_k$ denote the equilibria of (2) and (3) respectively. We first show that $\mathcal{E} \equiv \mathcal{E}_k$ so that the waveform relaxation does not introduce spurious equilibria when viewed as a mapping on $\mathbb{R}^+$.

THEOREM 3.1 *For any vector fields f and $\tilde{f}$ satisfying* ASSUMPTION 1.1 *it follows that $\mathcal{E} \equiv \mathcal{E}_k$.*

Proof. First let $v \in \mathcal{E}$ so that $f(v) = 0$. Since $\tilde{f}(u,u) = f(u)$ for all $u \in \mathbb{R}^m$, it follows that $\tilde{f}(v,v) = 0$. Thus

$$u_n^{(\ell)}(t) \equiv v, \quad \forall\, \ell = 1,2,\ldots,k, \ \ \forall\, n = 0,1,2,\ldots \ \ , \forall\, t \in \overline{I_n}$$

is a solution of (3) with $U = v$ so that $v \in \mathcal{E}_k$ follows. Thus $\mathcal{E} \subseteq \mathcal{E}_k$.

Secondly, let $u_n^{(k)} \equiv v \in \mathcal{E}_k$. It follows that

$$\dot{u}_n^{(k)}(t) = 0 \quad \forall\, t \in I_n \ .$$

From this we deduce that

$$\tilde{f}(u_n^{(k)}(t), u_n^{(k-1)}(t)) = 0 \quad \forall\, t \in I_n \ .$$

Thus

$$\tilde{f}(v, u_n^{(k-1)}(t)) = 0 \quad \forall\, t \in I_n\ .$$

Hence, since $u_n^{(\ell)}(t)$ is continuous in t and since $f(v, \cdot)$ is a continuous function, it follows that

$$\tilde{f}(v, u_n^{(k-1)}(nT)) = 0.$$

But $u_n^{(\ell)}(t) = u_{n-1}^{(k)}(nT) = v$ for $\ell = 1, \dots, k$ since $v \in \mathcal{E}_k$. Thus we have

$$\tilde{f}(v, v) = 0.$$

Since $f(v) = \tilde{f}(v, v)$ we deduce that $v \in \mathcal{E}$ and hence that $\mathcal{E}_k \subseteq \mathcal{E}$. The proof is complete. ■

THEOREM 3.1 should be interpreted with caution: since the natural dynamical system for (3) is found by considering the time T map, we are led to study, instead of (17), the fixed points of S^1 and S^1_k; thus we define

$$\mathcal{E}^* = \{v \in \mathbb{R}^m \mid S^1 v = v\}\ ,$$

$$\mathcal{E}^*_k = \{v \in \mathbb{R}^m \mid S^1_k v = v\}\ .$$

As we shall see, these sets are *not* necessarily the same. In fact we prove:

THEOREM 3.2 *For any vector fields f and $\tilde{f}$ satisfying* ASSUMPTION 1.1 *we have:*

(i) $\mathcal{E}^* \supseteq \mathcal{E}$; *if $v \in \mathcal{E}^*$ and v is an isolated point in $\mathcal{E}^*$, then $v \in \mathcal{E}$.*

(ii) $\mathcal{E}^*_k \supseteq \mathcal{E}$; *if $v \in \mathcal{E}^*_k$ and v is an isolated point in $\mathcal{E}^*_k$, then it is not necessarily the case that $v \in \mathcal{E}$.*

Proof. Clearly $\mathcal{E}^* \supseteq \mathcal{E}$ since $\mathcal{E}^*$ comprises equilibria together with points on periodic solutions of period T. If $v \in \mathcal{E}^*$ is isolated then it cannot be a point on a periodic solution. Hence $v \in \mathcal{E}$ and (i) follows.

Clearly $\mathcal{E}^*_k \supseteq \mathcal{E}_k \equiv \mathcal{E}$. Thus to complete the proof of (ii) we need only find a counter-example. Consider (ii) in the case $k = 2$ and $\tilde{f}(u, v) = f(v)$. Then, if $w^{(\ell)}(t) = u_0^{(\ell)}(t)$ we have

$$\frac{dw^{(\ell)}}{dt}(t) = f(w^{(\ell-1)}(t)), \quad w^{(\ell)}(0) = U,$$

$w^0(t) \equiv U$, $t \in I_1$ and

$$w^{(1)}(t) = U + tf(U).$$

Thus

$$w^{(2)}(t) = U + \int_0^t f(U + sf(U))ds.$$

Hence

$$S^1_2 U = U + \int_0^T f(U + sf(U))ds.$$

To find an isolated fixed point of $S_2^1 : \mathbb{R}^m \to \mathbb{R}^m$ we consider the function $f(x) = -x^3$. (Although this function is not globally Lipschitz we can modify it smoothly outside a bounded set in such a way that the following argument still holds.) Then

$$\begin{array}{rcl} S_2^1 U &=& U + \int_0^T -(U - sU^3)^3 ds \\ &=& U + \left[\frac{(U-sU^3)^4}{4U^3} \right]_0^T \\ &=& U + \frac{U}{4}(1 - TU^2)^4 - \frac{U}{4}. \end{array}$$

The fixed points of S_2^1 are thus

$$U = 0 \ , \ \ U = \sqrt{2/T} \ .$$

Clearly $0 \in \mathcal{E}$; but $\sqrt{2/T} \notin \mathcal{E}$ and it is an isolated point in $\mathcal{E}_k^*$ so that (ii) follows. ∎

Remark 3.3 The construction of the counter-example in proving (ii) of THEOREM 3.2 is motivated by results concerning spurious fixed points of Runge-Kutta methods; see [6] for example.

As a consequence of THEOREM 3.1 and THEOREM 3.2 we now know that equilibria of (2) are fixed points of S^1 and S_k^1. It is natural to consider the behaviour of these maps in the neighbourhood of such fixed points. By doing this we can make deductions about the solutions of (2) and (3) near to equilibria.

Let the sequences $\{u_n\}_{n=0}^{\infty}$ and $\{U_n\}_{n=0}^{\infty}$ be generated by the maps

$$u_{n+1} = S^1 u_n \tag{18}$$

and

$$U_{n+1} = S_k^1 U_n \tag{19}$$

and assume that (2) has a hyperbolic equilibrium point v. Thus $v \in \mathcal{E}^*$ and, furthermore, v is a hyperbolic fixed point of $S^1 : \mathbb{R}^m \to \mathbb{R}^m$. This follows since

$$L := DS^1(v) = \exp(Df(v)T)$$

and the hyperbolicity of v as an equilibrium of (2) implies that no point of the spectrum of $Df(v)$ lies on the imaginary axis; it follows that no point of the spectrum of L lies on the unit circle.

Let $\mathcal{P}$ and $\mathcal{Q}$ denote projections onto the subspaces Y and Z corresponding to spectrum of L outside and inside the unit circle respectively. Thus

$$Y = \mathcal{P}\mathbb{R}^m, \quad Z = \mathcal{Q}\mathbb{R}^m, \quad \mathbb{R}^m = Y \oplus Z.$$

It follows that there exists $a < 1$ and norms $\|\cdot\|_u$, $\|\cdot\|_s$ on $\mathbb{R}^m$ in which

$$\|L^{-1}y\|_u \leq a\|y\|_u \quad \forall\, y \in Y \ ,$$

$$\|Lz\|_s \leq a\|z\|_s \quad \forall\, z \in Z \ .$$

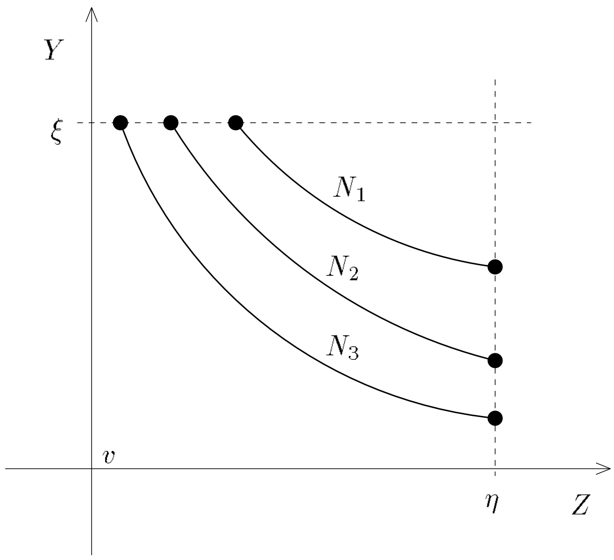

Figure 1: Pictorial representation of 3 solutions from COROLLARY 3.5. Here $0 < N_1 < N_2 < N_3 < \infty$.

If we define

$$\| \cdot \|_x = \max\{\|\mathcal{P} \cdot \|_u, \|\mathcal{Q} \cdot \|_s\}$$

then this is equivalent to the euclidean norm $\| \cdot \|$ on $\mathbb{R}^m$.

The local phase portrait of the mappings S^1 and S_k^1 may be found by solving (18), (19) subject to

$$\mathcal{P}[u_N - v] = \xi, \quad \mathcal{Q}[u_0 - v] = \eta, \tag{20}$$

$$\mathcal{P}[U_N - v] = \xi, \quad \mathcal{Q}[U_0 - v] = \eta. \tag{21}$$

Applying Theorems 4.13 and 4.17 of [11], which require the C^1-closeness established in PROPOSITION 2.4, we obtain the following result:

THEOREM 3.4 *Let* ASSUMPTION 1.1 *hold, let v be a hyperbolic equilibrium point of* (2) *and let $\xi \in Y$, $\eta \in Z$ satisfy $\|\xi\|_x, \|\eta\|_x \le \delta$. Then there are constants $C_6, C_7 > 0$ such that, for all such ξ and η, for all δ and $\varepsilon(k,T)$ sufficiently small, and all integers N, problems* (18), (20) *and* (19), (21) *have unique solutions satisfying*

$$\max_{0 \le n \le N} \|u_n - v\|_x , \max_{0 \le n \le N} \|U_n - v\|_x \le C_6 \delta;$$

furthermore,

$$\max_{0 \le n \le N} \|u_n - U_n\|_x \le C_7 \varepsilon(k,T).$$

A straightforward corollary is the following:

COROLLARY 3.5 *Let* ASSUMPTION 1.1 *hold, let v be a hyperbolic equilibrium point of* (2) *and let $\xi \in Y$ and $\eta \in Z$ satisfy $\|\xi\|_x, \|\eta\|_x \leq \delta$. Then there are constants $C_8, C_9 > 0$ such that, for all such ξ and η, for all δ and $\varepsilon(k,T)$ sufficiently small, and all integers N, equations* (2) *and* (3) *have solutions satisfying*

$$\mathcal{P}[u(NT) - v] = \xi\,, \quad \mathcal{Q}[u(0) - v] = \eta\,,$$

$$\mathcal{P}[u^{(k)}_{N-1}(NT) - v] = \xi\,, \quad \mathcal{Q}[u^{(k)}_1(0) - v] = \eta\,.$$

Furthermore,

$$\max_{0 \leq n \leq N-1} \sup_{0 \leq t \leq T} \|u(nT+t) - v\|_x\,, \quad \max_{0 \leq n \leq N-1} \sup_{0 \leq t \leq T} \|u^{(k)}_n(nT+t) - v\|_x \leq C_8\delta$$

and

$$\max_{0 \leq n \leq N-1} \sup_{0 \leq t \leq T} \|u(nT+t) - u^{(k)}_n(nT+t)\|_x \leq C_9\varepsilon(k,T)\,.$$

Remark 3.6 The previous result allows us to interpret certain solutions of (2) and (3) as being uniformly close, over arbitrarily long time intervals, in the neighbourhood of hyperbolic equilibria. This improves upon COROLLARY 2.6 which is not uniformly valid in time. The key to the improved result in the neighbourhood of equilibria is that the solutions of (2) and (3) have different initial conditions. Hence we have a shadowing-type result.

Figure 1 illustrates three solutions of (2), with fixed ξ and η, corresponding to three different values of N. Notice that, the larger N, the closer the solution lies to the equilibrium point v. In fact, as $N \to \infty$, the solution must approach arbitrarily close to v at some point on its trajectory.

Next we turn our attention to unstable manifolds. In the following we will need the Hausdorff semi-distance $dist(A,B)$ and the Hausdorff distance $d(A,B)$. Recall that

$$dist(A,B) = 0 \Leftrightarrow \overline{A} \subseteq \overline{B}$$

and

$$d(A,B) = 0 \Leftrightarrow \overline{A} \equiv \overline{B}\,;$$

see [10], for example.

We define the set

$$\overline{Y} = \{y \in Y \mid \|y\|_x \leq \delta\}.$$

Spør Andrew om normen her!

Straightforward application of Theorems 5.3 and 5.5 in [11], using the C^1-closeness of PROPOSITION 2.4, yields the following result:

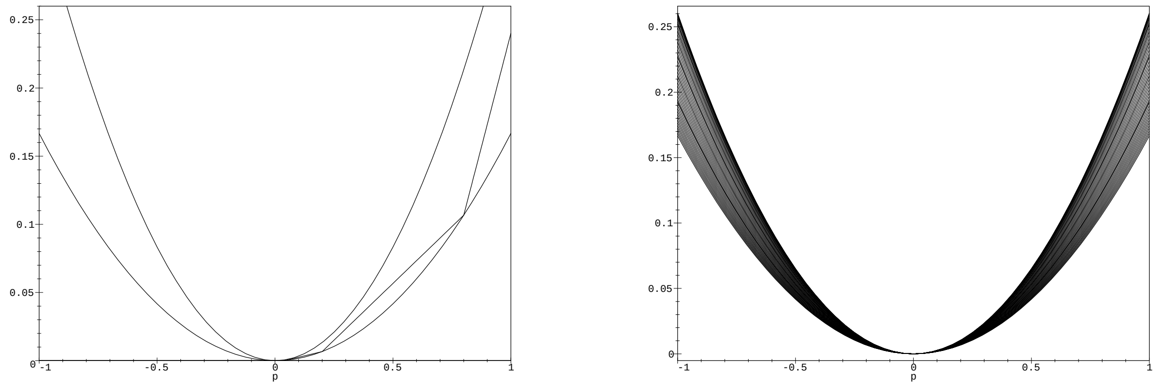

Figure 2: Illustrates Remark 3.8

THEOREM 3.7 *Let* ASSUMPTION 1.1 *hold. Then, for all δ and $\varepsilon(k,T)$ sufficiently small, there exist functions $\Phi, \Phi_k \in C(\overline{Y}, Z)$ such that the mappings S^1 and S^1_k have local unstable manifolds*

$$\mathcal{M} = \{x \in \mathbb{R}^m \mid \|\mathcal{P}x\|_x \leq \delta, \mathcal{Q}x = \Phi(\mathcal{P}x)\}\ ,$$

$$\mathcal{M}_k = \{x \in \mathbb{R}^m \mid \|\mathcal{P}x\|_x \leq \delta, \mathcal{Q}x = \Phi_k(\mathcal{P}x)\}\ .$$

Spør Andrew om normene her!

Furthermore, there is a constant $C_{10} > 0$ such that $d(\mathcal{M}, \mathcal{M}_k) \leq C_{10}\varepsilon(k,T)$.

Remark 3.8 Although the set $\mathcal{M}_k$ is (locally) invariant under S^1_k, it is *not* locally invariant under $\mathscr{S}_k(t)$. To see this, consider the equations

$$\begin{aligned} p_t &= p \\ q_t &= -q + p^2. \end{aligned} \tag{22}$$

It is straightforward to verify that the unstable manifold of the origin for (22) is the set

$$\mathcal{M} = \left\{(p,q) \in \mathbb{R}^2 \mid q = \frac{1}{3}p^2\right\}.$$

Hence this is the unstable manifold for S^1 and it is invariant under $\mathscr{S}(t)$ for any $t \geq 0$.

Now, for comparison, consider approximation of (22) by (3) with the Picard splitting $\tilde{f}(v,w) = f(w)$ and $k = 1$. Then, for $t \in [0,T]$ and $u = (p,q)^T$, we have that $\mathscr{S}_1(t)u = v$ where

$$v = ((1+t)p, (1-t)q + tp^2)^T.$$

It is straightforward to verify that the set

$$\mathcal{M}_1 = \left\{(p,q) \in \mathbb{R}^2 \mid q = \frac{1}{(3+T)}p^2\right\}$$

is invariant under the map $S_1^1 \equiv \mathscr{S}_1(T)$. However, this set is not invariant under $\mathscr{S}_1(t)$ in general; indeed, for $t \in [0, T]$,

$$\mathscr{S}_1(t)\mathcal{M}_1 = \left\{ (P, Q) \in \mathbb{R}^2 \mid Q = \frac{(1 + 2t + tT)}{(3 + T)(1 + t)^2} P^2 \right\}.$$

This is illustrated in Figure 2. The left figure denotes $\mathcal{M}$ (upper parabola), $\mathcal{M}_1$ (lower parabola) and the set $\mathscr{S}_1(t)U$ for some $U \in \mathcal{M}_1$ (piecewise linear curve). The right figure shows $\bigcup_{t\in[0,T]} \mathscr{S}_1(t)\mathcal{M}_1$ which clearly differs substantially from $\mathcal{M}_1$.

4 Periodic solutions and attractors

In the previous section we gave a detailed analysis of the behaviour of the waveform relaxation (1) as an approximation to (2), in the neighbourhood of a hyperbolic equilibrium point. In this section we consider more complicated invariant sets, namely periodic solutions and attractors.

If equation (2) has a hyperbolic periodic solution (that is, all its Floquet multipliers lie off the unit circle except for a simple multiplier at 1; see [3]), then the map S^1 will have a closed invariant curve Γ. In fact, in the terminology of [5], S^1 is $r-$*normally* hyperbolic at Γ for some $r \geq 1$. Application of Theorem 4.1 in [5], which requires the C^1-closeness established in PROPOSITION 2.4, gives the following result:

THEOREM 4.1 *Let* ASSUMPTION 1.1 *hold. Assume that* (2) *has a hyperbolic periodic solution comprising the set* Γ. *Then, for all* $\varepsilon(k, T)$ *sufficiently small, there is a constant* $C_{11} > 0$ *such that* S_k^1 *has a closed invariant curve* Γ_k *satisfying*

$$d(\Gamma, \Gamma_k) \leq C_{11}\varepsilon(k, T).$$

Remark 4.2 Although Γ_k is invariant under S_k^1 it is not, in general, invariant under $\mathscr{S}_k(t)$. To illustrate this we consider the equation

$$z_t = (i + 1 - |z|^2)z.$$

This has the hyperbolic periodic solution $z(t) = e^{it}$. Now consider again the Picard splitting with $k = 1$ so that

$$\mathscr{S}_1(t)z = z[(1 + t - t|z|^2) + it], \quad t \in [0, T].$$

For this example we have

$$\Gamma = \{z \in \mathbb{C} \mid |z|^2 = 1\}.$$

To calculate an invariant curve for S_1^1 set $w(t) = \mathscr{S}_1(t)z$ and note that

$$|w(t)|^2 = |z|^2[(1 + t - t|z|^2)^2 + t^2]. \tag{23}$$

Thus $|w(T)|^2 = |z|^2$ if

$$(1 + T - T|z|^2)^2 + T^2 = 1.$$

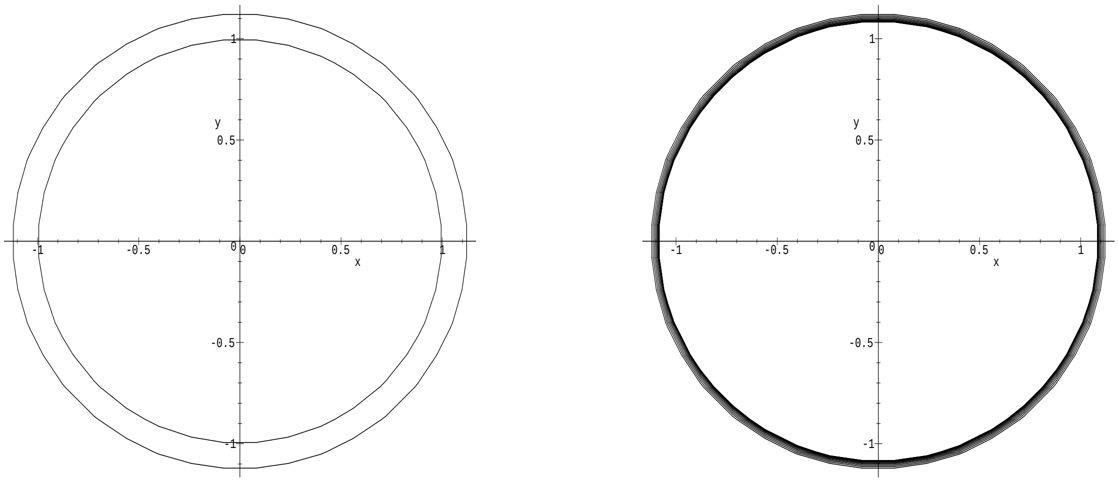

Figure 3: Illustrates Remark 4.2

If $T < 1$ this gives two values for $|z|$; to define Γ_1 we chose the value which converges to 1 as $T \to 0$. This gives

$$\Gamma_1 = \left\{ z \in \mathbb{C} \mid |z|^2 = \frac{1 + T - (1 - T^2)^{\frac{1}{2}}}{T} \right\} .$$

Using (23) we see that, for $t \in [0, T]$,

$$\mathscr{S}_1(t)\Gamma_1 = \left\{ Z \in \mathbb{C} \mid |Z|^2 = \frac{1 + T - (1 - T^2)^{\frac{1}{2}}}{T} \left(\left[1 - \frac{t}{T} \left\{ 1 - (1 - T^2)^{\frac{1}{2}} \right\} \right]^2 + t^2 \right) \right\} .$$

Thus Γ_1 is not invariant under $\mathscr{S}_1(t)$ for $t \in (0, T)$. Figure 3 illustrates this: the left figure shows Γ (inner circle) and Γ_1 (outer circle). The right figure shows $\bigcup_{t \in [0,T]} \mathscr{S}_1(t)\Gamma_1$.

Recall that an attractor is a compact, invariant set which attracts an open neighbourhood of itself. This forms a useful abstraction of the notion of a stable object which is observed after a long period of time in a dynamical system; in particular complicated objects such as the strange attractor for the Lorenz equations, together with simpler objects such as asymptotically stable equilibria and periodic solutions, are included in the definition. We now prove:

THEOREM 4.3 *Let* ASSUMPTION 1.1 *hold. Suppose that the semigroup generated by* (2) *has an attractor* $\mathcal{A}$. *Then* S_k^1 *has an attractor* $\mathcal{A}_k$ *for all* k *sufficiently large (resp. T sufficiently small), and*

$$dist(\mathcal{A}_k, \mathcal{A}) \to 0$$

as $k \to \infty$ *(resp.* $T \to 0$*).*

Proof. Since $\mathcal{A}$ is an attractor for the continuous semigroup generated by (2), it is also an attractor for S^1. Applying Theorem 2.5.4 in [4], noting that (ii) holds in its statement by virtue of THEOREM 2.5, the result follows. ■

Remark 4.4 The hypotheses of ASSUMPTION 1.1 can be considerably weakened for this theorem to be proved since only C^0-closeness of S^1 and S_k^1 is required to apply the required theorem in [4]. Thus only (7), (8) and (9) are required.

Remark 4.5 As we have seen with the simple invariant sets comprising unstable manifolds and periodic solutions, the invariance of a set $\mathcal{M}$ under $\mathscr{S}(T)$ does not imply invariance under $\mathscr{S}(t)$ for $t \in (0,T)$. To illustrate this we consider the approximation of the Lorenz equations

$$x_t = \sigma(y - x) \; ,$$

$$y_t = rx - y - xz \; ,$$

$$z_t = xy - bz \; ,$$

by use of the Picard splitting with $k = 1$ (Euler's method) and $T = 0.02446$. Figures 4 and 5 show solutions of the equations, with parameters set at $\sigma = 10, r = 28$ and $b = \frac{8}{3}$, and initial data $x = 0.0, y = 1.0, z = 0.0$. In both cases the initial transient is not plotted. In Figure 4 the solution is only plotted at integer multiples of T yielding the attractor $\mathcal{A}_1$. In Figure 5 we show $\bigcup_{t\in[0,T]} \mathscr{S}_1(t)\mathcal{A}_1$. Notice the difference between these two objects.

5 Conclusion

Theory and examples have been used to show that waveform relaxation does not form a dynamical system on $\mathbb{R}^+$; the numerical examples illustrating this point are somewhat crude, being based on the Picard splitting of f and using only one sweep of the iteration, but they illustrate the underlying principle clearly. We have shown that, if windowed waveform relaxation *is* to be viewed as a dynamical system, then the map between integer multiples of the window length T is the appropriate approximate dynamical system to compare with the underlying continuous dynamical system. Using this theory we have proved the convergence of a variety of invariant sets of (2) under the approximation (3).

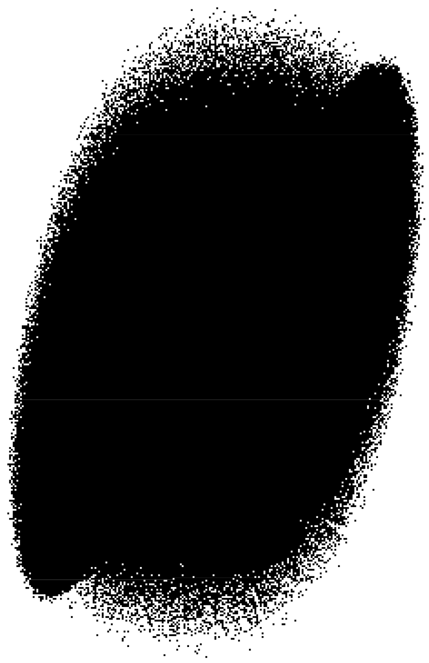

Figure 4: $\mathcal{A}_1$, the attractor for S_1^1 applied to the Lorenz equations

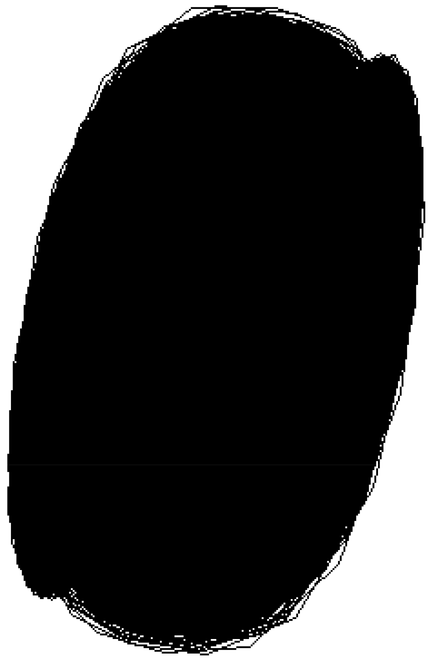

Figure 5: $\mathscr{S}_1(t)\mathcal{A}_1$; see Remark 4.5

References

[1] A. Bellen and M. Zennaro, *The use of Runge-Kutta formulae in waveform relaxation methods*, Appl. Numer. Math., 11 (1993), pp. 95–114.

[2] M. Bjørhus, *A note on the convergence of discretized dynamic iteration*, BIT, 35 (1995), pp. 291–296.

[3] J. K. Hale, *Ordinary differential equations*, Wiley, Chichester, 1969.

[4] ——, *Asymptotic behaviour of dissipative systems*, no. 25 in Mathematical surveys and Monographs, AMS, Rhode Island, 1988.

[5] M. Hirsch, C. Pugh, and M. Shub, *Invariant manifolds*, Springer-Verlag, New York, 1977.

[6] A. Iserles, *Stability and dynamics of numerical methods for nonlinear ordinary differential equations*, IMA J. Num. Anal., 10 (1990), pp. 1–30.

[7] E. Lelarasmee, A. E. Ruehli, and A. L. Sangiovanni-Vincentelli, *The waveform relaxation method for time-domain analysis of large scale integrated circuits*, IEEE Trans. on CAD of IC and Syst., 1 (1982), pp. 131–145.

[8] U. Miekkala and O. Nevanlinna, *Convergence of dynamic iteration methods for initial value problems*, SIAM J. Sci. Stat. Comput., 8 (1987), pp. 459–482.

[9] O. Nevanlinna, *Remarks on Picard-Lindelöf iteration*, BIT, 29 (1989), pp. 328–346, 535–562.

[10] A. M. Stuart, *Numerical analysis of dynamical systems*, in Acta Numerica, Cambridge University Press, 1994.

[11] ——, *Perturbation theory for infinite dimensional dynamical systems*, in Advances in Numerical Analysis, M. Ainsworth, J. Levesley, W. Light, and M. Marletta, eds., Oxford University Press, 1995.