On domain decomposition and subdomain iteration

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Abstract

An abstract formulation of domain decomposition methods is presented, and the notion of a reformulation of the original problem is introduced. The results obtained in the abstract case are used in the PDE context to provide sufficient conditions for a decomposed problem to be a reformulation of the corresponding original problem. Several examples are given. When a reformulation is established, one can employ a subdomain iteration method to solve the decomposed problem. A convergence result for one-dimensional hyperbolic systems is presented.

1 Introduction

For many purposes it is useful, and even necessary, to decompose the domain of a system of partial differential equations into subdomains. Often this domain decomposition is motivated by the possibilities it opens for in connection with the numerical approximation of the problem, such as parallel processing, treatment of complex geometries, and reduction in computer memory requirements. In other cases the decomposition is motivated by underlying physical principles, as in the case of a boundary layer, or in the coupling of different media.

In the first part of this paper, we investigate under which circumstances, concerning the geometry and the interface conditions, we can say that a decomposed problem is equivalent to the original problem with respect to a given function space. By this we mean that each solution of the original problem should also solve the decomposed problem, and that each solution of the decomposed problem should be the restriction of exactly one solution of the original problem (for admissible data). Hence we can concentrate only on solving the decomposed problem, as all the solutions of the original problem can be found this way, and moreover, no spurious solutions will exist.

The following example illustrates the problem. Let the original problem be to find solutions of the heat equation in the strip $-1 < x < 1$, $t > 0$, which we denote by G, with initial and boundary conditions for $t = 0$ and $x = \pm 1$, respectively. Let $G_1 = G \cap \{x < 0\}$ and $G_2 = G \cap \{x > 0\}$, and let the decomposed problem be to find solutions v_1 and v_2 of the heat equation in the strips G_1 and G_2, respectively, with initial and boundary conditions for $t = 0$, $x = \pm 1$, and with the additional interface condition $v_1 = v_2$ for $x = 0$, $t > 0$.

However, it is not difficult to find a solution of this decomposed problem, with zero data (hence admissible for the original problem), that does not correspond to a solution of the original problem. Such a solution is

$$v_1 = (1+x) - w|_{G_1} \ , \quad \text{and} \quad v_2 = (1-x) - w|_{G_2} \ ,$$

where w is the solution of the original problem with initial data $1-|x|$ and zero boundary conditions. Clearly, this solution is not the restriction to G_1 and G_2 of a solution of the original problem with the same data, as this would be identically zero.

The problem in this example is obviously that we did not impose sufficiently many interface conditions to avoid this spurious solution of the decomposed problem. The v_1 and v_2 do not have the same partial derivative with respect to x at the interface, hence it is impossible that they are the restrictions of the same continuously differentiable function in G. However, an additional interface condition which requires that these derivatives coincide will not allow such spurious solutions to exist. In many ways this example illustrates one of the conclusions of this paper, namely that if one chooses non-characteristic interfaces, and lets the interface conditions express that the (generalized) Cauchy data on both sides of each interface coincide, the decomposed problem is a reformulation of the original problem for classical solutions.

In the next section, we present an abstract formulation of the domain decomposition approach, and define what we mean by a reformulation of the original problem. An abstract result providing a necessary and sufficient condition for the decomposed problem to be a reformulation of the original problem is then given. In Section 3 we show that differential and trace operators satisfy the assumptions made in the abstract case, so that we can use the abstract result in Sections 4 and 5 to give sufficient conditions for having a reformulation in $C^k(\overline{G})$ and $H^1(G)$, respectively. We present several examples in order to illustrate the results.

The last part of the paper concerns a particular solution technique for the decomposed problem, called the subdomain iteration method. After setting up the iterative scheme in the abstract framework used earlier, and giving some examples, we present in Section 7 a convergence result for the subdomain iteration method applied to one-dimensional hyperbolic systems.

Domain decomposition and subdomain iteration methods have been presented in a general setting before by Quarteroni [11], in which both time-dependent and stationary problems are discussed. Numerical experiments using both spectral collocation and finite element methods are reported. Moreover, in [10], Quarteroni derives a convergence result for the fully discrete subdomain iteration applied to a one-dimensional 2×2 hyperbolic system. Convergence results for a semi-discrete and fully discrete subdomain iteration method for hyperbolic systems in any number of dimensions are presented by the author in [2].

2 Abstract formulation of domain decomposition

We start by a definition that provides us with objects, associated with a subset E of $\mathbb{R}^n$, that can be restricted to open subsets of E.

DEFINITION 2.1 *Let X be a set, and let $E \subset \mathbb{R}^n$. If for each $V \subset E$ open (in the topology induced by $\mathbb{R}^n$) there is a mapping $R_V : X \to R_V X$, and R_E is the identity mapping on X, we say that $\{R_V\}$ is a family of restrictions on X relative to E. We say that X is restrictable relative to E if there exists a family of restrictions on X relative to E.*

Clearly, the space of all functions in E and, if E is open, the space of distributions in E, are restrictable relative to E by letting R_V represent the usual restriction of a function (resp. distribution) to a subset.

Remark 2.2 The definition assumes only what we need in order to make sense to the abstract problem. The assumptions are clearly too weak to characterize the usual restrictions of a function defined in E to open subsets of E. For example we see that the choice $R_E : x \mapsto x$ and $R_V : x \mapsto \{\{\}\}$ (a set with only one element) for $V \neq E$ is a family of restrictions on any set X relative to E.

Let $G \subset \mathbb{R}^n$ be open. Let X, Y be restrictable relative to G. To make the notation more intuitive, we denote the restriction operator by a vertical line on the right; that is, $R_V x = x|_V$, etc. Let P be a mapping

$$P : X \to Y \ . \tag{1}$$

Let J be some indexing set. For each $j \in J$, let $\Sigma^j \subset \partial G$ and let Z^j be restrictable relative to Σ^j. Let B^j be a mapping

$$B^j : X \to Z^j \ . \tag{2}$$

We assume that P (resp. B^j) is a local (resp. strictly local) mapping, according to the following definition. This assumption guarantees uniqueness of the restricted mappings P_i, B_i^j defined below. For $E \subset F \subset \mathbb{R}^n$, $int_F(E)$ denotes the interior of E in F, where F is equipped with the topology induced by $\mathbb{R}^n$.

DEFINITION 2.3 *Let $E \subset \mathbb{R}^n$ be open, and let $F \subset \overline{E}$. Let X and Y be restrictable relative to E and F, respectively. We say that a mapping $T : X \to Y$ is local if, for all $V \subset E$ open, we have*

$$u_1|_V = u_2|_V \implies (Tu_1)|_{V\cap F} = (Tu_2)|_{V\cap F} \ .$$

We say that T is strictly local if, for all $V \subset E$ open, we have

$$u_1|_V = u_2|_V \implies (Tu_1)|_{int_F(\overline{V}\cap F)} = (Tu_2)|_{int_F(\overline{V}\cap F)} \ .$$

Remark 2.4 The definition makes sense because $V \cap F$ is always open in F in the topology induced by $\mathbb{R}^n$, since V is open in $\mathbb{R}^n$. Similarly, $int_F(\overline{V} \cap F)$ is open in F by definition.

Remark 2.5 An important observation is that if X and Y are vector spaces, if T is linear, and if the restriction operators are linear, then T is local if and only if $u|_V = 0$ implies $(Tu)|_{V\cap F} = 0$ for all V open; hence it is a condition on the behavior of supp(u) under T. Similarly for T strictly local. This conforms with the usual definition of local operators (see e.g. Dautray, Lions [5, Ch.V]).

Let $\mathscr{U} \subset X$, and consider the following problem:

$$(\mathbf{O}) \begin{cases} \text{Find } u \in \mathscr{U} \text{ satisfying} \\ \\ \quad Pu = f \quad \text{in} \quad G \ \ (\ i.e.,\ in\ \ Y\ )\ , \\ \quad B^j u = g^j \quad \text{on} \quad \Sigma^j \ \ (\ i.e.,\ in\ \ Z^j\ )\ \text{ for } \ j \in J\ . \end{cases}$$

Problem $(\mathbf{O})$ will also be referred to as the *original problem.*

Let $\{G_i\}_{i\in I}$ be a family of disjoint open subsets of G. Let $P_i : X|_{G_i} \to Y|_{G_i}$ be the restriction of P to G_i (i.e., to $X|_{G_i}$), defined by

$$P_i v = (P\tilde{v})|_{G_i}\ , \quad \text{for} \quad \tilde{v} \in X \quad \text{s.t.} \quad \tilde{v}|_{G_i} = v\ . \tag{3}$$

Since P is local, this determines P_i uniquely: Choosing $V = G_i$ in DEFINITION 2.3, we see that $\tilde{v}_1|_{G_i} = \tilde{v}_2|_{G_i}$ implies $(P\tilde{v}_1)|_{G_i} = (P\tilde{v}_2)|_{G_i}$.

For $i \in I$ and $j \in J$, let $\Sigma_i^j = int_{\Sigma^j}(\overline{G_i} \cap \Sigma^j)$, hence Σ_i^j is open in Σ^j. Let $B_i^j : X|_{G_i} \to Z^j|_{\Sigma_i^j}$ be the restriction of B^j to G_i (i.e., to $X|_{G_i}$), defined by

$$B_i^j v = (B^j \tilde{v})|_{\Sigma_i^j}\ , \quad \text{for} \quad \tilde{v} \in X \quad \text{s.t.} \quad \tilde{v}|_{G_i} = v\ . \tag{4}$$

This determines B_i^j uniquely since B^j is strictly local. Namely, by choosing $V = G_i$, we see that $\tilde{v}_1|_{G_i} = \tilde{v}_2|_{G_i}$ implies $(B^j\tilde{v}_1)|_{\Sigma_i^j} = (B^j\tilde{v}_2)|_{\Sigma_i^j}$.

For $i, i' \in I$, $i \neq i'$, let $J_{ii'}$ be an indexing set. For $j \in J_{ii'}$, let $\Sigma_{ii'}^j \subset \partial G_i \cap \partial G_{i'} \cap G$ and let $Z_{ii'}^j$ be restrictable relative to $\Sigma_{ii'}^j$. Let $M_{ii'}^j$ be a strictly local mapping

$$M_{ii'}^j : X \to Z_{ii'}^j\ . \tag{5}$$

We define the restriction of $M_{ii'}^j$ to G_i (resp. $G_{i'}$), also denoted by $M_{ii'}^j$, by

$$M_{ii'}^j v = M_{ii'}^j \tilde{v}\ , \quad \text{for} \quad \tilde{v} \in X \quad \text{s.t.} \quad \tilde{v}|_{G_i} = v \quad (\text{resp. } \tilde{v}|_{G_{i'}} = v)\ . \tag{6}$$

Again, this defines the restriction uniquely by choosing $V = G_i$ (resp. $V = G_{i'}$), since $\overline{G_i} \cap \Sigma_{ii'}^j = \Sigma_{ii'}^j$ and $int_{\Sigma_{ii'}^j}(\Sigma_{ii'}^j) = \Sigma_{ii'}^j$.

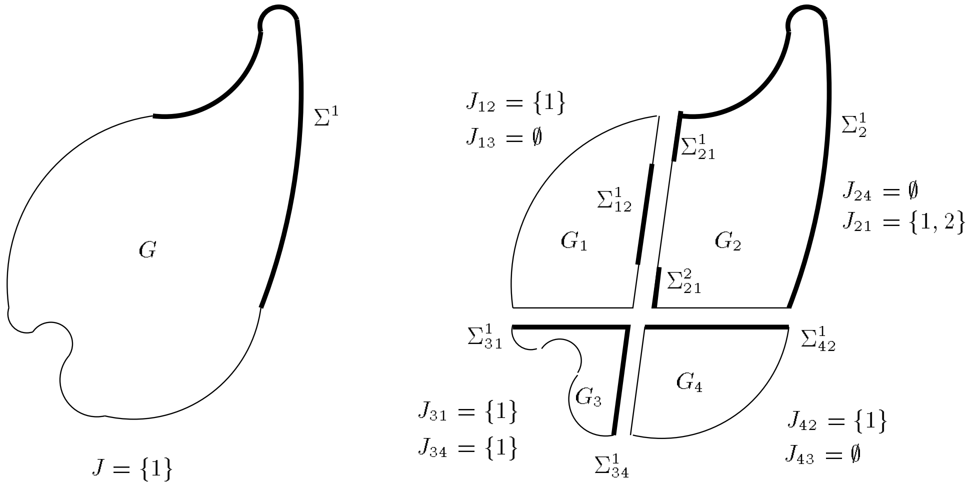

Figure 1: A decomposition of G with associated subsets.

Let $\prod_{i\in I} \mathscr{U}|_{G_i}$ denote the Cartesian product of the $\mathscr{U}|_{G_i}$; i.e., $(v_i) \in \prod_{i\in I} \mathscr{U}|_{G_i}$ means that $v_i \in \mathscr{U}|_{G_i}$ for each $i \in I$. Consider the following problem:

$$
(\mathbf{D}) \left\{ \begin{array}{lll} \multicolumn{3}{l}{\text{Find } (v_i) \in \prod_{i\in I} \mathscr{U}|_{G_i} \text{ satisfying}} \\ \\ P_i v_i = f_i & \text{in} & G_i \ \ (\textit{ i.e., in } \ Y|_{G_i}\ ) \ , \\ B_i^j v_i = g_i^j & \text{on} & \Sigma_i^j \ \ (\textit{ i.e., in } \ Z^j|_{\Sigma_i^j}\ ) \ \text{for } j \in J \ , \\ M_{ii'}^j v_i = M_{i'i}^j v_{i'} & \text{on} & \Sigma_{ii'}^j \ \ (\textit{ i.e., in } \ Z_{ii'}^j\ ) \ \text{ for } \ i' \neq i \ \text{ and } \ j \in J_{ii'} \ . \end{array} \right.
$$

This problem will also be referred to as the *decomposed problem.*

DEFINITION 2.6 *Let $\mathscr{U} \subset X$. The data for* $(\mathbf{D})$ *is said to be $\mathscr{U}$-admissible if $f_i = f|_{G_i}$ and $g_i^j = g^j|_{\Sigma_i^j}$ for $f \in P(\mathscr{U})$ and $g^j \in B^j(\mathscr{U})$.*

For the decomposition of the original problem to be useful, we must ensure that problem $(\mathbf{D})$ is a reformulation of problem $(\mathbf{O})$ according to the following definition:

DEFINITION 2.7 *Let $\mathscr{U} \subset X$. We say that* $(\mathbf{D})$ *is a reformulation of* $(\mathbf{O})$ *in $\mathscr{U}$ if the following conditions are satisfied whenever f_i and g_i^j are $\mathscr{U}$-admissible:*

1. *If u is a solution of* $(\mathbf{O})$*, then $(u|_{G_i})$ is a solution of* $(\mathbf{D})$.

2. *If (v_i) is a solution of* $(\mathbf{D})$*, then there is a unique solution u of* $(\mathbf{O})$ *satisfying $v_i = u|_{G_i}$.*

Remark 2.8 In other words, saying that $(\mathbf{D})$ is a reformulation of problem $(\mathbf{O})$ in $\mathscr{U}$ is the same as saying that the mapping $u \mapsto (u|_{G_i})$ is a bijection from the subset of $\mathscr{U}$ consisting of solutions of $(\mathbf{O})$ to the subset of $\prod_{i\in I} \mathscr{U}|_{G_i}$ consisting of solutions of $(\mathbf{D})$ with $\mathscr{U}$-admissible data.

The following result shows that we will have a reformulation if and only if the pieces v_i can be 'glued' together uniquely. It also serves the purpose of summing up all assumptions we have made so far.

THEOREM 2.9 *Let $G \subset \mathbb{R}^n$ be open, and let $\{G_i\}_{i \in I}$ be a family of disjoint open subsets of G. For each $j \in J$ (resp. $j \in J_{ii'}$), let $\Sigma^j \subset \partial G$ (resp. $\Sigma^j_{ii'} \subset \partial G_i \cap \partial G_{i'} \cap G$). Let X and Y (resp. Z^j,$Z^j_{ii'}$) be restrictable relative to G (resp. Σ^j,$\Sigma^j_{ii'}$). Let P (resp. B^j,$M^j_{ii'}$) be a local mapping (resp. strictly local mappings) satisfying* (1) *(resp.* (2),(5)*). Let $\mathscr{U} \subset X$. Then a necessary and sufficient condition for* (**D**) *to be a reformulation of* (**O**) *in $\mathscr{U}$, is that, for any (v_i) solving* (**D**) *with $\mathscr{U}$-admissible data, we can show the existence of a unique element of $\mathscr{U}$ whose restriction to each G_i is v_i.*

Proof. Let us show that the condition is sufficient by verifying that conditions *1.* and *2.* of DEFINITION 2.7 are satisfied: Let u solve (**O**). Then, by (3), (4), and (6), we have

$$P_i u|_{G_i} = (Pu)|_{G_i} = f|_{G_i} \ ,$$

$$B^j_i u|_{G_i} = (B^j u)|_{\Sigma^j_i} = g^j|_{\Sigma^j_i} \ ,$$

$$M^j_{ii'} u|_{G_i} = M^j_{ii'} u = M^j_{ii'} u|_{G_{i'}} \ .$$

Hence $(u|_{G_i})$ solves (**D**) with the data $f|_{G_i}$ and $g^j|_{\Sigma^j_i}$. On the other hand, let (v_i) solve (**D**) with data $f_i = f|_{G_i}$ and $g^j_i = g^j|_{\Sigma^j_i}$ for some $f \in P(\mathscr{U})$ and $g^j \in B^j(\mathscr{U})$. By assumption, there is a unique $u \in \mathscr{U}$ satisfying $u|_{G_i} = v_i$. Clearly u is a solution of a problem of type (**O**) corresponding to data $\hat{f} := Pu$ and $\hat{g}^j := B^j u$. Moreover, we can let $f = \hat{f}$ and $g^j = \hat{g}^j$, since

$$\hat{f}|_{G_i} = (Pu)|_{G_i} = P_i v_i = f_i \ ,$$

$$\hat{g}^j|_{\Sigma^j_i} = (B^j u)_{\Sigma^j_i} = B^j_i v_i = g^j_i \ .$$

That the condition is also necessary, follows directly from the second part of the proof of the sufficiency. Namely, if (**D**) is a reformulation of (**O**), then we know that there is at least one $u \in \mathscr{U}$ satisfying $u|_{G_i} = v_i$. However, we have just seen that any such u solves (**D**), hence there can not be more than one. ■

Remark 2.10 Note that, for $u \in X$, the interface conditions are always satisfied for $(u|_{G_i})$.

COROLLARY 2.11 *A necessary condition for having a reformulation in $\mathscr{U}$ is that $(u_1|_{G_i}) = (u_2|_{G_i})$ in $\prod_{i \in I} \mathscr{U}|_{G_i}$ implies $u_1 = u_2$ in $\mathscr{U}$.*

Proof. Assume the condition is not satisfied, and let u_1 and u_2 be different elements of $\mathscr{U}$ such that $(u_1|_{G_i}) = (u_2|_{G_i})$. But then the latter solve problem (**D**) with the same data, hence the condition of THEOREM 2.9 is violated. ■

Remark 2.12 In the case where X is a space of functions or distributions, this means that it is impossible to establish a reformulation in $\mathscr{U} \subset X$ if there exist distinct elements of $\mathscr{U}$ which differ only on a subset of $(\cup_i G_i)^c$. Hence we must work in spaces that behave sufficiently nice on this set (which includes both the Σ^j and the interfaces $\Sigma^j_{ii'}$).

For linear operators, we have the following result. It is used in the next section to show that certain operators are local and strictly local. If a topological vector space (TVS) X is restrictable, let K_V denote the kernel of the restriction R_V, that is,

$$K_V = \{x \in X \mid x|_V = 0\} \ .$$

LEMMA 2.13 *Let $E \subset \mathbb{R}^n$ be open, and let $F \subset \overline{E}$. Let X and Y be TVS, restrictable relative to E and F, respectively, with linear restriction operators. Moreover, for each $U \subset F$ open (in the topology induced by $\mathbb{R}^n$), let $Y|_U$ be Hausdorff, and let the restriction operator $R_U : Y \to Y|_U$ be continuous. Let $T : X \to Y$ be continuous and linear, and let $X_0 \subset X$ be a dense subset. If T is local (resp. strictly local) on X_0, and the set $K_V \cap X_0$ is dense in K_V for each $V \subset E$ open, then $T : X \to Y$ is local (resp. strictly local).*

Proof. For $V \subset E$ open, let $U = V \cap F$ (resp. $U = int_F(\overline{V} \cap F)$). Let

$$T_U = R_U \circ T : X \to Y|_U \ .$$

From Remark 2.5 it follows that it is sufficient to show that $T_U(K_V) = \{0\}$ for all $V \in E$ open. We know that $T_U(K_V \cap X_0) = \{0\}$ since T is local (resp. strictly local) on X_0. Since $K_V \cap X_0$ is dense in K_V and T_U is continuous, we have $T_U(K_V) \subset T_U(\overline{K_V \cap X_0}) \subset \overline{T_U(K_V \cap X_0)} = \{0\}$. But $T_U(0) = 0$, so $T_U(K_V) = \{0\}$; that is, T is local (resp. strictly local). ∎

3 Differential and trace operators in $C^k(\overline{G})^r$ and $H^1(G)^r$

In the remainder of this paper we will work with linear differential operators and trace operators in the spaces $C^k(\overline{G})^r = C^k(\overline{G}; \mathbb{C}^r)$ and $H^1(G)^r = H^1(G; \mathbb{C}^r)$. In this section it is verified that everything fit into the abstract framework of the previous section. Recall that $C^k(\overline{G})$ is the subspace of $C^k(G)$ consisting of those u for which $D^\alpha u$ has a continuous extension $\widetilde{D^\alpha u}$ to $\overline{G}$ for $|\alpha| \leq k$ (see e.g., [1]). We use the following notation: The generic element of $\mathbb{R}^n$ is $(x_1, \ldots, x_n)$, $D = (D_1, \ldots, D_n)$ where $D_j = \partial/\partial x_j$, and $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index. We will use matrix-vector notation when appropriate.

Let $E \subset \mathbb{R}^n$ be open. For any set W it is clear that the space W^E, that is, the space of all mappings from E to W, is restrictable relative to E by defining the restriction $u|_V$ pointwise by $(u|_V)(x) = u(x)$ for $x \in V$.

Let $G \subset \mathbb{R}^n$ be open. If P is a differential operator of order m in G with coefficients in $(\mathbb{C}^{r\times r})^G$, then P maps $C^k(G)^r$ into $(\mathbb{C}^r)^G$ when $k \geq m$. Since the differentiation is local, it follows immediately that P is local whenever $k \geq m$.

Now, let G be bounded and have the segment property (see e.g., Wloka [15, Def. 2.1]), and let P have coefficients in $L^\infty(G)^{r\times r}$. Taking derivatives in the sense of distributions, we have that P maps $H^k(G)^r$ continuously into $L^2(G)^r$ when $k \geq m$. Let $H^k(G)^r|_V$ be normed in the natural way. Since $X_0 = C^k(\overline{G})^r$ is a dense subspace of $H^k(G)^r$ (see [15, Th.

3.6]) and P is local on $C^k(\overline{G})^r$ (distributional derivatives and classical derivatives coincide), we can, according to LEMMA 2.13, conclude that $P : H^k(G) \to L^2(G)$ is local if we can establish that for all $V \subset G$ open,

$$K_V \cap X_0 = \{u \in C^k(\overline{G}) \mid u|_V = 0\} \text{ is dense in } K_V = \{u \in H^k(G) \mid u|_V = 0\} \, . \qquad (7)$$

But this follows by standard approximation arguments.

We will now look at trace operators. Again let $G \subset \mathbb{R}^n$ be arbitrary and open, and let $\Sigma \subset \overline{G}$. Let $\mathscr{S} \subset \mathbb{R}^n$ be an open neighborhood of Σ, and let B be a differential operator of order m in $\mathscr{S}$ with coefficients in $C^0(\mathscr{S})^{s\times r}$, $1 \leq s \leq r$. We say that B is a *trace operator* on Σ (with s components), and define its action on $u \in C^k(\overline{G})^r$ by

$$Bu = \sum_{|\alpha|\leq m} \left(b_\alpha \widetilde{D^\alpha u}\right)|_\Sigma \, .$$

Clearly, B maps $C^k(\overline{G})^r$ into $C^0(\Sigma)^s$ when $k \geq m$. If $u \in C^k(\overline{G})^r$ satisfies $u|_V = 0$ for $V \subset G$ open, we have $\widetilde{D^\alpha u} \equiv 0$ in $\overline{V}$ for $|\alpha| \leq k$. Since $int_\Sigma(\overline{V} \cap \Sigma) \subset \overline{V}$ it follows that $Bu|_{int_\Sigma(\overline{V}\cap\Sigma)} = 0$; hence $B : C^k(\overline{G})^r \to C^0(\Sigma)^s$ is strictly local for $k \geq m$.

Let G be a bounded Lipschitz domain, and let Σ either be a subset of ∂G or a subset of the boundary of a Lipschitz subdomain of G. Let B be of order 0, that is, $Bu = b_0 \tilde{u}|_\Sigma$. We know that B can be extended uniquely to a continuous linear operator B from $H^1(G)^r$ to $L^2(\Sigma)^s$ by the density of $C^1(\overline{G})^r$ in $H^1(G)^r$. Since B is strictly local on $X_0 = C^1(\overline{G})^r$, we can again conclude that $B : H^1(G)^r \to L^2(\Sigma)^s$ is strictly local because of (7).

Remark 3.1 It is also possible to treat cases in which the order of B is greater than zero, but then one has to impose stronger smoothness requirements on Σ; see Wloka [15, Th. 8.7] for the special case where B is expressed in terms of higher order normal derivatives.

We can summarize these considerations in the following results, which enable us to apply the abstract theory to concrete cases.

LEMMA 3.2 *Let $G \subset \mathbb{R}^n$ be open, and let $r, k \geq 1$. A differential operator of order $m \leq k$ in G with coefficients in $(\mathbb{C}^{r\times r})^G$ is a local mapping from $C^k(G)^r$ to $(\mathbb{C}^r)^G$. If $\Sigma \subset \overline{G}$, then a trace operator of order $m \leq k$ on Σ with $s \leq r$ components is a strictly local mapping from $C^k(\overline{G})^r$ to $C^0(\Sigma)^s$.*

LEMMA 3.3 *Let $G \subset \mathbb{R}^n$ be bounded and have the segment property, and let $r, k \geq 1$. A differential operator of order $m \leq k$ in G with coefficients in $L^\infty(G)^{r\times r}$ is a local mapping from $H^k(G)^r$ to $L^2(G)^r$. If Σ is a subset of the boundary of a Lipschitz subdomain of G (possibly G itself), then a trace operator of order zero on Σ with $s \leq r$ components is a strictly local mapping from $H^1(G)^r$ to $L^2(\Sigma)^s$.*

We will now give an important example of a trace operator. Let $\Sigma \subset \overline{G}$ be such that we can define a continuous mapping $\nu : \Sigma \to \mathbb{R}^n$ such that $\nu(x)$ is a unit normal vector of

Σ at x. Define the *m-th order normal derivative* $D_\nu^m : C^k(\overline{G})^r \to C^0(\Sigma)^r$, $m \leq k$, by

$$D_\nu^m u = \sum_{|\alpha|=m} \binom{m}{\alpha} \nu^\alpha \widetilde{D^\alpha u} \ .$$

Since the definition of D_ν^m does not involve a differential operator of order m in a neighborhood of Σ with continuous coefficients, it is not obvious that D_ν^m is a trace operator. However, if Σ is a subset of the boundary of a C^1 domain (see e.g., [1]), we can, by way of a partition of unity subordinate to a locally finite covering of Σ, find a continuous extension of ν and thus provide such an operator. So $D_\nu^m : C^k(\overline{G})^r \to C^0(\Sigma)^r$ is in this case a trace operator with r components when $m \leq k$, hence also strictly local by LEMMA 3.2.

4 Reformulation in $C^k(\overline{G})^r$

Let $G \subset \mathbb{R}^n$ be open, and let $\{G_i\}_{i\in I}$ be a decomposition of G, that is, a family of disjoint open subsets of G with the property

$$\overline{G} = \overline{\bigcup_{i\in I} G_i} \ .$$

Let Σ^j, Σ_i^j, and $\Sigma_{ii'}^j$ be as defined in Section 2. We will in this section present sufficient conditions for (**D**) to be a reformulation of (**O**) in $C^k(\overline{G})^r$ in the case of differential and trace operators.

We will make the following assumptions, which by LEMMA 3.2 ensure that P (resp. B^j,$M_{ii'}^j$) is a local mapping (resp. are strictly local mappings) on $C^k(\overline{G})^r$.

ASSUMPTION 4.1 *Let P be a differential operator in G of order $m \leq k$ with coefficients in $C^{k-m}(G)^{r\times r}$, and let B^j (resp. $M_{ii'}^j$) be trace operators on Σ^j (resp. $\Sigma_{ii'}^j$) of order $m^j \leq k$ (resp. $m_{ii'}^j \leq k$) with $\leq r$ components.*

To obtain a reformulation it follows from THEOREM 2.9 that the interface conditions must be chosen so that it is always possible to recompose a solution $(v_i) \in \prod_{i\in I} C^k(\overline{G})^r|_{G_i}$ of (**D**) with $C^k(\overline{G})^r$-admissible data, to a unique $u \in C^k(\overline{G})^r$ satisfying $u|_{G_i} = v_i$. This recomposition problem has been resolved by the author in the case where $\{G_i\}$ is a hyposmooth decomposition of G (see [1]). We recall some definitions.

DEFINITION 4.2 *Let $G \subset \mathbb{R}^n$ be open. A hyposmooth decomposition of G of order k is a family $\{G_i\}_{i\in I}$ consisting of the connected components of*

$$G \setminus \partial A_1 \setminus \partial A_2 \cdots \setminus \partial A_R \ ,$$

where $A_1, \ldots, A_R$ are C^k domains. A hyposmooth decomposition of G is said to be non-characteristic with respect to a differential operator P, if the $\partial A_j \cap G$ are non-characteristic for P.

Remark 4.3 The class of hyposmooth decompositions of G is general enough to include most decompositions one is likely to encounter in practical situations. Moreover, a decomposition which is not hyposmooth can in many cases be made into a hyposmooth decomposition of order k simply by subtracting the boundaries of a finite number of C^k domains (cf. Remark 4.7). Some examples of such decompositions are given in [1].

Given a hyposmooth decomposition $\{G_i\}$ of G, the sets $\partial G_i \cap G$ consist of subsets of the smooth surfaces ∂A_j. The subset of the interface $\partial G_i \cap \partial G_{i'} \cap G$ belonging only to the smooth surface generating it is called $\Gamma_{ii'}$; it is defined by the relation

$$\Gamma_{ii'} = \Gamma_{i'i} = \partial G_i \cap \partial G_{i'} \cap G \setminus \left(\bigcup_{k \neq i,i'} \partial G_k \right) .$$

Trace operators that are restrictions of trace operators defined on the smooth surfaces ∂A_j which generate the hyposmooth decomposition are important (see Figure 2 for an illustration).

DEFINITION 4.4 *Let $\{G_i\}$ be a hyposmooth decomposition of G. If a mapping*

$$\mathscr{M} : C^k(G)^r \to C^0(\bigcup \Gamma_{ii'})^s$$

satisfies, for each $j = 1, \ldots, R$,

$$(\mathscr{M} u)|_{\partial A_j} = B^{(j)} u ,$$

where $B^{(j)}$ is a trace operator of order k on $\partial A_j \cap G$ with s components, we say that $\mathscr{M}$ is a compatible interface operator of order k with s components. We say that $\{G_i\}$ is non-characteristic with respect to $\mathscr{M}$ if, for each $j = 1, \ldots, R$, $\partial A_j \cap G$ is non-characteristic for $B^{(j)}$.

Since $u \mapsto (\mathscr{M} u)|_{\partial A_j}$ is a trace operator, $u \mapsto (\mathscr{M} u)|_{\Gamma_{ii'}}$ is also a trace operator, hence strictly local. Hence the restrictions of this mapping to $C^k(G)^r|_{G_i}$ and $C^k(G)^r|_{G_i'}$, both denoted by $\mathscr{M}|_{ii'}$, are uniquely defined.

If $E \subset \mathbb{R}^n$ is an open neighborhood of $\bigcup(\partial A_j \cap G)$, a differential operator of order k in E with coefficients in $C^0(E)^{s \times r}$ is a compatible interface operator of order k with s components. An example of a compatible interface operator which is not of this type is the normal derivative which was introduced above.

We can now state a result providing us with sufficient conditions for a reformulation in smooth spaces. We say that ∂G is *regular* with respect to a hyposmooth decomposition if the boundaries of any two different subdomains G_i, $G_{i'}$ does not share a subset of ∂G; the exception being points in the adherence of $\Gamma_{ii'}$.

THEOREM 4.5 *Let $k, m, r \geq 1$, and let* ASSUMPTION 4.1 *hold. Let $G \subset \mathbb{R}^n$ be open, and let $\{G_i\}_{i \in I}$ be a locally finite hyposmooth decomposition of G of order k, non-characteristic with respect to P. Let ∂G be regular.*

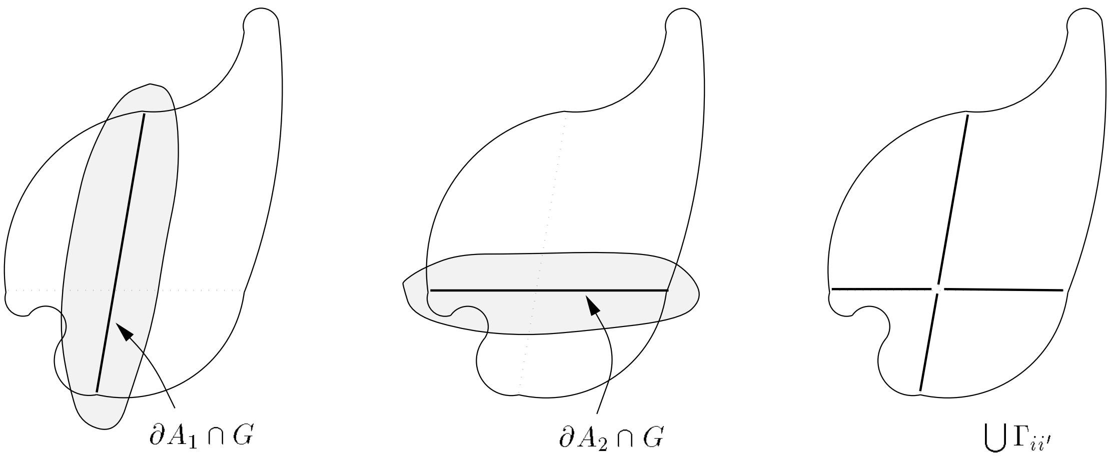

Figure 2: A compatible interface operator $\mathscr{M}$ is defined via trace operators $B^{(j)}$ on the smooth surfaces $\partial A_j \cap G$.

If, for each $\ell = 0, \dots, m-1$, there is a compatible interface operator $\mathscr{M}_\ell$ of order ℓ with r components, $\{G_i\}$ non-characteristic with respect to $\mathscr{M}_\ell$, such that for all $i' \neq i$ the interface conditions on $\Gamma_{ii'}$

$$\begin{cases} M^j_{ii'} v_i = M^j_{ii'} v_{i'} & \text{on } \Sigma^j_{ii'} \text{ for } j \in J_{ii'} \\ M^j_{i'i} v_i = M^j_{i'i} v_{i'} & \text{on } \Sigma^j_{i'i} \text{ for } j \in J_{i'i} \end{cases}$$

can be written equivalently as

$$\mathscr{M}_\ell|_{ii'} v_i = \mathscr{M}_\ell|_{ii'} v_{i'} \quad \text{on } \Gamma_{ii'} \text{ for } 0 \leq \ell \leq m-1\ , \tag{8}$$

then problem (**D**) *is a reformulation of problem* (**O**) *in $C^k(\overline{G})^r$.*

COROLLARY 4.6 *The result still holds if the conditions* (8) *are replaced by*

$$\sum_{\ell=0}^{m-1} a_{k\ell} \mathscr{M}_\ell|_{ii'} v_i = \sum_{\ell=0}^{m-1} a_{k\ell} \mathscr{M}_\ell|_{ii'} v_{i'} \quad \text{on } \Gamma_{ii'} \text{ for } 0 \leq k \leq m-1\ ,$$

where $a_{k\ell} : \bigcup \Gamma_{ii'} \to \mathbb{C}^{r \times r}$ are matrix valued functions such that the $mr \times mr$ matrix $(a_{k\ell})$ is non-singular for $x \in \bigcup \Gamma_{ii'}$.

Proof. It is sufficient to show the corollary, which is the most general statement. According to THEOREM 2.9, all we have to do is to show that our assumptions imply that for any (v_i) solving (**D**) with $C^k(\overline{G})^r$-admissible data, there is a unique element of $u \in C^k(\overline{G})^r$ satisfying $u|_{G_i} = v_i$. But this follows immediately by Corollary 5.2 in [1]. ■

Remark 4.7 The assumptions made on the decomposition $\{G_i\}$ can be weakened to allow for decompositions that can be *made into* a hyposmooth decomposition by subtracting the boundaries of a finite number of C^k domains from G. For more details we refer to the last section in [1].

In the remainder of this section we present some applications of THEOREM 4.5. The choice of problems is partly motivated by the type of problems studied in [11]. As always, ν refers to a unit normal vector of a surface.

Example 4.8 (The Poisson equation) *Consider the Poisson equation in an open subset G of $\mathbb{R}^n$ with Dirichlet conditions on the entire boundary ∂G:*

$$(\mathbf{O1}) \begin{cases} \text{Find } u \in C^k(\overline{G}) \text{ satisfying} \\ \\ \Delta u = f \quad \text{in} \quad G \ , \\ \ u = g \quad \text{on} \quad \partial G \ . \end{cases}$$

Let $\{G_i\}_{i\in I}$ be a locally finite hyposmooth decomposition of G of order k, such that ∂G itself is regular. We can always assume that $I \subset \mathbb{N}$ (see [1, Prop. 2.3]). Then, if a, b are complex-valued functions, the decomposed problem

$$(\mathbf{D1}) \begin{cases} \text{find } (v_i) \in \prod_{i\in I} C^k(\overline{G})|_{G_i} \text{ satisfying} \\ \\ \Delta v_i = f & \text{in} \quad G_i \ , \\ v_i = g & \text{on} \quad int_{\partial G}(\partial G_i \cap \partial G) \ , \\ v_i + aD_\nu v_i = v_{i'} + aD_\nu v_{i'} & \text{on} \quad \Gamma_{ii'} \quad \text{if} \quad i < i' \ , \\ bv_i + D_\nu v_i = bv_{i'} + D_\nu v_{i'} & \text{on} \quad \Gamma_{ii'} \quad \text{if} \quad i > i' \end{cases}$$

is a reformulation of $(\mathbf{O1})$ in $C^k(\overline{G})$ for $k \geq 2$, if

$$ab \neq 1 \quad \text{on} \quad \bigcup \Gamma_{ii'} \ .$$

This follows from COROLLARY 4.6 *by letting $\mathscr{M}_0 = 1$ and $\mathscr{M}_1 = D_\nu$. Note that in the special case $a \equiv b \equiv 0$, we have a Dirichlet condition from one side of each interface, and a Neumann condition from the other.*

Remark 4.9 It is clear that the specific type of boundary conditions chosen is not important; in particular, $(\mathbf{D1})$ will be a reformulation of $(\mathbf{O1})$ in $C^k(\overline{G})$ also when $u = g$ on ∂G is replaced by a Neumann condition or mixed Dirichlet and Neumann conditions, for example. This comment applies also in the general case.

Remark 4.10 Note that in this and the following examples, the interface conditions (8) needed at each interface $\Gamma_{ii'}$ are split into two parts, each of which is assigned to one of the two subproblems in the adjacent subdomains. For notational convenience we have simply chosen to assign the first part to the subdomain with lowest index i, and the second part to the remaining subdomain. Clearly, any other assignment scheme can be used.

The reason for this splitting of the interface conditions between the two adjacent subdomains, is to ensure that each subproblem is of roughly the same type as the original problem when considered as a separate problem. This is essential in the case where we wish to solve the decomposed problem using techniques that involve the solution of similar problems posed in the subdomains. An example of such a technique is the subdomain iteration method.

Example 4.11 (Linear elasticity) *Consider the equations for a homogeneous isotropic elastic surface in an open subset G of $\mathbb{R}^2$, assumed to be fixed on the entire boundary ∂G for simplicity:*

$$(\mathbf{O2}) \begin{cases} \textit{Find } u \in C^k(\overline{G})^2 \textit{ satisfying} \\ \\ \sum_{k=1}^2 D_k \sigma_{jk}(u) = f_j \quad \textit{in} \quad G \ \textit{ for } \ j = 1,2 \ , \\ \qquad u = 0 \qquad\qquad \textit{on} \quad \partial G \ . \end{cases}$$

Here $u = (u_1, u_2)^T$ and $\sigma_{jk}(u) = \lambda(div\ u)\delta_{jk} + \mu\varepsilon_{jk}(u)$ for some material constants λ and μ. The strain tensor ε is defined by

$$\varepsilon_{jk}(u) = \frac{1}{2}\left(D_k u_j + D_j u_k\right) \ ;$$

see Duvaut [6], for instance. Let $\{G_i\}_{i\in I}$ be a locally finite hyposmooth decomposition of G of order k, such that ∂G is regular. Let $\sigma(u)$ denote the 2×2 matrix with entries $\sigma_{jk}(u)$. Then, if $A_1, \ldots, A_4$ are functions taking values in $\mathbb{C}^{2\times 2}$, the decomposed problem

$$(\mathbf{D2}) \begin{cases} \textit{find } (v_i) \in \prod_{i\in I} C^k(\overline{G})^2|_{G_i} \textit{ satisfying} \\ \\ \qquad \sum_{k=1}^2 D_k\sigma_{jk}(v_i) = f_j & \textit{in} \quad G_i \ \textit{ for } \ j = 1,2 \ , \\ \qquad\qquad v_i = 0 & \textit{on} \quad int_{\partial G}(\partial G_i \cap \partial G) \ , \\ A_1 v_i + A_2\sigma(v_i)\,\nu = A_1 v_{i'} + A_2\sigma(v_{i'})\,\nu & \textit{on} \quad \Gamma_{ii'} \ \textit{ if } \ i < i' \ , \\ A_3 v_i + A_4\sigma(v_i)\,\nu = A_3 v_{i'} + A_4\sigma(v_{i'})\,\nu & \textit{on} \quad \Gamma_{ii'} \ \textit{ if } \ i > i' \end{cases}$$

is a reformulation of ($\mathbf{O2}$) *in $C^k(\overline{G})^2$ for $k \geq 2$, if the material constants satisfy the relation*

$$\mu(\lambda + \mu) \neq 0 \ ,$$

and the A_j satisfy

$$\det \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \neq 0 \quad on \quad \bigcup \Gamma_{ii'} \ .$$

This follows from COROLLARY 4.6 *by letting $\mathscr{M}_0 = I$ and $\mathscr{M}_1 = \sigma(u)\,\nu$. Note that in the special case $A_1 \equiv A_4 \equiv I$ and $A_2 \equiv A_3 \equiv 0$ the interface conditions amounts to requiring continuity of u from one side of each interface, and continuity of the density of the surface forces from the other.*

Example 4.12 (The heat equation) *Let $\Omega \subset \mathbb{R}^n$ be open, and let $0 < T \leq \infty$. Denote the generic element of $\mathbb{R}^{n+1}$ by $(x,t) = (x_1, \ldots, x_n, t)$, and consider the heat equation in $G = \Omega \times (0,T)$:*

$$(\mathbf{O3}) \begin{cases} \textit{Find } u \in C^k(\overline{G}) \textit{ satisfying} \\ \\ D_t u - \Delta u = f \quad \textit{in} \quad G \ , \\ \qquad u = u_0 \qquad \textit{on} \quad H = \Omega \times \{0\} \ , \\ \qquad u = g \qquad\ \textit{on} \quad \Sigma = \partial\Omega \times (0,T) \ . \end{cases}$$

Let $\{G_i\}_{i\in I}$ be a locally finite hyposmooth decomposition of G of order k, non-characteristic with respect to $P = D_t - \sum D_j^2$; this is equivalent to requiring that the normal vectors ν of the surfaces $\partial A_j \cap G$ defining the hyposmooth decomposition must never be parallel to the t-axis. Assume further that ∂G is regular. Then, if a, b are complex-valued functions, the decomposed problem

$$(\mathbf{D3}) \begin{cases} \text{find } (v_i) \in \prod_{i\in I} C^k(\overline{G})|_{G_i} \text{ satisfying} \\ \\ \quad D_t v_i - \Delta v_i = f \quad & \text{in} \quad G_i \ , \\ \quad v_i = u_0 & \text{on} \quad int_H(\partial G_i \cap H) \ , \\ \quad v_i = g & \text{on} \quad int_\Sigma(\partial G_i \cap \Sigma) \ , \\ \quad v_i + aD_\nu v_i = v_{i'} + aD_\nu v_{i'} & \text{on} \quad \Gamma_{ii'} \quad \text{if} \quad i < i' \ , \\ \quad bv_i + D_\nu v_i = bv_{i'} + D_\nu v_{i'} & \text{on} \quad \Gamma_{ii'} \quad \text{if} \quad i > i' \end{cases}$$

is a reformulation of $(\mathbf{O3})$ *in $C^k(\overline{G})$ for $k \geq 2$, if*

$$ab \neq 1 \quad on \quad \bigcup \Gamma_{ii'} \ .$$

Again we note that if $a \equiv b \equiv 0$ we have a generalized Neumann condition from one side of each interface and a Dirichlet condition from the other.

Example 4.13 (First order systems) *Let Ω and T be as before, let $r \geq 1$, and consider the first order system in $G = \Omega \times (0, T)$:*

$$(\mathbf{O4}) \begin{cases} \text{Find } u \in C^k(\overline{G})^r \text{ satisfying} \\ \\ \quad D_t u + \sum_{j=1}^n A_j D_j u + Cu = f \quad & \text{in} \quad G \ , \\ \quad u = u_0 & \text{on} \quad H = \Omega \times \{0\} \ , \\ \quad B^j u = g^j & \text{on} \quad \Sigma^j \subset \partial\Omega \times (0, T) \quad \text{for} \quad j \in J \ . \end{cases}$$

Assume $A_j, C \in C^{k-1}(G)^{r\times r}$, and $B^j \in C^0(\mathbb{R}^{n+1})^{r^j\times r}$ with $1 \leq r^j \leq r$. Let $\{G_i\}_{i\in I}$ be a locally finite hyposmooth decomposition of G of order k, non-characteristic with respect to $P = D_t + \sum A_j D_j + C$; this is equivalent to requiring $\det(\nu_t + \sum A_j \nu_j) \neq 0$ for all normal vectors ν of the surfaces $\partial A_j \cap G$ defining the hyposmooth decomposition. Assume further that ∂G is regular. Let M and r be functions taking values in $\mathbb{C}^{r\times r}$ and $\{0, \ldots, r\}$, respectively. Let M_1 denote the matrix consisting of the r_1 first rows of M, and let M_2 denote the matrix consisting of the $r_2 = r - r_1$ last rows of M; that is

$$M(x,t) = \begin{array}{cl} \begin{pmatrix} M_1(x,t) \\ M_2(x,t) \end{pmatrix} & \begin{array}{l} \}\, r_1(x,t) \\ \}\, r_2(x,t) \end{array} \end{array} .$$

Then the decomposed problem

$$(\mathbf{D4})\begin{cases} \textit{find } (v_i) \in \prod_{i\in I} C^k(\overline{G})^r|_{G_i} \textit{ satisfying} & \\ & \\ D_t v_i + \sum_{j=1}^n A_j D_j v_i + C v_i = f & \textit{in} \quad G_i\ , \\ v_i = u_0 & \textit{on} \quad int_H(\partial G_i \cap H)\ , \\ B^j v_i = g^j & \textit{on} \quad int_{\Sigma^j}(\partial G_i \cap \Sigma^j) \quad \textit{for} \ \ j \in J\ , \\ M_1 v_i = M_1 v_{i'} & \textit{on} \quad \Gamma_{ii'} \quad \textit{if} \ \ i < i'\ , \\ M_2 v_i = M_2 v_{i'} & \textit{on} \quad \Gamma_{ii'} \quad \textit{if} \ \ i > i' \end{cases}$$

is a reformulation of $(\mathbf{O4})$ *in* $C^k(\overline{G})^r$ *for* $k \geq 1$*, if*

$$\det M \neq 0 \quad \textit{on} \quad \bigcup \Gamma_{ii'}\ .$$

Remark 4.14 Consider the case where the decomposition consists of cylinders, that is, when each G_i is of the form $\Omega_i \times (0,T)$ for some $\Omega_i \subset \mathbb{R}^n$. In the hyperbolic case, where the characteristic matrix $A(\xi) = \sum_j A_j \xi_j$ has only real eigenvalues and linearly independent eigenvectors for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$, one can choose M to be the matrix consisting of the left eigenvectors of $A(\nu)$, where ν is the normal vector of $\Gamma_{ii'}$ pointing from G_i to $G_{i'}$ when $i < i'$. In this case it is natural to partition M such that M_1 (resp. M_2) consists of the left eigenvectors belonging to negative (resp. positive) eigenvalues. In this way one can consider the interface conditions as specifying 'incoming information' through the interface.

5 Reformulation in $H^1(G)$

Let $G \subset \mathbb{R}^n$ be open, and let $\{G_i\}_{i=1}^N$ be a decomposition of G. We will now present sufficient conditions for $(\mathbf{D})$ to be a reformulation of $(\mathbf{O})$ in $H^1(G)$ in the case of first order differential operators and zero order trace operators. By LEMMA 3.3, the following assumptions ensure that P (resp. B^j,$M^j_{ii'}$) is a local mapping (resp. are strictly local mappings) on $H^1(\overline{G})^r$.

ASSUMPTION 5.1 *Let G and the G_i be bounded Lipschitz domains. Let P be a first order differential operator in G with coefficients in $L^\infty(G)^{r\times r}$, and let B^j (resp. $M^j_{ii'}$) be trace operators on Σ^j (resp. $\Sigma^j_{ii'}$) of order zero with $\leq r$ components.*

We have the following result.

THEOREM 5.2 *Let $G \subset \mathbb{R}^n$ be open, let $\{G_i\}_{i=1}^N$ be a decomposition of G. Let $r \geq 1$, and let* ASSUMPTION 5.1 *hold. If for all $i \neq i'$ the interface conditions on $\Gamma_{ii'}$*

$$\begin{cases} M^j_{ii'} v_i = M^j_{ii'} v_{i'} & \textit{a.e. on} \ \ \Sigma^j_{ii'} \quad \textit{for} \ \ j \in J_{ii'} \\ M^j_{i'i} v_i = M^j_{i'i} v_{i'} & \textit{a.e. on} \ \ \Sigma^j_{i'i} \quad \textit{for} \ \ j \in J_{i'i} \end{cases}$$

can be written equivalently as

$$v_i = v_{i'} \quad \textit{a.e. on} \ \ \partial G_i \cap \partial G_{i'} \cap G\ ,$$

then $(\mathbf{D})$ *is a reformulation of* $(\mathbf{O})$ *in* $H^1(G)^r$*.*

Proof. Since the boundary of a Lipschitz domain has Lebesgue measure 0 (see e.g., Wloka [15, Th. 2.7]), it is clear that the Lebesgue measure of $\bigcup_{i=1}^{N} \partial G_i$ is zero. Hence a function u in $L^2(G)^r$ is uniquely defined by its restrictions $u|_{G_i}$. Assume that $(v_i) \in \prod_{i=1}^{N} H^1(G)^r|_{G_i}$ solves (**D**), and let $u \in L^2(G)^r$ be defined by $u|_{G_i} = v_i$. By LEMMA 5.3 below, applied to each component of u, it follows that $u \in H^1(G)^r$. The result follows by THEOREM 2.9. ∎

The following result was needed in the proof. It represents a slight generalization of Theorem 1.4–3 in Raviart, Thomas [13].

LEMMA 5.3 *Let $G \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $\{G_i\}_{i=1}^{N}$ be a decomposition of G, such that the G_i are bounded Lipschitz domains. Let $u \in L^2(G)$ satisfy $u|_{G_i} \in H^1(G_i)$ and*

$$u|_{G_i} = u|_{G_{i'}} \quad \text{a.e. on} \quad \partial G_i \cap \partial G_{i'} \cap G\ . \tag{9}$$

Then $u \in H^1(G)$.

Proof. We will need Greens' formula for a bounded Lipschitz domain A:

$$\int_A (D_j u) v \, dx = - \int_A u(D_j v)\, dx + \int_{\partial A} u v \nu_j \, da\ ,$$

see Ciarlet [4, Th. 6.1–9]. It holds in particular for $u \in H^1(A)$ and $v \in \mathscr{D}(\mathbb{R}^n)$. ν is the exterior unit normal vector of ∂A, which exists da-a.e. on ∂A.

Let $1 \le j \le n$ and let $w_j \in L^2(G)$ be defined by $w_j|_{G_i} = D_j(u|_{G_i})$. We will show that $D_j u = w_j$ in $\mathscr{D}'(G)$. Let $\varphi \in \mathscr{D}(G)$, then

$$\langle w_j, \varphi \rangle = \int_G w_j \varphi \, dx = \sum_{i=1}^{N} \int_{G_i} (D_j u)\varphi \, dx = - \sum_{i=1}^{N} \int_{G_i} u(D_j \varphi)\, dx + \sum_{i=1}^{N} \int_{\partial G_i} u \varphi \nu_j \, da\ .$$

The integrand in the last term is non-zero only on the interfaces $\partial G_i \cap \partial G_{i'} \cap G$, and therefore the resulting integrals cancel pairwise because of assumption (9). Hence

$$\langle w_j, \varphi \rangle = - \sum_{i=1}^{N} \int_{G_i} u(D_j \varphi)\, dx = -\langle u, D_j \varphi \rangle = \langle D_j u, \varphi \rangle\ ,$$

and $D_j u = w_j$ in $\mathscr{D}'(G)$; that is, $D_j u = w_j$ in $L^2(G)$. In other words $u \in H^1(G)$. ∎

Example 5.4 (First order systems) *Consider problem* (**O4**)*, with $C^k(\overline{G})^r$ replaced by $H^1(G)^r$. Given that the geometry of G and its decomposition $\{G_i\}_{i=1}^{N}$, and the smoothness of the coefficients A_j,C, are as specified in* ASSUMPTION 5.1, *it follows that the decomposed problem* (**D4**)*, with $C^k(\overline{G})^r$ replaced by $H^1(G)^r$ is a reformulation of* (**O4**) *in $H^1(G)^r$, under the same condition on M as before.*

6 Subdomain iteration

In the case where we have established that $(\mathbf{D})$ is a reformulation of $(\mathbf{O})$ in $\mathscr{U}$, we can concentrate our efforts on how to effectively solve problem $(\mathbf{D})$. This will in general be highly problem dependent. One possibility, however, seems to be of particular interest when considering the increased use of parallel computers in numerical calculations. This is what we will refer to as the *subdomain iteration* method, and it appears naturally by decoupling the subproblems through an iteration.

Let us present the method in the abstract framework of Section 2. For each $i \in I$, let $\mathscr{V}_i \supset X|_{G_i}$, such that P_i, B_i^j can be extended to $\mathscr{V}_i$, and let $W_{ii'}^j \supset Z_{ii'}^j$. We say that a mapping $K_{ii'}^j$ that satisfies

$$K_{ii'}^j : \mathscr{V}_i \times \mathscr{V}_i \to W_{ii'}^j$$

and

$$K_{ii'}^j : \mathscr{V}_{i'} \times \mathscr{V}_{i'} \to W_{ii'}^j$$

is a *splitting* of $M_{ii'}^j$, if $K_{ii'}^j(v,v) = M_{ii'}^j v$ for all $v \in X|_{G_i}$ and for all $v \in X|_{G_{i'}}$. Simple examples of splittings of $M_{ii'}^j$ are given by

$$K_{ii}^j(v,w) \equiv M_{ii'}^j v \quad \text{and} \quad K_{ii}^j(v,w) \equiv M_{ii'}^j w \ ,$$

in the case where $M_{ii'}^j$ can be extended to $\mathscr{V}_i$ and $\mathscr{V}_{i'}$. Let $K_{ii'}^j$ and $L_{ii'}^j$ be splittings of $M_{ii'}^j$, and consider the following iteration:

$$(\mathbf{I}) \begin{cases} \text{For } \ell = 1,2,3,\dots \text{ find } (v_i^{(\ell)}) \in \prod_{i\in I} \mathscr{V}_i \text{ satisfying} & \\ \\ \qquad P_i v_i^{(\ell)} = f_i & \text{in } \ G_i \ , \\ \qquad B_i^j v_i^{(\ell)} = g_i^j & \text{on } \ \Sigma_i^j \ \text{ for } j \in J \ , \\ K_{ii'}^j(v_i^{(\ell)}, v_i^{(\ell-1)}) = L_{ii'}^j(v_{i'}^{(\ell)}, v_{i'}^{(\ell-1)}) & \text{on } \ \Sigma_{ii'}^j \ \text{ for } \ i' \neq i \ \text{ and } \ j \in J_{ii'} \ , \end{cases}$$

where the $v_i^{(0)}$ are given. Of course, a decoupling only occurs if $L_{ii'}^j(v_{i'}^{(\ell)}, v_{i'}^{(\ell-1)}) \equiv M_{ii'}^j v_i^{(\ell-1)}$. If (under suitable assumptions) we have $v_i^{(\ell)} \to w_i$ in $\mathscr{V}_i$ and $w_i \in \mathscr{U}|_{G_i}$ for all $i \in I$, then (w_i) is a solution of $(\mathbf{D})$ (corresponding to exactly one solution of $(\mathbf{O})$ in the case of a reformulation).

Remark 6.1 Even if the decomposed problem $(\mathbf{D})$ is well-posed, that is, it has a unique solution for all $\mathscr{U}$-admissible data, it might happen that some of the problems arising during the iteration do not have a (unique) solution. Hence a natural choice for $\mathscr{V}_i$ is the smallest space such that the problem in G_i is well-posed. Moreover, the initial iterates would have to be chosen according to this choice.

Example 6.2 (The Poisson equation) *The following is a subdomain iteration scheme for problem* (**D1**) *with* $a \equiv b \equiv 0$ *which employs a simple weighing between the present and the previous iterates:*

$$
(\mathbf{I1}) \quad \left\{ \begin{array}{lll}
\multicolumn{3}{l}{\text{For } \ell = 1,2,3,\ldots \text{ find } (v_i^{(\ell)}) \in \prod_{i\in I} \mathscr{V}_i\,,\ \mathscr{V}_i \supset C^2(\overline{G})|_{G_i},\ \text{satisfying}} \\[2ex]
\Delta v_i^{(\ell)} = f & \text{in} & G_i\,, \\
v_i^{(\ell)} = g & \text{on} & int_{\partial G}(\partial G_i \cap \partial G), \\
\theta_1 v_i^{(\ell)} + (1-\theta_1) v_i^{(\ell-1)} = \theta_2 v_{i'}^{(\ell)} + (1-\theta_2) v_{i'}^{(\ell-1)} & \text{on} & \Gamma_{ii'} \ \text{if } i < i'\,, \\
\theta_3 D_\nu v_i^{(\ell)} + (1-\theta_3) D_\nu v_i^{(\ell-1)} = \theta_4 D_\nu v_{i'}^{(\ell)} + (1-\theta_4) D_\nu v_{i'}^{(\ell-1)} & \text{on} & \Gamma_{ii'} \ \text{if } i > i'\,.
\end{array} \right.
$$

The choice $\theta_2 = 0$ *and* $\theta_3 = \theta_4 = 1$ *yields the scheme analyzed by Funaro, Quarteroni, and Zanolli [7] for the Helmholtz equation in the two rectangles* $G_1 = (-a,0)\times(-1,1)$ *and* $G_2 = (0,b)\times(-1,1)$*;* $\mathscr{V}_i$ *being appropriate Sobolev spaces. This choice of the relaxation parameters decouples the two problems in a sequential manner, in the sense that in each iteration, the left subproblem can be solved independently of the right problem. It is shown that with no relaxation (*$\theta_1 = 1$*), the iteration converges if and only if* $a > b$*, that is, if the area of the domain with a Dirichlet interface condition is larger than that of the domain with the Neumann interface condition. In the case of internal relaxation (*$\theta_1 \neq 1$*), one can always choose* θ_1 *such that the iteration converges; moreover, if* $a = b$ *there is a* θ_1 *which yields convergence in two iterations. The authors report analogous results when the problems are discretized by the spectral collocation method.*

The numerical analysis of subdomain iteration methods for elliptic problems, and particularly in the context of the finite difference and finite element method, has been an active field of research in the last decade. This includes not only the type of method discussed here, but also cases where the subdomains are allowed to overlap. A survey of this topic can be found in Chan and Mathew [3].

For time-dependent problems there has been less activity. For parabolic problems, time is often discretized immediately to reduce the problem to one of elliptic type (see e.g., Chan and Mathew [3], Israeli, Vozovoi, and Averbuch [8]). However, the continuous subdomain iteration method for the one-dimensional heat equation can probably be analyzed in a similar fashion as in the Funaro, Quarteroni, and Zanolli paper. In particular convergence has been established for the heat equation if one use a similar iterative method to the one described in Example 6.2, with $\theta_1 = 1$, except that the subdomains have a certain amount of overlap at the interfaces [14]. As mentioned in the introduction, Quarteroni [10, 11] has presented both theoretical and numerical results in the case of hyperbolic systems.

Example 6.3 (First order systems) *The following subdomain iteration scheme for problem* (**D4**) *decouples all the subproblems in each iteration:*

$$(\mathbf{I4})\quad \begin{cases} \textit{For } \ell = 1,2,3,\dots \textit{ find } (v_i^{(\ell)}) \in \prod_{i\in I} \mathscr{V}_i\,,\ \mathscr{V}_i \supset C^1(\overline{G})^r|_{G_i},\ \textit{satisfying} \\ \\ D_t v_i^{(\ell)} + \sum_{j=1}^n A_j D_j v_i^{(\ell)} + C v_i^{(\ell)} = f \quad \textit{in} \quad G_i\,, \\ v_i^{(\ell)} = u_0 \quad \textit{on} \quad int_H(\partial G_i \cap H)\,, \\ B^j v_i^{(\ell)} = g^j \quad \textit{on} \quad int_{\Sigma^j}(\partial G_i \cap \Sigma^j) \ \textit{for}\ j \in J\,, \\ M_1 v_i^{(\ell)} = M_1 v_{i'}^{(\ell-1)} \quad \textit{on} \quad \Gamma_{ii'} \ \textit{if}\ i < i'\,, \\ M_2 v_i^{(\ell)} = M_2 v_{i'}^{(\ell-1)} \quad \textit{on} \quad \Gamma_{ii'} \ \textit{if}\ i > i'\,. \end{cases}$$

We will study the convergence properties of this method in the next section for the case $n = 1$, with constant coefficients and characteristic boundary and interface conditions.

7 Application: One-dimensional hyperbolic systems

In this section, we will show a convergence result for one-dimensional hyperbolic systems. The setting is as follows: Denote the generic element of $\mathbb{R}^2$ by (x,t), let $G = (a,b)\times(0,\infty)$, and let $G_i = (a_i,b_i)\times(0,\infty)$ for $i = 1,\dots,M$, with $a_i < b_i$, $b_i = a_{i+1}$, and $a_1 = a$, $b_M = b$. We would like to compute solutions of (**D4**), and hence of (**O4**), by way of the iterative scheme (**I4**). We make the following assumptions: The coefficients are constant, the system is hyperbolic, that is, A has only real eigenvalues and a complete set of eigenvectors, and the interfaces are non-characteristic, that is, $\det A \neq 0$. Moreover, we assume characteristic boundary and interface conditions, that is, the incoming characteristic combinations are specified; cf. Remark 4.14.

Without loss of generality, we will in the following assume $A = \Lambda$, a diagonal matrix, and let Λ be partitioned as

$$\Lambda = \begin{pmatrix} \Lambda_+ & 0 \\ 0 & \Lambda_- \end{pmatrix}\,,$$

where Λ_+ (resp. Λ_-) contains the r_+ positive (resp. $r_- = r - r_+$ negative) elements of Λ. In general, we let B_+ (resp. B_-) denote the matrix consisting of the r_+ first (resp. r_- last) rows of a matrix B. Let u be partitioned likewise as

$$u = \begin{pmatrix} u_+ \\ u_- \end{pmatrix}\,.$$

The original problem is the following:

$$(\mathbf{O4'})\quad \begin{cases} \text{Find } u \in C^k(\overline{G})^r \text{ satisfying} \\ \\ D_t u + \Lambda D_x u + C u = f \quad \text{for} \quad a < x < b,\ t > 0\,, \\ u = u_0 \quad \text{for} \quad a < x < b,\ t = 0\,, \\ u_+ = g^1 \quad \text{for} \quad x = a,\ t > 0\,, \\ u_- = g^2 \quad \text{for} \quad x = b,\ t > 0\,. \end{cases}$$

By Example 4.13, we know that this problem is equivalent to the following decomposed problem:

$$
(\mathbf{D4'}) \begin{cases}
\text{Find } (v_i) \in C^k(\overline{G})^r|_{G_1} \times \cdots \times C^k(\overline{G})^r|_{G_M} \text{ satisfying} \\
\\
D_t v_i + \Lambda D_x v_i + C v_i = f & \text{for} \quad a_i < x < b_i,\ t > 0\ , \\
v_i = u_0 & \text{for} \quad a_i < x < b_i,\ t = 0\ , \\
v_{i+} = v_{i-1+} & \text{for} \quad x = a_i,\ t > 0\ ,\ i \neq 1\ , \\
v_{i-} = v_{i+1-} & \text{for} \quad x = b_i,\ t > 0\ ,\ i \neq M\ , \\
v_{1+} = g^1 & \text{for} \quad x = a,\ t > 0\ , \\
v_{M-} = g^2 & \text{for} \quad x = b,\ t > 0\ .
\end{cases}
$$

To solve (**D4'**) we will use the subdomain iteration of type (**I4**) in the spaces $C^1(\overline{G_i})^r$, given by

$$
(\mathbf{I4'}) \begin{cases}
\text{For } \ell = 1, 2, 3, \ldots \text{ find } (v_i^{(\ell)}) \in C^1(\overline{G_1})^r \times \cdots \times C^1(\overline{G_M})^r \text{ satisfying} \\
\\
D_t v_i^{(\ell)} + \Lambda D_x v_i^{(\ell)} + C v_i^{(\ell)} = f & \text{for} \quad a_i < x < b_i,\ t > 0\ , \\
v_i^{(\ell)} = u_0 & \text{for} \quad a_i < x < b_i,\ t = 0\ , \\
v_{i+}^{(\ell)} = v_{i-1+}^{(\ell-1)} & \text{for} \quad x = a_i,\ t > 0\ ,\ i \neq 1\ , \\
v_{i-}^{(\ell)} = v_{i+1-}^{(\ell-1)} & \text{for} \quad x = b_i,\ t > 0\ ,\ i \neq M\ , \\
v_{1+}^{(\ell)} = g^1 & \text{for} \quad x = a,\ t > 0\ , \\
v_{M-}^{(\ell)} = g^2 & \text{for} \quad x = b,\ t > 0\ .
\end{cases}
$$

This splitting decouples the subproblems, hence we are left to solve M independent hyperbolic initial-boundary value problems in each iteration. In the context of numerical approximations, this would suggest a parallel implementation (cf. [2]).

To obtain smooth solutions, we have to ensure compatibility between the initial and boundary data.

DEFINITION 7.1 *Let $s \geq 0$. We say that the data for problem* (**O4'**) *are C^s compatible if the $2(s+1)$ conditions*

$$(D_t^j g^1)(0) = F_j(a)\ ,\ (D_t^j g^2)(0) = F_j(b)\ ,\ j = 0, \ldots, s \tag{10}$$

are satisfied, where $F_0 = u_0$ and $F_j = -(\Lambda D_x + C)F_{j-1} + (D_t^{j-1} f)(\cdot, 0)$.

If $u \in C^k(\overline{G})^r$ is a solution of (**O4'**), then the corresponding data are necessarily C^k compatible. On the other hand, for compatible data $g^1, g^2 \in C^k([0,\infty))^{r\pm}$, $u_0 \in C^k([a,b])^r$, and $f \in C^k(\overline{G})^r$, there exists a unique solution of problem (**O4'**) in $C^k(\overline{G})^r$ (see e.g., Kreiss and Lorenz [9]).

Remark 7.2 For variable coefficients and in several dimensions, similar well-posedness results are true. In this case, the compatibility conditions also involve time derivatives of the coefficient matrices (see e.g., Rauch and Massey [12]).

We will need to refer to $r \times r$ matrices having a particular sparsity structure:

DEFINITION 7.3 *We say that an $r \times r$ matrix B decouples to the left if the lower left $r_- \times r_+$ submatrix is zero. Similarly, we say that B decouples to the right if the upper right $r_+ \times r_-$ submatrix is zero. If B decouples both to the left and to the right, we say that B is completely decoupled.*

We have the following result.

THEOREM 7.4 *Let $u \in C^k(\overline{G})^r$ be a solution of the constant coefficient hyperbolic system* (**O4'**). *Let $b_i - a_i = 1$, and let the $v_i^{(0)}$ be chosen such that for $\ell = 1$ the data for the M initial-boundary value problems in* (**I4'**) *are C^1 compatible.*

If there is a number γ_0 such that $e^{-\Lambda^{-1}(sI+C)}$ decouples either to the left or to the right for all Re $s \geq \gamma_0$, then (**I4'**) *will produce u in at most $2M-1$ iterations; that is,*

$$v_i^{(\ell)} = v_i = u|_{G_i} \quad \textit{for} \quad \ell \geq 2M-1 \; .$$

If there is a number γ_1 such that $e^{-\Lambda^{-1}(sI+C)}$ is completely decoupled for all Re $s \geq \gamma_1$, then (**I4'**) *will produce u in at most M iterations; that is,*

$$v_i^{(\ell)} = v_i = u|_{G_i} \quad \textit{for} \quad \ell \geq M \; .$$

Remark 7.5 We note that the conditions on the structure of the matrix exponential are satisfied at least in the following cases: If C is completely decoupled, then $e^{-\Lambda^{-1}(sI+C)}$ is completely decoupled for any s. In particular, this is the case if C is diagonal or if C is zero; in which case the result can be proven simply by integration along characteristics. Similarly, if C decouples to the left (resp. right), then the exponential also decouples to the left (resp. right) for any s.

COROLLARY 7.6 *The result holds also for the general constant coefficient hyperbolic system, with characteristic boundary and interface conditions. This follows since $SA = \Lambda S$, by defining a new variable $v = Su$. In this case, the compatibility conditions* (10) *must be modified the obvious way.*

Even if the v_i are elements of $C^k(\overline{G_i})^r$, some of the $v_i^{(\ell)}$ may not be. However, the following result lets us control the smoothness of the $v_i^{(\ell)}$ by choosing the initial iterates properly.

COROLLARY 7.7 *If the $v_i^{(0)}$ are chosen such that for $\ell = 1$ the data for the M initial-boundary value problems in* (**I4'**) *are C^s compatible, then $v_i^{(\ell)} \in C^{\min\{s,k\}}(\overline{G_i})^r$ for $\ell \geq 1$.*

Proof. We will argue by looking at the errors $e_i^{(\ell)} = v_i - v_i^{(\ell)}$ which satisfy the iterative scheme

$$
(\mathbf{E4'})\quad \begin{cases}
\text{For } \ell = 1,2,3,\dots \text{ find } (e_i^{(\ell)}) \in C^1(\overline{G_1})^r \times \cdots \times C^1(\overline{G_M})^r \text{ satisfying} \\
\\
D_t e_i^{(\ell)} + \Lambda D_x e_i^{(\ell)} + C e_i^{(\ell)} = 0 \quad \text{for} \quad a_i < x < b_i,\ t > 0\ , \\
e_i^{(\ell)} = 0 \quad \text{for} \quad a_i < x < b_i,\ t = 0\ , \\
e_{i+}^{(\ell)} = e_{i-1+}^{(\ell-1)} \quad \text{for} \quad x = a_i,\ t > 0\ ,\ i \neq 1\ , \\
e_{i-}^{(\ell)} = e_{i+1-}^{(\ell-1)} \quad \text{for} \quad x = b_i,\ t > 0\ ,\ i \neq M\ , \\
e_{1+}^{(\ell)} = 0 \quad \text{for} \quad x = a,\ t > 0\ , \\
e_{M-}^{(\ell)} = 0 \quad \text{for} \quad x = b,\ t > 0\ .
\end{cases}
$$

Since (v_i) is a solution of (**D4'**), and since the $v_i^{(0)}$ satisfy the C^1 compatibility conditions by assumption, it follows that the $e_i^{(0)}$ satisfy the C^1 compatibility conditions. Hence $e_i^{(1)}$ and thus $v_i^{(1)}$ exist, and they are elements of $C^1(\overline{G_i})^r$ by the comment following DEFINITION 7.1. Continuing in the same manner, we see that $e_i^{(\ell)}$ and $v_i^{(\ell)}$ exist and are elements of $C^1(\overline{G_i})^r$ for all $i = 1,\dots,M$ and $\ell \geq 1$.

Without loss of generality, we can assume that $e^{-\Lambda^{-1}(sI+C)}$ decouples to the left for all Re $s \geq \gamma_0$, and that the $e_i^{(0)}$ are exponentially bounded with respect to t. Applying the Laplace transform in the t variable, we see that the transformed functions $\hat{e}_i^{(\ell)} = \hat{e}_i^{(\ell)}(x,s)$ satisfy the following scheme for all Re s greater that some number γ:

$$
\begin{aligned}
D_x \hat{e}_1^{(\ell)} + \Lambda^{-1}(sI+C)\hat{e}_1^{(\ell)} &= 0 && \text{for } a_1 < x < b_1 \\
\hat{e}_{1+}^{(\ell)} &= 0 && \text{for } x = a_1 \\
\hat{e}_{1-}^{(\ell)} &= \hat{e}_{2-}^{(\ell-1)} && \text{for } x = b_1\ ,
\end{aligned}
$$

$$
\left.\begin{aligned}
D_x \hat{e}_i^{(\ell)} + \Lambda^{-1}(sI+C)\hat{e}_i^{(\ell)} &= 0 && \text{for } a_i < x < b_i \\
\hat{e}_{i+}^{(\ell)} &= \hat{e}_{i-1+}^{(\ell-1)} && \text{for } x = a_i \\
\hat{e}_{i-}^{(\ell)} &= \hat{e}_{i+1-}^{(\ell-1)} && \text{for } x = b_i
\end{aligned}\right\}\ i = 2,\dots,M-1\ ,
$$

$$
\begin{aligned}
D_x \hat{e}_M^{(\ell)} + \Lambda^{-1}(sI+C)\hat{e}_M^{(\ell)} &= 0 && \text{for } a_M < x < b_M \\
\hat{e}_{M+}^{(\ell)} &= \hat{e}_{M-1+}^{(\ell-1)} && \text{for } x = a_M \\
\hat{e}_{M-}^{(\ell)} &= 0 && \text{for } x = b_M\ .
\end{aligned}
$$

Hence, in each iteration, $\hat{e}_i^{(\ell)}$ satisfy a two-point boundary value problem.

We know that the solution of a problem

$$
\begin{aligned}
D_x v + \Lambda^{-1}(sI+C)v &= 0 && \text{for } 0 < x < 1\ , \\
v_+ &= g_+ && \text{for } x = 0\ , \\
v_- &= g_- && \text{for } x = 1\ ,
\end{aligned}
$$

is given by $v(x) = e^{-\Lambda^{-1}(sI+C)x} z$, where z satisfies

$$
\begin{pmatrix} I & 0 \\ \multicolumn{2}{c}{\left[e^{-\Lambda^{-1}(sI+C)}\right]_-} \end{pmatrix} \begin{pmatrix} z_+ \\ z_- \end{pmatrix} = \begin{pmatrix} g_+ \\ g_- \end{pmatrix} .
$$

Let Re $s \geq \max\{\gamma_0, \gamma\}$. Since the matrix exponential decouples to the left, we can write it as

$$e^{-\Lambda^{-1}(sI+C)} = \begin{pmatrix} G_{11}(s) & G_{12}(s) \\ 0 & G_{22}(s) \end{pmatrix}$$

and solve for z to get v in the form

$$v(x) = e^{-\Lambda^{-1}(sI+C)x} \begin{pmatrix} I & 0 \\ 0 & G_{22}^{-1} \end{pmatrix} \begin{pmatrix} g_+ \\ g_- \end{pmatrix} .$$

Hence the values of v_- and v_+ at $x = 0$ and $x = 1$, respectively, are given by

$$v_-(0) = G_{22}^{-1} g_- \quad \text{and} \quad v_+(1) = G_{11} g_+ + G_{12} G_{22}^{-1} g_- \, . \tag{11}$$

Let $V^\ell = (V_+^\ell, V_-^\ell)^T = \left(\hat{e}_{1+}^{(\ell)}(b_1), \ldots, \hat{e}_{M-1+}^{(\ell)}(b_{M-1}), \hat{e}_{2-}^{(\ell)}(a_2), \ldots, \hat{e}_{M-}^{(\ell)}(a_M)\right)^T$. By (11), applied to the two-point boundary value problems above, it follows that we can define a mapping Q taking $V^{\ell-1}$ into V^ℓ (hence of Steklov-Poincare type), that is

$$V^\ell = Q V^{\ell-1} = Q^\ell V^0 \, ,$$

and that the matrix representation of Q is of the form

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ 0 & Q_{22} \end{pmatrix} .$$

Let X, Y, Z denote the $M-1 \times M-1$ connectivity matrices corresponding to the blocks that constitute the (super-)blocks Q_{11}, Q_{12}, Q_{21}, respectively. We see from (11) that

$$X = Z^T = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & & & & \vdots \\ 0 & 1 & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ldots & 1 & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & 1 & 0 \end{pmatrix} , \quad Y = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 1 & 0 & & & \vdots \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 & \vdots \\ \vdots & & & 0 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{pmatrix} .$$

We then have that the connectivity matrix corresponding to the blocks of Q^ℓ is given by

$$\begin{pmatrix} X^\ell & Y_\ell \\ 0 & X^{T\ell} \end{pmatrix} ,$$

where $Y_1 = Y$, and $Y_j = XY_{j-1} + Y_{j-1}X^T$ for $j \geq 2$. Clearly, X^ℓ and $X^{T\ell}$ are zero when $\ell \geq M-1$. Moreover, since X (resp. X^T) acting on a matrix from the left (resp. right) amounts to shifting the rows (resp. columns) one step down (resp. to the right), padding with zeros in the new row (resp. column), it is clear that Y_ℓ is zero when $\ell \geq 2(M-1)$. In other words, $Q^\ell = 0$ when $\ell \geq 2(M-1)$.

Now, since $V^\ell = Q^\ell V^0$ is exactly the data for the $\hat{e}_i^{(\ell+1)}$, it follows immediately that $\hat{e}_i^{(\ell+1)} \equiv 0$ for $i = 1, \ldots, M$ when $\ell \geq 2(M-1)$. Since this holds for any s in a half-space to the right of Re $s \geq \gamma$, an application of the inverse Laplace transform establishes the first part of the result.

If $e^{-\Lambda^{-1}(sI+C)}$ decouples to the right instead, everything is of course completely analogous. On the other hand, if it is completely decoupled, then $G_{12}(s) = 0$, and from (11) it is easy to see that the connectivity matrix Y is zero. Since X^ℓ and $X^{T\ell}$ are zero when $\ell \geq M-1$, it follows that $Q^\ell = 0$ when $\ell \geq M-1$. Again this is the same as saying that $\hat{e}_i^{(\ell+1)} \equiv 0$ for $i = 1, \ldots, M$ when $\ell \geq M-1$, and the last part of the result is established like before.

COROLLARY 7.7 follows by the same argument we used in order to establish that all iterates are elements of $C^1(\overline{G_i})^r$. ■

Remark 7.8 Since the proof utilizes the nilpotency of Q, it is clear that if the initial iterates are chosen in such a way that V^0 is in the range of Q^p for some $p \geq 1$, the iteration will converge in p less iterations than predicted by the theorem.

Remark 7.9 If $d_i := b_i - a_i$ is not equal to one for all $i = 1, \ldots, M$, the condition on $e^{-\Lambda^{-1}(sI+C)}$ in the theorem should be replaced by the same conditions on the matrices $e^{-\Lambda^{-1}(sI+C)d_i}$.

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