The whole set of symmetry preserving discrete versions of a heat transfer equation with a source. [1]

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Abstract

In this letter we present the set of invariant difference equations and meshes which preserve the Lie group symmetries of the equation

$$u_t = (K(u)u_x)_x + Q(u).$$

All special cases of $K(u)$ and $Q(u)$ that extend the symmetry group are considered. Numerical realization of some solutions of a linear heat equation is developed. The relations between invariant discrete models for heat equation and for Burgers equation are shown.

Contents

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Introduction

Symmetries are intrinsic and fundamental features of the differential equations of mathematical physics. Consequently, they should be retained when discrete analogs of such equations are constructed.

The group properties of a heat transfer equation with a source

$$u_t = (K(u)u_x)_x + Q(u), \tag{1}$$

were considered in [4]. This classification contains the result of L.V.Ovsyannikov [15] for (1) with $Q \equiv 0$ as well as symmetries of the linear case, which were known by S.Lie. All choices of $K(u)$, $Q(u)$ which extend the symmetry group admitted by equation (1) were identified.

For all cases of $K(u)$ and $Q(u)$ in (1) we construct difference equations and meshes (lattices) which admit the same Lie group of point transformations as their continuous limits.

We recall that Lie point symmetries yield a number of useful properties of differential equations [13,16,10]:

a) A group action transforms the complete set of solutions into itself; so it is possible to obtain new solutions from a given one;

b) There exists a standard procedure to obtain the whole set of invariants and differential invariants for a symmetry group; it yields the invariant representation of the differential equations and the forms of invariant solutions in which they could be found (symmetry reduction of PDE);

c) For ODEs the known symmetry yields the reduction of the order; if the dimension of symmetry is equal to (or greater than) the order of ODE, then we have a complete integrability;

d) The invariance of PDEs is a necessary condition for the application of Noether's theorem on variational problems to obtain conservation laws.

e) It should be noticed that Lie point transformations have a clear geometrical interpretation and one can construct the orbits of a group in a finite dimensional space of independent and dependent variables;

The structure of the admitted group essentially effects the construction of equations and grids. Group transformations can break the geometric structure of the mesh that influences the approximation and other properties of a difference equation. Early contributions to the construction of the difference grids based on the symmetries of the initial difference model were done in [6,8]. Classes of transformations that conserve uniformity, orthogonality, and other properties of meshes will be defined below.

In accordance with equation (1) we consider Lie point transformations in a space with two independent variables: t and x. Let

$$X = \xi^t \frac{\partial}{\partial t} + \xi^x \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \dots, \tag{2}$$

be an operator of a one-parameter transformation group. Dots denote prolongation of the operator on other variables used in the given differential equation:

$$F(x, t, u_t, u, u_x, u_{xx}) = 0. \tag{3}$$

The group generated by (2) transforms a point $(x, t, u, u_t, u_x, u_{xx})$ to a new one $(x^*, t^*, u^*, u_t^*, u_x^*, u_{xx}^*)$ together with equation (3). This situation changes when applying Lie points transformations to difference equations. Let

$$F(z) = 0 \tag{4}$$

be a difference equation defined on some finite set of points $z^1, z^2, \dots$ (difference stencil) on a mesh. In contrast to the point $(x, t, u, u_t, u_x, u_{xx})$, the "difference point" - the difference stencil has its own geometrical structure.

Let

$$\Omega(z, h) = 0 \tag{5}$$

be an equation that define a difference stencil and a mesh. As an example it will be a uniform mesh if the left step (spacing) equals the right step:

$$h^{+} = h^{-}. \tag{6}$$

The invariance of the difference equation (4) depends on the invariance of (5), since the latter must be included in the general condition of invariance:

$$\left\{ \begin{array}{rcl} XF(z)|_{(4)(5)} & = & 0, \\ X\Omega(z,h)|_{(4)(5)} & = & 0. \end{array} \right. \tag{7}$$

Relations between the two conditions in (7) depends on whether Ω, ξ^t, ξ^x depend on solution or not. If $\Omega_u = \xi^t_u = \xi^x_u = 0$, then the conditions (7) could be considered independently.

Thus, what makes our approach [5-8] special is the inclusion of the second equation of (7) in the conditions of invariance, admitting the whole set of properties a)-e) stated for equations (4) and (5).

There exist a few ways to avoid transformations of the difference stencil and, consequently, transformations of a mesh. One way is to restrict transformations to the case when independent variables are not changed: $\xi^t = \xi^x = 0$, yielding any mesh invariant (see [12]). But this restriction is very strong and would exclude most symmetries of physical problems.

Another approach is connected with application of evolutionary vector fields. It is known [10,16] that symmetry operator (2) could be represented as

$$\bar{X} = \left(\eta - \xi^t u_t - \xi^x u_x\right) \frac{\partial}{\partial u} + \ldots \tag{8}$$

Operator (8) could be viewed as representative of a factor algebra by the ideal

$$\xi^t(z)D_t + \xi^x(z)D_x, \tag{9}$$

where $D_t = \dfrac{\partial}{\partial t} + u_t \dfrac{\partial}{\partial u} + \ldots, \quad D_x = \dfrac{\partial}{\partial x} + u_x \dfrac{\partial}{\partial u} \ldots$, both admitted by all equations (3). For differential equations this approach gives an equivalent result but with some losses in points a) - e). In particular we lose the geometric sense even for point transformations due to the realization of group transformations in infinite dimensional spaces $(t, x, u_t, u_x,\ u_{tx}, u_{xx}, u_{tt}, \ldots)$, there is no procedure for calculating invariants of operators of the type given by (8), etc.

As it was shown in [5], the exact representation of the operator of total differentiation D_x in a space of difference variables is given by the following operators (the same is also true for D_t):

$$\begin{aligned} D^{+} &= \frac{\partial}{\partial x} + \underset{+h}{\tilde{D}}(u)\frac{\partial}{\partial u} + \ldots, \\ D^{-} &= \frac{\partial}{\partial x} + \underset{-h}{\tilde{D}}(u)\frac{\partial}{\partial u} + \ldots, \end{aligned} \tag{10}$$

where $\underset{+h}{\tilde{D}} = \sum\limits_{n=1}^{\infty} \dfrac{(-h)^{n-1}}{n} \underset{+h}{D}^{n}, \quad \underset{-h}{\tilde{D}} = \sum\limits_{n=1}^{\infty} \dfrac{h^{n-1}}{n} \underset{-h}{D}^{n}$, $\underset{+h}{D}, \underset{-h}{D}$ - are right and left difference operators on a uniform mesh. As difference operators are related to corresponding shift operators $\underset{+h}{S}, \underset{-h}{S}$:

$$\underset{\pm h}{D} = \pm \frac{(\underset{\pm h}{S} - 1)}{h},$$

we obtain another representation for $\underset{\pm h}{\tilde{D}}$:

$$\underset{+h}{\tilde{D}} = \sum_{n=1}^{\infty} \frac{(-h)^{n-1}}{n} \underset{+h}{D}^{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{nh} (\underset{+h}{S} - 1)^{n}, \tag{11}$$

$$\tilde{\underset{-h}{D}} = \sum_{n=1}^{\infty} \frac{(1 - \underset{-h}{S})^n}{nh}. \tag{12}$$

The group transformation for operators (10) can be obtained by the exponential mapping or by means of the so called Newton series (see [5]). It is important to notice that every difference equation on a regular mesh admits the operators D^+ and D^-, which do not change a mesh. It was shown [5], that a family of operators $\xi(z)D^\pm$ forms an ideal in the Lie algebra of operators (2), since it is possible to rewrite these operators as evolutionary vector fields

$$\bar{X} = \left(\eta - \xi^t D_t^+(u) - \xi^x D_x^+(u)\right) \frac{\partial}{\partial u} + \ldots \tag{13}$$

(We follow here the right semiaxis representation D^+. The same formulation can be done with help of left semiaxis representation D^-.)

It is important to notice that the representation (10) is true only for regular (uniform) meshes. Thus, the evolutionary vector fields (13) are applicable only for the groups which do not change uniformity of a mesh. So, one can not apply them for modern numerical methods with moving meshes, self adaptive meshes, multi-grid methods etc.

Let us consider a one-parameter transformation group which is generated by the operator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \ldots \tag{14}$$

We prolong (14) on the right and left steps h^+, h^- by means of relations $h^+ = x^+ - x$ and $h^- = x - x^-$, where $f^\pm \equiv \underset{\pm h}{S}(f)$:

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \ldots + [\underset{+h}{S}(\xi) - \xi] \frac{\partial}{\partial h^+} + [\xi - \underset{-h}{S}(\xi)] \frac{\partial}{\partial h^-} \tag{15}$$

From (15) it is easy to obtain the invariance condition for a uniform mesh in a given direction. Let (6) be invariant with respect to (15), then

$$\underset{+h}{S}(\xi) - 2\xi + \underset{-h}{S}(\xi) = 0 \quad \text{or} \quad \underset{+h}{D}\,\underset{-h}{D}(\xi) = 0. \tag{16}$$

Condition (16) is a strong limitation on the admitted group. In addition the coefficients of (13) are the power series of $\underset{\pm h}{D}$ or $\underset{\pm h}{S}$, since one should consider the whole set of mesh points and not a stencil only.

Let us illustrate the above with a simple example. The ODE

$$u_{xx} = u^2, \tag{17}$$

can be viewed as the stationary case of equation (1) with $K(u) = 0$ and $Q(u) = u^2$. Equation (17) has the following Lie point symmetries:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}; \tag{18}$$

As a difference analog of (17) we consider

$$\frac{u^+ - 2u + u^-}{h^2} = u^2. \tag{19}$$

on a uniform mesh

$$h^+ = h^-, \tag{20}$$

where $u^+ = \underset{+h}{S}(u), \quad u^- = \underset{-h}{S}(u)$.

Equations (19)-(20) use a three-point stencil or subspace $(x, x^+, x^-, u, u^+, u^-)$ and operators (18) have the following prolongation for the shifted points of the difference stencil:

$$
\begin{aligned}
X_1 &= \frac{\partial}{\partial x} + \frac{\partial}{\partial x^+} + \frac{\partial}{\partial x^-}, \\
X_2 &= x\frac{\partial}{\partial x} + x^+\frac{\partial}{\partial x^+} + x^-\frac{\partial}{\partial x^-} - 2u\frac{\partial}{\partial u} - 2u^+\frac{\partial}{\partial u^+} - 2u^-\frac{\partial}{\partial u^-} + h^+\frac{\partial}{\partial h^+} + h^-\frac{\partial}{\partial h^-}.
\end{aligned}
\tag{21}
$$

Symmetry algebra (18) acts on the space (t, x), so the coefficients of (21) have the same form in different points of the stencil. The prolongation forms for h^+ and h^- are easily derived from the relations $h^+ = x^+ - x, \quad h^- = x^- - x$.

It is easy to verify that the equations (19)-(20) are invariant under the action of (21):

$$
\begin{aligned}
X_2 \left(\frac{u^+ - 2u + u^-}{h^2} - u^2 \right)_{|_{(19)}} &= 0, \\
X_2 \left(h^+ - h^- \right)_{|_{(20)}} &= 0,
\end{aligned}
\tag{22}
$$

(Operator X_1 leaves (19)-(20) unchanged).

It follows that the system (19)-(20) has the same Lie point symmetry as its continuous limit. Notice that the invariance conditions (22) are mutually independent.

As for the continuous case one can easily calculate the finite difference invariants for (21) by solving the standard system:

$$
X_i I^\tau (x, h^+, h^-, u, u^+, u^-) = 0, \quad i = 1, 2. \quad \tau = 1, 2, 3, 4. \tag{23}
$$

The solution of (23) yields the whole set of difference invariants

$$
I^1 = \frac{h^+}{h^-}, \quad I^2 = \frac{u^+}{u}, \quad I^3 = \frac{u^-}{u}, \quad I^4 = (h^+)^2 u; \tag{24}
$$

It follows that the difference model (19)-(20) can be represented by means of the invariants (24) as $I^2 + I^3 - 2 = I^4$ and $I^1 = 1$.

Let us now consider the evolutionary vector fields for the difference equation (19) (we consider the right semiaxis representation D^+). We prolong the operator (10) on all points of a given stencil $(x, x^+, x^-, u, u^+, u^-)$:

$$
D^+ = \frac{\partial}{\partial x} + \frac{\partial}{\partial x^+} + \frac{\partial}{\partial x^-} + u_x\frac{\partial}{\partial u} + u_x^+\frac{\partial}{\partial u^+} + u_x^-\frac{\partial}{\partial u^-}, \tag{25}
$$

where $u_x \equiv \sum_{n=1}^{\infty} \frac{(-h)^{n-1}}{n} \underset{+h}{D}^n(u), \qquad u_x^+ \equiv \sum_{n=1}^{\infty} \frac{(-h)^{n-1}}{n} \underset{+h}{D}^n(u^+)$ and $u_x^- \equiv \sum_{n=1}^{\infty} \frac{(-h)^{n-1}}{n} \underset{+h}{D}^n(u^-)$. The evolution vector fields (13) will have the following forms:

$$
\begin{aligned}
\underset{h}{\bar{X}}_1 &= -X_1 + D^+ = u_x\frac{\partial}{\partial u} + u_x^+\frac{\partial}{\partial u^+} + u_x^-\frac{\partial}{\partial u^-}, \\
\underset{h}{\bar{X}}_2 &= -X_2 + xD^+ = -h^+\frac{\partial}{\partial h^+} - h^-\frac{\partial}{\partial h^-} + \\
&(2u + xu_x)\frac{\partial}{\partial u} + (2u^+ + xu_x^+)\frac{\partial}{\partial u^+} + (2u^- + xu_x^-)\frac{\partial}{\partial u^-}.
\end{aligned}
\tag{26}
$$

It is not easy to check the invariance conditions of equation (19) for the operators (26) because one should use not only the equation (19), but all its sequences obtained by shifting to the right. A harder question is how to produce the finite difference invariants (24) by means of (26). That is why we prefer the first classical way for Lie point symmetries and include a mesh in the invariance condition (7) (a very similar approach for the semi-discretized nonlinear heat equation was introduced recently in [3]). An additional convincing reason to apply the classical representation of Lie point symmetries springs from comparison of unsuitable evolutionary vector fields approaches to invariant variational problems, developed in [5], and a clear classical way to construct Noether type theorems for difference equations [7].

Another approach to symmetry of difference equations on a fixed uniform mesh was introduced in [9]. However, that way is only applicable to linear equations; moreover, it requires knowing a complete set of solutions of difference equations. The newly introduced approach [11] deals with evolutionary vector fields on a uniform mesh. The advantage of the last two approaches seems to be in the potential of finding non-point symmetries of difference equations which are not available in their continuous limits.

Returning to equation (1), we notice that a transformation defined by (2) conserves uniformity of a grid in t and x directions, if

$$\underset{+\tau}{D}\underset{-\tau}{D}(\xi^t) = 0, \tag{27}$$

$$\underset{+h}{D}\underset{-h}{D}(\xi^x) = 0, \tag{28}$$

where $\underset{+\tau}{D}$ and $\underset{-\tau}{D}$ are right and left difference operators in t direction.

Conditions (27) and (28) are not sufficient to describe the invariance of an orthogonal mesh. For an orthogonal mesh to conserve its orthogonality under the transformation, it is necessary and sufficient that [6,8]:

$$\underset{+h}{D}(\xi^t) = -\underset{+\tau}{D}(\xi^x). \tag{29}$$

When condition (29) is not satisfied for a given group, the flatness of the layer of a grid in some direction is rather important. For evolution equations it is significant to have flat time layers, since otherwise, after transformations, some domains of a space could be in the future, while others in the past. We have a simple criterion [6,8] for the preserving the flatness of the layer of a grid in the time direction under the action of a given operator (2):

$$\underset{+h}{D}\underset{+\tau}{D}(\xi^t) = 0. \tag{30}$$

So, the conditions (27)–(30) provide a geometry of grids that is based on the Lie group symmetry. These conditions will be used in what follows.

Let us note that group classification for the equation (1) was done in paper [1] up to equivalent transformations:

$$\bar{t} = at + e, \qquad \bar{x} = bx + f, \qquad \bar{u} = cu + g, \qquad \bar{k} = \frac{b^2}{a}k, \qquad \bar{q} = \frac{c}{a}q, \tag{31}$$

where a, b, c, e, f, and g are arbitrary constants, $abc \neq 0$. These transformations don't change differential structure of the equation (1), transforming corresponding admitted group into another one.

§1. An arbitrary heat transfer coefficient $K(u)$.

1.1. If $K(u)$ and $Q(u)$ are arbitrary, the equation (1) admits only a two parameter group of translations. This group is defined by the following infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}; \tag{32}$$

which generate the translation of independent variables.

In this case almost no limits are imposed on a grid and a difference equation. In particular, we can use an orthogonal grid in the plane (x,t) which is regular in both directions because the conditions (27)-(29) are valid for the operators (32).

The group with operators (32) in the subspace $(x, t, h, \tau, u, u_-, u_+, \hat{u}, \hat{u}_-, \hat{u}_+)$ corresponding to the stencil shown on Fig. 1,

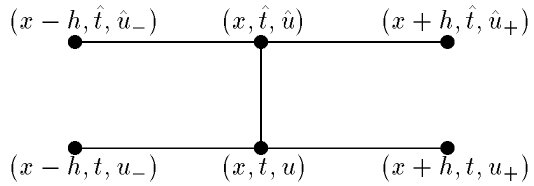

Figure 1: The stencil of the orthogonal mesh.

has 8 invariants:

$$\tau, \quad h, \quad u, \quad u_+, \quad u_-, \quad \hat{u}, \quad \hat{u}_-, \quad \hat{u}_+.$$

That is why any difference approximation of the equation (1) by the above invariants could give difference equation which admits operators (32). For example, the explicit model

$$\frac{\hat{u}-u}{\tau} = \frac{1}{h}\left(K\left(u_+\right) \underset{h}{u_x} - K\left(u\right) \underset{h}{u_{\bar{x}}}\right) + Q(u), \tag{33}$$

where $K(u)$ and $Q(u)$ represent any approximation of the corresponding coefficients by invariants and $\underset{h}{u_x} = \dfrac{u_+ - u}{h}$, $\underset{h}{u_{\bar{x}}} = \dfrac{u - u_-}{h}$ are right and left difference derivatives by x correspondingly, admits operators (32).

1.2. If $K(u)$ is arbitrary function and $Q(u) \equiv 0$, the equation

$$u_t = (K(u)u_x)_x \tag{34}$$

admits a three-parameter algebra of operators (see [15]):

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{35}$$

This case is almost analogous to the previous one. The operators (35) don't violate conditions of invariant orthogonality (29) and invariant uniformity of a grid (27),(28). Thus, in this case we could use the orthogonal grid shown on Fig. 1. Any approximation of the equation (34) by the seven invariants

$$\frac{h^2}{\tau}, \quad u, \quad u_+, \quad u_-, \quad \hat{u}, \quad \hat{u}_-, \quad \hat{u}_+ \tag{36}$$

gives an invariant model for the equation (34). In particular the explicit model (33) with $Q \equiv 0$:

$$\frac{\hat{u}-u}{\tau} = \frac{1}{h}\left(K\left(u_+\right) \underset{h}{u_x} - K\left(u\right) \underset{h}{u_{\bar{x}}}\right) \tag{37}$$

can be used.

§2. The exponential heat transfer coefficient $K = e^u$.

In this paragraph we consider three cases of group classification for $K = e^u$, in accordance with [4] and [15].

2.1. If $Q = 0$ then the equation

$$u_t = (e^u u_x)_x \tag{38}$$

admits a four-dimensional algebra of infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_4 = t\frac{\partial}{\partial t} - \frac{\partial}{\partial u}. \tag{39}$$

As in the cases considered above, conditions of invariant uniformity and invariant orthogonality are valid. A difference model for the equation (38) can be constructed by approximation of the initial differential equation with the help of difference invariants:

$$e^u\frac{\tau}{h^2}, \quad (\hat{u} - u), \quad (u_+ - u), \quad (u - u_-), \quad (\hat{u}_+ - \hat{u}), \quad (\hat{u} - \hat{u}_-). \tag{40}$$

An example is the simple explicit difference model:

$$\frac{\hat{u} - u}{\tau} = \frac{1}{h}\left(\exp\left(\frac{u_+ + u}{2}\right)\underset{h}{u_x} - \exp\left(\frac{u + u_-}{2}\right)\underset{h}{u_{\bar{x}}}\right). \tag{41}$$

So, in this case one has a lot of freedom to construct invariant schemes starting from the set (40).

2.2. For $Q = \delta = \pm 1$ we have possibility to exclude the constant source from the equation

$$u_t = (e^u u_x)_x + \delta \tag{42}$$

by change of variables:

$$\begin{cases} \bar{u} = u - \delta t; \\ \bar{t} = \delta(e^{\delta t} - 1). \end{cases} \tag{43}$$

Equation (42) will be transformed into equation (38) by this change, but the uniformity of the grid in the t-direction is destroyed. Equation (42) admits the four-dimensional algebra of infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = e^{-\delta t}\frac{\partial}{\partial t} + \delta e^{-\delta t}\frac{\partial}{\partial u}, \quad X_4 = x\frac{\partial}{\partial x} + 2\frac{\partial}{\partial u}, \tag{44}$$

and we can easily see that the operator X_3 doesn't conserve uniformity of a grid in the time direction. The finite-difference invariants:

$$\frac{e^u(e^{\delta\tau} - 1)}{h^2}, \quad (\hat{u} - u - \delta\tau), \quad (u_+ - u), \quad (u - u_-), \quad (\hat{u}_+ - \hat{u}), \quad (\hat{u} - \hat{u}_-) \tag{45}$$

permit us to construct the following variant of difference model for the equation (42):

$$\frac{\delta(\hat{u} - u) - \tau}{e^{\delta\tau} - 1} = \frac{1}{h}\left(\exp\left(\frac{u_+ + u}{2}\right)\underset{h}{u_x} - \exp\left(\frac{u + u_-}{2}\right)\underset{h}{u_{\bar{x}}}\right) \tag{46}$$

on an invariant orthogonal grid on time interval $[0, T]$, given by the formula:

$$t_n = \delta \ln\left(1 + \frac{n}{k}(e^{\delta T} - 1)\right), \quad n = 0, .., k, \tag{47}$$

where k is the number of time steps of the grid. We note that the exchange of variables (43) transforms the model (46) into the model (41), while chosen nonuniform grid is transforming into the uniform grid on time interval $[0, \delta(e^{\delta T} - 1)]$.

2.3. If $Q = \pm e^{\alpha u}$, $\alpha \neq 0$ the equation

$$u_t = (e^u u_x)_x \pm e^{\alpha u} \tag{48}$$

admits 3 infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2\alpha t\frac{\partial}{\partial t} + (\alpha-1)x\frac{\partial}{\partial x} - 2\frac{\partial}{\partial u}. \tag{49}$$

These operators satisfy the conditions of orthogonality and uniformity of invariant grids and we will consider the stencil of Fig.1. Any approximation of the equation (48) by the finite-difference invariants

$$\frac{\tau^{\frac{\alpha-1}{2\alpha}}}{h}, \quad e^{\alpha u}\tau, \quad (\hat{u}-u), \quad (u_+-u), \quad (u-u_-), \quad (\hat{u}_+-\hat{u}), \quad (\hat{u}-\hat{u}_-) \tag{50}$$

gives a variant of a difference model for the equation (48), admitting the symmetries (49), for example, we obtain the following model:

$$\frac{\hat{u}-u}{\tau} = \frac{1}{h}\left(\exp\left(\frac{u_++u}{2}\right)u_{\underset{h}{x}} - \exp\left(\frac{u+u_-}{2}\right)u_{\underset{h}{\bar{x}}}\right) \pm e^{\alpha u}. \tag{51}$$

2.4. In accordance with group classification [4] we will also consider the case $Q = \pm e^u + \delta$, $\delta = \pm 1$. As in the case (2.2) we have the possibility to exclude the constant source from the equation

$$u_t = (e^u u_x)_x \pm e^u + \delta \tag{52}$$

by the change of variables (43). The equation (52) will be transformed into the equation (48). The equation (52) admits the following infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = e^{-\delta t}\frac{\partial}{\partial t} + \delta e^{-\delta t}\frac{\partial}{\partial u}. \tag{53}$$

Finite-difference invariants of (54):

$$e^u(e^{\delta\tau}-1), \quad h, \quad (\hat{u}-u-\delta\tau), \quad (u_+-u), \quad (u-u_-), \quad (\hat{u}_+-\hat{u}), \quad (\hat{u}-\hat{u}_-) \tag{54}$$

permit to construct the following variant of the difference model

$$\frac{\delta(\hat{u}-u)-\tau}{e^{\delta\tau}-1} = \frac{1}{h}\left(\exp\left(\frac{u_++u}{2}\right)u_{\underset{h}{x}} - \exp\left(\frac{u+u_-}{2}\right)u_{\underset{h}{\bar{x}}}\right) \pm e^u \tag{55}$$

on the invariant grid (47). The model for the considered equation can be obtained from the model (51) by the transformation (43).

§3. The power heat transfer coefficient: $K = u^\sigma$, $\sigma \neq 0, -\frac{4}{3}$.

With $K = u^\sigma$ the classification depends only on the source.

3.1. Let us start with the simplest case $Q \equiv 0$:

$$u_t = (u^\sigma u_x)_x. \tag{56}$$

Symmetries of the equation (56) are described by the four-dimensional algebra of infinitesimal operators (see [15]):

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_4 = \sigma x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}. \tag{57}$$

With any σ the operators (57) conserve uniformity and orthogonality of a grid. The finite-difference invariants corresponding to the stencil of Fig.1 are

$$u^\sigma\frac{\tau}{h^2}, \quad \frac{\hat{u}}{u}, \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}. \tag{58}$$

They permit us to write, for example, the following variant of the difference model on the orthogonal uniform mesh in both directions:

$$\frac{\hat{u}-u}{\tau}=\frac{1}{h}\left(\left(\frac{u_{+}+u}{2}\right)^{\sigma}\underset{h}{u_{x}}-\left(\frac{u+u_{-}}{2}\right)^{\sigma}\underset{h}{u_{\bar{x}}}\right). \tag{59}$$

Orthogonal grid is not the only possible way for discrete modeling. By using the equation (56) as an example we show how to introduce moving mesh of the form shown in Fig.2.

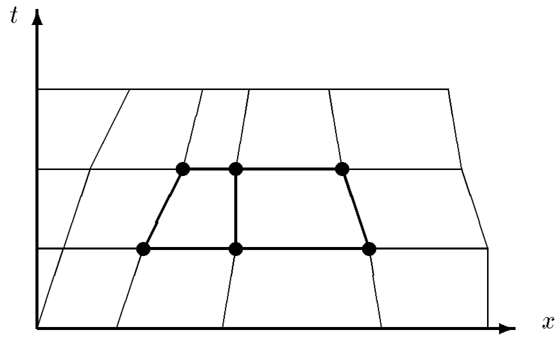

Figure 2: An evolutionary mesh with flat time-layers.

One can use an adaptive grid defined by an evolution equation (see also [3]):

$$\frac{dx}{dt}=\varphi(t,x,u,u_{x}). \tag{60}$$

For such kind of adaptation the heat transfer equation will take the form

$$\frac{du}{dt}=(u^{\sigma}u_{x})_{x}+\varphi(t,x,u,u_{x})u_{x}. \tag{61}$$

Different requirements could be imposed on the function φ. If we require invariance of equation (60) with respect to the whole set of operators (57) our freedom to chose φ is limited by the function $\varphi=Cu^{\sigma-1}u_{x},\quad C=const.$

Below we show how to introduce Lagrangian type of evolution $\frac{dx}{dt}$. Let us note that the equation (56) has a form of the conservation law which presents the conservation of heat. Hence we can search for a moving mesh of Lagrange type which will be moved in accordance with heat diffusion. We should find an evolution $\frac{dx}{dt}$ which satisfy the equation: $\displaystyle\frac{d}{dt}\int_{x1(t)}^{x2(t)}udx=0.$

For the equation (56) simple computations give

$$\frac{d}{dt}\int_{x1}^{x2}udx=\int_{x1}^{x2}\frac{\partial u}{\partial t}dx+\left[u\frac{dx}{dt}\right]_{x1}^{x2}=\left[u^{\sigma}u_{x}+u\frac{dx}{dt}\right]_{x1}^{x2}=0.$$

Thus we get the evolution $\frac{dx}{dt}=-u^{\sigma-1}u_{x}$. Note that this evolution is invariant with respect to the operators (57). Our initial differential equation (56) can now be presented in the form of the system

$$\left\{\begin{array}{l}\dfrac{dx}{dt}=-u^{\sigma-1}u_{x};\\[2ex]\dfrac{1}{u}\dfrac{du}{dt}=(u^{\sigma-1}u_{x})_{x},\end{array}\right. \tag{62}$$

or in the form of conservation laws

$$\begin{cases} \dfrac{dx}{dt} = -\dfrac{1}{\sigma}(u^\sigma)_x; \\ \\ \dfrac{d\ln u}{dt} = \dfrac{1}{\sigma}(u^\sigma)_{xx}. \end{cases} \tag{63}$$

For difference modeling of system (62) we can consider the stencil:

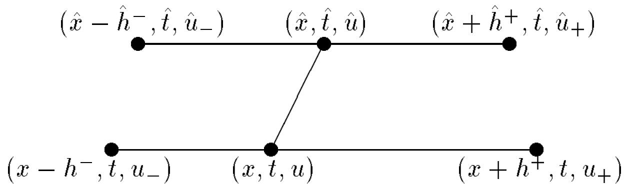

Figure 3: The stencil of the evolutionary mesh.

In the space of the variables $(x, t, h^+, h^-, \hat{h}^+, \hat{h}^- \Delta x, \tau, u, u_+, u_-, \hat{u}, \hat{u}_+, \hat{u}_-)$ corresponding to the chosen stencil of Fig.3 there are 10 finite-difference invariants:

$$u^\sigma \frac{\tau}{h^{+2}}, \quad \frac{\hat{u}}{u}, \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}, \quad \frac{h^-}{h_+}, \quad \frac{\hat{h}^-}{h^+}, \quad \frac{\hat{h}^+}{h^+}, \quad \frac{\Delta x}{h^+}. \tag{64}$$

Approximating the system (62) by these invariants we can get, for example, the system of two explicit equations (compare with [3]):

$$\begin{cases} \dfrac{\Delta x}{\tau} = -\dfrac{1}{2\sigma}\left(\dfrac{u_+^\sigma - u^\sigma}{h^+} + \dfrac{u^\sigma - u_-^\sigma}{h^-}\right); \\ \\ \dfrac{1}{\tau}\ln\left(\dfrac{\hat{u}}{u}\right) = \dfrac{1}{\sigma}\dfrac{2}{h^+ + h^-}\left(\dfrac{u_+^\sigma - u^\sigma}{h^+} - \dfrac{u^\sigma - u_-^\sigma}{h^-}\right); \end{cases} \tag{65}$$

The first equation of system (62) shows that the evolution of x depends on the solution. The model (65) may be inconvenient for computations because steplength will be changed automatically and the nature of this process is not clear. In order to avoid this indeterminacy we introduce a new space variable whose values will characterize the evolution trajectories of x. Let us now consider a new variable s defined by the system:

$$\begin{cases} s_x = u; \\ \\ s_t = u^\sigma u_x; \end{cases}$$

It is easy to see that each trajectory of x is prescribed by a fixed value of s since

$$\frac{ds}{dt} = s_t + s_x \frac{dx}{dt} = u^\sigma u_x - u\frac{1}{\sigma}(u^\sigma)_x = 0.$$

In a new coordinate system with the pair of independent variables (t, s) equation (56) gets the following form

$$\left(\frac{1}{u}\right)_t = -(u^\sigma u_s)_s \tag{66}$$

and the former space variable x satisfies

$$
\begin{cases} x_s = \dfrac{1}{u}; \\ \\ x_t = -u^\sigma u_s. \end{cases} \tag{67}
$$

For discrete modeling of equation (56) one can use equation (66) in new independent variables (t, s) to describe diffusion process and the second equation of the system (67) to trace the evolution of the coordinate x.

Equation (66) together with system (67) admits the following 5-dimensional group of transformations

$$
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{\partial}{\partial s}, \quad X_4 = 2t\frac{\partial}{\partial t} + s\frac{\partial}{\partial s} + x\frac{\partial}{\partial x}, \tag{68}
$$
$$
X_5 = (\sigma + 2)s\frac{\partial}{\partial s} + \sigma x\frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}.
$$

In the new variables (t, s) the stencil becomes orthogonal and there is no need to consider a nonuniform grid in the space variable. There are following invariants for this set of operators in the space $(t, \tau, s, h_s, x, h_x^+, h_x^-, \hat{h}_x^+, \hat{h}_x^- \Delta x, u, u_+, u_-, \hat{u}, \hat{u}_+, \hat{u}_-)$ corresponding to the orthogonal stencil in (t, s) extended by additional dependent variable x:

$$
u^\sigma \frac{\tau}{h_x^{+2}}, \quad \frac{\hat{u}}{u}, \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}, \quad \frac{h_x^-}{h_x^+}, \quad \frac{\hat{h}_x^-}{h_x^+}, \quad \frac{\hat{h}_x^+}{h_x^+}, \quad \frac{\Delta x}{h_x^+}, \quad \frac{h_s}{h_x^+}.
$$

By means of these invariants we get an explicit approximation of (66) having the form of a conservation law:

$$
\frac{1}{\tau}\left(\frac{1}{\hat{u}} - \frac{1}{u}\right) = -\frac{\alpha}{\sigma + 1}\left(\frac{u_+^{\sigma+1} - 2u^{\sigma+1} + u_-^{\sigma+1}}{h_s^2}\right) - \frac{1-\alpha}{\sigma+1}\left(\frac{\hat{u}_+^{\sigma+1} - 2\hat{u}^{\sigma+1} + \hat{u}_-^{\sigma+1}}{h_s^2}\right), \tag{69}
$$

where $0 \leq \alpha \leq 1$. Note that in the coordinates (t, s) variable x is introduced by system (67) as a potential for equation (66). We can get the same in the discrete case if we introduce x with the help of the system

$$
\begin{cases} \dfrac{h_x^+}{h_s} = \dfrac{1}{2}\left(\dfrac{1}{u} + \dfrac{1}{u_+}\right); \\ \\ \dfrac{\Delta x}{\tau} = -\dfrac{\alpha}{\sigma+1}\left(\dfrac{u_+^{\sigma+1} - u_-^{\sigma+1}}{2h_s}\right) - \dfrac{1-\alpha}{\sigma+1}\left(\dfrac{\hat{u}_+^{\sigma+1} - \hat{u}_-^{\sigma+1}}{2h_s}\right). \end{cases} \tag{70}
$$

It is easy to check that for x we have

$$
\frac{1}{\tau}\left(\frac{\hat{h}_x^+}{h_s} - \frac{h_x^+}{h_s}\right) = \frac{1}{h_s}\left(\frac{\Delta x^+}{\tau} - \frac{\Delta x}{\tau}\right)
$$

because of (69).

In computations only equation (69) and the second equation of (70) are needed. The first equation of the system (70) is only needed to establish the connection between solutions $u(x)$ and $u(s)$ for a fixed time. The fact that x is introduced as a discrete potential means consistency of computations of x by second equation of (70) with the connection between x and s for a time layer. If we take x for which $\dfrac{h_x^+}{h_s} = \dfrac{1}{2}\left(\dfrac{1}{u} + \dfrac{1}{u_+}\right)$ holds on the initial time layer, than the same relation will hold on any time layer.

3.2. $Q = \delta u$, $\delta = \pm 1$. In this case the symmetry of the equation

$$u_t = (u^\sigma u_x)_x + \delta u \tag{71}$$

is described by the following infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \sigma x \frac{\partial}{\partial x} + 2u\frac{\partial}{\partial u}, \qquad X_4 = e^{-\delta\sigma t}\frac{\partial}{\partial t} + \delta e^{-\delta\sigma t} u \frac{\partial}{\partial u}. \tag{72}$$

By the change of variables

$$\begin{cases} \bar{u} = ue^{-\delta t}; \\ \bar{t} = \dfrac{\delta}{\sigma}(e^{\delta\sigma t} - 1); \end{cases} \tag{73}$$

equation (71) is transformed into equation (56).

The finite-difference invariants

$$\frac{u^\sigma(e^{\delta\sigma\tau} - 1)}{h^2}, \quad \left(\delta \ln\frac{\hat{u}}{u} - \tau\right), \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}} \tag{74}$$

give the following possibility for an explicit difference model

$$\frac{\sigma u}{e^{\delta\sigma\tau} - 1}\left(\delta \ln\frac{\hat{u}}{u} - \tau\right) = \frac{1}{h}\left(\left(\frac{u_+ + u}{2}\right)^\sigma u_{\underset{h}{x}} - \left(\frac{u + u_-}{2}\right)^\sigma u_{\underset{h}{\bar{x}}}\right) \tag{75}$$

on the invariant grid with time interval $[0, T]$, defined by the expression:

$$t_n = \frac{\delta}{\sigma}\ln\left(1 + \frac{n}{k}(e^{\delta\sigma T} - 1)\right), \quad n = 0, ..., k. \tag{76}$$

It is important to note that the change of variables (73) transforms the chosen nonuniform grid into an uniform grid on the time interval $[0, \frac{\delta}{\sigma}(e^{\delta\sigma T} - 1)]$, while the model (75) is transformed into equation (59).

3.3. $Q = \pm u^{\sigma+1} + \delta u$, $\delta = \pm 1$.
The equation

$$u_t = (u^\sigma u_x)_x \pm u^n, \quad \sigma, n = const, \tag{77}$$

admits a 3-parameter symmetry group. A possible representation of this group is by the following infinitesimal operators [4]:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2(n-1)t\frac{\partial}{\partial t} + (n - \sigma - 1)x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}. \tag{78}$$

The set (78) satisfies all the conditions (27)–(30). So we can use an orthogonal grid that is uniform in both t and x directions. By considering the set of operators (78) in the space $(t, \hat{t}, x, h^+, h^-, u, u_+, u_-,$
$\hat{u}, \hat{u}_+, \hat{u}_-)$ that corresponds to the stencil shown in Fig. 1 we find 7 difference invariants of the Lie algebra:

$$\frac{\tau^{\frac{n-\sigma-1}{2(n-1)}}}{h}, \quad \tau u^{n-1}, \quad \frac{\hat{u}}{u}, \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}}. \tag{79}$$

The small number of symmetry operators provides us with a large number of difference invariants. Thus we are left with some additional degrees of freedom in invariant difference modeling of (77). By means of the discrete invariants we obtain the following explicit scheme:

$$\frac{\hat{u}-u}{\tau} = \frac{1}{h}\left(\left(\frac{u_+ + u}{2}\right)^{\sigma} \underset{h}{u}_x - \left(\frac{u+u_-}{2}\right)^{\sigma} \underset{h}{u}_{\bar{x}}\right) \pm u^n, \tag{80}$$

where $\underset{h}{u}_x = \dfrac{u_+ - u}{h}$, $\underset{h}{u}_{\bar{x}} = \dfrac{u-u_-}{h}$.

This scheme is not unique and one could construct other invariant difference equations, for example an implicit scheme could be

$$\frac{\hat{u}-u}{\tau} = \frac{1}{h^2}(\hat{u}_+^{\sigma+1} - 2\hat{u}^{\sigma+1} + \hat{u}_-^{\sigma+1}) + \hat{u}^n. \tag{81}$$

Notice that the continuous limit of the last difference equation

$$u_t = (u^{\sigma+1})_{xx} + u^n. \tag{82}$$

is equivalent to the equation (77) up to the scaling of x. But scheme (81) is not equivalent to scheme (80) because there is no point transformation that relates them. In the work [18] Samarskii et. all considered the case $n = \sigma+1$ and found a finite-difference blow-up solution for the equation (81) that is invariant with respect to the operator

$$X_3^* = (t-T_o)\frac{\partial}{\partial t} - \frac{1}{\sigma}u\frac{\partial}{\partial u}, \tag{83}$$

where T_o is constant. The operator (83) defines a subgroup which is equivalent to the self-similar subgroup with the operator X_3 of the set (78). Let us find the solution of the problem (81) in the invariant form

$$u = \left(1-\frac{t}{T_o}\right)^{-\dfrac{1}{\sigma}} \theta(x).$$

This solution is sought ([18]) on the set of infinite number of time intervals on $[0, T_o]$:

$$\tau_j = T_o\frac{\rho^{\sigma}+1}{\rho^{\sigma}}\rho^{-\sigma j}, \tag{84}$$

where $\rho > 1$ is constant.

For the function $\theta(x)$ we have the equation

$$(\theta^{\sigma+1})_{\bar{x}x} + \theta^{\sigma+1} = \frac{1}{\sigma}\theta. \tag{85}$$

The solution of the problem (85) was found in [18] for the case $\sigma = 2$. Let us fix an arbitrary $M > 0$ and let $h = 2\sin\left(\dfrac{3\pi}{2(M+1)}\right)$. In this case the localization length equals

$$l_h = \frac{3\pi h}{2}\left(\arcsin\frac{h}{2}\right)^{-1}, \quad 0 < h \leq 2$$

(see [18]) and one can verify that the solution of (85) in the points $x_k = kh$ is

$$\theta_{kh} = \sqrt{2}\left(3\left(1-\frac{4}{h^2}\sin^2\frac{a_h h}{2}\right)\right)^{-1/2}\sin(a_h kh), \quad k = 0, 1, ..., M+1, \tag{86}$$

where $a_h = \pi/l_h$.

The obtained function u gives the blow up solution of the problem in the case $\sigma = 2$, $\beta = 3$, $l = l_h$. This solution tends to infinity in all points of the space grid, conserving the structure. As $h \to 0$, the difference solution (86) tends to the solution of the ordinary differential equation:

$$\theta(x) = (3/4)^{1/2}\sin(x/3), \quad 0 < x < l_0 = 3\pi.$$

3.4. $Q = \pm u^{\sigma+1} + \delta u$, $\delta = \pm 1$. The equation

$$u_t = (u^\sigma u_x)_x \pm u^{\sigma+1} + \delta u \tag{87}$$

is connected with equation (77) by the transformation (73). The infinitesimal operators admitted by the equation are

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = e^{-\delta\sigma t}\frac{\partial}{\partial t} + \delta e^{-\delta\sigma t} u\frac{\partial}{\partial u}. \tag{88}$$

With the help of invariants for operators (88)

$$u^\sigma(e^{\delta\sigma\tau} - 1), \quad h, \quad \left(\delta \ln\frac{\hat{u}}{u} - \tau\right), \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}} \tag{89}$$

we have the following example of an explicit difference model:

$$\frac{\sigma u}{e^{\delta\sigma\tau} - 1}\left(\delta \ln\frac{\hat{u}}{u} - \tau\right) = \frac{1}{h}\left(\left(\frac{u_+ + u}{2}\right)^\sigma u_{\underset{h}{x}} - \left(\frac{u + u_-}{2}\right)^\sigma u_{\underset{h}{\bar{x}}}\right) \pm u^{\sigma+1}. \tag{90}$$

This model should be considered on the invariant grid (76). By (73) the difference model (90) on grid (76) is transformed into the model (80) on an uniform in time direction grid.

§4. The special case of power heat transfer coefficient: $K = u^{-\frac{4}{3}}$.

4.1. If $Q \equiv 0$, then the symmetry of the equation

$$u_t = (u^{-\frac{4}{3}} u_x)_x \tag{91}$$

is described by the five-dimensional algebra of infinitesimal operators (see [15]):

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\ X_4 = 2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u}, \qquad X_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}. \end{gathered} \tag{92}$$

These operators conserve orthogonality and uniformity of a grid in the time direction. The operator X_5 conserve uniformity in the t-direction, but doesn't conserve uniformity of the grid in the x-direction; however orthogonality is not disturbed. We will consider the stencil of the form:

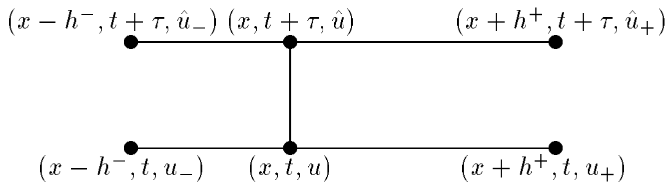

Figure 4: The stencil of nonuniform mesh.

The finite-difference invariants corresponding to the stencil of Fig.2

$$\frac{\hat{u}}{u}, \quad \frac{\hat{u}_+}{u_+}, \quad \frac{\hat{u}_-}{u_-}, \quad u_+^{1/3}u^{1/3}\frac{h^+}{\sqrt{\tau}}, \quad u_-^{1/3}u^{1/3}\frac{h^-}{\sqrt{\tau}}, \quad \frac{u^{2/3}}{\sqrt{\tau}}\left(\frac{h^+h^-}{h^+ + h^-}\right) \tag{93}$$

give among others the explicit difference model:

$$\frac{\hat{u}-u}{\tau} = -\frac{h^+ + h^-}{6h^+h^-}\left(\frac{u_+^{-1/3} - u^{-1/3}}{h^+} - \frac{u^{-1/3} - u_-^{-1/3}}{h^-}\right). \tag{94}$$

As an invariant grid we chose the one, defined by the conditions:

$$h^+ = \varepsilon_o^+ \frac{\sqrt{\tau}}{(uu_+)^{1/3}}, \qquad h^- = \varepsilon_o^- \frac{\sqrt{\tau}}{(uu_-)^{1/3}}, \tag{95}$$

where ε_o^+ and ε_o^- are constants. One can chose the "local" constants ε_o^+ and ε_o^- depending on the solution in such a way that the spacing in the x–direction become equal on the all time layers. Such a choice gives us the possibility to conserve the orthogonal structure of an invariant grid. Let $h^+ = h^-$ then

$$\varepsilon_o^+ = \varepsilon_o^- \left(\frac{u_+}{u_-}\right)^{1/3}. \tag{96}$$

Let us show the choice of "local" constants ε_o^+, ε_o^- for orthogonal and uniform spacing of x for an invariant solution of the equation (94). The stationary solution of the initial differential equation

$$u = (C_0 x + C_1)^{-3}$$

is an exact solution of (94) on uniform grid with space step h. For the constants we find the values

$$\varepsilon_o^+ = \frac{h}{\sqrt{\tau}(C_0 x + C_1)(C_0(x+h) + C_1)},$$
$$\varepsilon_o^- = \frac{h}{\sqrt{\tau}(C_0 x + C_1)(C_0(x-h) + C_1)}.$$

In all points of this part we will use invariant grids, defined in the same way.

4.2. In the case $Q = \delta u$, $\delta = \pm 1$ equation

$$u_t = (u^{-\frac{4}{3}} u_x)_x + \delta u \tag{97}$$

admits

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2x\frac{\partial}{\partial x} - 3u\frac{\partial}{\partial u},$$
$$X_4 = e^{\frac{4\delta t}{3}}\frac{\partial}{\partial t} + \delta e^{\frac{4\delta t}{3}} u\frac{\partial}{\partial u}, \qquad X_5 = x^2\frac{\partial}{\partial x} - 3xu\frac{\partial}{\partial u}. \tag{98}$$

By the change of variables (73) this equation can be transformed into the equation (91). Let us write out the difference invariants for the set of operators (98):

$$\left(\delta \ln\frac{\hat{u}}{u} - \tau\right), \quad u^{2/3}\left(\frac{h^+h^-}{h^+ + h^-}\right)\frac{1}{\sqrt{(e^{\delta\sigma\tau} - 1)}},$$
$$\frac{u_+^{1/3}u^{1/3}h^+}{\sqrt{(e^{\delta\sigma\tau} - 1)}}, \quad \frac{u_-^{1/3}u^{1/3}h^-}{\sqrt{(e^{\delta\sigma\tau} - 1)}}, \quad \frac{\hat{u}_+^{1/3}\hat{u}^{1/3}h^+}{\sqrt{(e^{\delta\sigma\tau} - 1)}}, \quad \frac{\hat{u}_-^{1/3}\hat{u}^{1/3}h^-}{\sqrt{(e^{\delta\sigma\tau} - 1)}}. \tag{99}$$

These invariants can be used for construction of a difference model for the equation (97). We show the explicit variant of the difference model:

$$\frac{\sigma u}{e^{\delta\sigma\tau} - 1}\left(\delta \ln\frac{\hat{u}}{u} - \tau\right) = -\frac{h^+ + h^-}{6h^+h^-}\left(\frac{u_+^{-1/3} - u^{-1/3}}{h^+} - \frac{u^{-1/3} - u_-^{-1/3}}{h^-}\right). \tag{100}$$

As a grid we will chose the grid defined by conditions

$$h^+ = \varepsilon_o^+ \frac{\sqrt{(e^{\delta\sigma\tau} - 1)}}{(uu_+)^{1/3}}, \qquad h^- = \varepsilon_o^- \frac{\sqrt{(e^{\delta\sigma\tau} - 1)}}{(uu_-)^{1/3}}.$$

4.3. $Q = \pm u^n$, $n \neq -\frac{1}{3}$. The equation

$$u_t = (u^{-\frac{4}{3}} u_x)_x \pm u^n \tag{101}$$

admits infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2(n-1)t\frac{\partial}{\partial t} + (n + \frac{1}{3})x\frac{\partial}{\partial x} - 2u\frac{\partial}{\partial u}. \tag{102}$$

Although this equation is specified in group classification (see [4]), it is a particular case of the equation (77), — there is no extension of the admitted group. That's why as an invariant difference model for the equation (101) we can use the model (80) with parameter $\sigma = -\frac{4}{3}$, corresponding to the given equation.

4.4. If $Q = \alpha u^{-\frac{1}{3}}$, $\alpha = \pm 1$, then the variant of the difference model for the equation

$$u_t = (u^{-\frac{4}{3}} u_x)_x \pm u^{-\frac{1}{3}} \tag{103}$$

depends on the sign of the coefficient α. Equation (103) admits a five-dimensional algebra of infinitesimal operators, namely

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = \frac{4}{3}t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}, \tag{104}$$

$$X_4 = e^{2\sqrt{\alpha/3}x}\frac{\partial}{\partial x} - \sqrt{3\alpha}e^{2\sqrt{\alpha/3}x}u\frac{\partial}{\partial u}, \quad X_5 = e^{-2\sqrt{\alpha/3}x}\frac{\partial}{\partial x} + \sqrt{3\alpha}e^{-2\sqrt{\alpha/3}x}u\frac{\partial}{\partial u}.$$

a.) The case $\alpha = 1$.

By the exchange of variables

$$\begin{cases} \bar{u} = uch^3\left(\dfrac{x}{\sqrt{3}}\right); \\ \\ \bar{x} = \sqrt{3}th\left(\dfrac{x}{\sqrt{3}}\right) \end{cases} \tag{105}$$

we transfer the considered equation into equation (91) (see [1]). With the help of difference invariants

$$\frac{\hat{u}}{u}, \quad \frac{\hat{u}_+}{u_+}, \quad \frac{\hat{u}_-}{u_-}, \quad \sqrt{\tau}u^{-2/3}\left(\frac{1}{th\left(\frac{h^+}{\sqrt{3}}\right)} + \frac{1}{th\left(\frac{h^-}{\sqrt{3}}\right)}\right), \tag{106}$$

$$\frac{1}{\sqrt{\tau}}u^{1/3}u_+^{1/3}sh\left(\frac{h^+}{\sqrt{3}}\right), \quad \frac{1}{\sqrt{\tau}}u^{1/3}u_-^{1/3}sh\left(\frac{h^-}{\sqrt{3}}\right)$$

we can construct a difference model. Let us write out one of the possible variants of the difference model — an explicit model:

$$\frac{\hat{u} - u}{\tau} = -\frac{1}{18}\left(\frac{1}{th\left(\frac{h^+}{\sqrt{3}}\right)} + \frac{1}{th\left(\frac{h^-}{\sqrt{3}}\right)}\right) \times \tag{107}$$

$$\times\left(\frac{u_+^{-1/3}-u^{-1/3}ch\left(\frac{h^+}{\sqrt{3}}\right)}{sh\left(\frac{h^+}{\sqrt{3}}\right)}-\frac{u^{-1/3}ch\left(\frac{h^-}{\sqrt{3}}\right)-u_-^{-1/3}}{sh\left(\frac{h^-}{\sqrt{3}}\right)}\right)$$

on the invariant grid:

$$\varepsilon_o^+=\frac{1}{\sqrt{\tau}}u^{1/3}u_+^{1/3}sh\left(\frac{h^+}{\sqrt{3}}\right),\qquad \varepsilon_o^-=\frac{1}{\sqrt{\tau}}u^{1/3}u_-^{1/3}sh\left(\frac{h^-}{\sqrt{3}}\right).$$

b.) The case $\alpha=-1$.

By the exchange of variables

$$\begin{cases}\bar{u}=u\cos^3\left(\frac{x}{\sqrt{3}}\right);\\ \\ \bar{x}=\sqrt{3}tg\left(\frac{x}{\sqrt{3}}\right);\end{cases}\tag{108}$$

we can transfer the given equation into the equation (91) (see [1]). The set of finite-difference invariants:

$$\frac{\hat{u}}{u},\quad\frac{\hat{u}_+}{u_+},\quad\frac{\hat{u}_-}{u_-},\quad\sqrt{\tau}u^{-2/3}\left(\frac{1}{tg\left(\frac{h^+}{\sqrt{3}}\right)}+\frac{1}{tg\left(\frac{h^-}{\sqrt{3}}\right)}\right),\tag{109}$$

$$\frac{1}{\sqrt{\tau}}u^{1/3}u_+^{1/3}\sin\left(\frac{h^+}{\sqrt{3}}\right),\quad\frac{1}{\sqrt{\tau}}u^{1/3}u_-^{1/3}\sin\left(\frac{h^-}{\sqrt{3}}\right)$$

provides us with a possibility to construct an invariant difference scheme and mesh. For example, one can use an explicit difference model:

$$\frac{\hat{u}-u}{\tau}=-\frac{1}{18}\left(\frac{1}{tg\left(\frac{h^+}{\sqrt{3}}\right)}+\frac{1}{tg\left(\frac{h^-}{\sqrt{3}}\right)}\right)\times\tag{110}$$

$$\times\left(\frac{u_+^{-1/3}-u^{-1/3}\cos\left(\frac{h^+}{\sqrt{3}}\right)}{\sin\left(\frac{h^+}{\sqrt{3}}\right)}-\frac{u^{-1/3}\cos\left(\frac{h^-}{\sqrt{3}}\right)-u_-^{-1/3}}{\sin\left(\frac{h^-}{\sqrt{3}}\right)}\right)$$

and the grid, defined by the conditions:

$$\varepsilon_o^+=\frac{1}{\sqrt{\tau}}u^{1/3}u_+^{1/3}\sin\left(\frac{h^+}{\sqrt{3}}\right),\qquad \varepsilon_o^-=\frac{1}{\sqrt{\tau}}u^{1/3}u_-^{1/3}\sin\left(\frac{h^-}{\sqrt{3}}\right).$$

We stress that the obtained difference models (107) and (110) are connected with difference model (94) for equation (91) by the changes of variables (105) and (108) correspondingly as the initial differential equations.

4.5. $Q=\alpha u^{-\frac{1}{3}}+\delta u$, $|\alpha|=|\delta|=1$. As in the previous point, two cases of parameter α in the equation

$$u_t=(u^\sigma u_x)_x\pm u^{\sigma+1}+\delta u\tag{111}$$

should be considered separately and two difference models should be constructed. Let us write out the infinitesimal operators, admitted by equation (111):

$$X_1=\frac{\partial}{\partial t},\qquad X_2=\frac{\partial}{\partial x},\qquad X_3=e^{\frac{4\delta t}{3}}\frac{\partial}{\partial t}+\delta e^{\frac{4\delta t}{3}}u\frac{\partial}{\partial u},$$

$$X_4 = e^{2\sqrt{\alpha/3}x}\frac{\partial}{\partial x} - \sqrt{3\alpha}e^{2\sqrt{\alpha/3}x}u\frac{\partial}{\partial u}, \tag{112}$$
$$X_5 = e^{-2\sqrt{\alpha/3}x}\frac{\partial}{\partial x} + \sqrt{3\alpha}e^{-2\sqrt{\alpha/3}x}u\frac{\partial}{\partial u}.$$

a.) The case $\alpha = 1$.

Exchange (73) is transforming considered equation into equation (103), and exchange (105) — into equation (91).

We write out the set of finite-difference invariants for the equation (111) with $\alpha = 1$:

$$\left(\delta \ln \frac{\hat{u}}{u} - \tau\right), \quad \sqrt{(e^{\delta\sigma\tau} - 1)}u^{-2/3}\left(\frac{1}{th\left(\frac{h^+}{\sqrt{3}}\right)} + \frac{1}{th\left(\frac{h^-}{\sqrt{3}}\right)}\right), \tag{113}$$
$$\frac{u^{1/3}u_+^{1/3}}{\sqrt{(e^{\delta\sigma\tau} - 1)}}sh\left(\frac{h^+}{\sqrt{3}}\right), \quad \frac{u^{1/3}u_-^{1/3}}{\sqrt{(e^{\delta\sigma\tau} - 1)}}sh\left(\frac{h^-}{\sqrt{3}}\right),$$
$$\frac{\hat{u}^{1/3}\hat{u}_+^{1/3}}{\sqrt{(e^{\delta\sigma\tau} - 1)}}sh\left(\frac{h^+}{\sqrt{3}}\right), \quad \frac{\hat{u}^{1/3}\hat{u}_-^{1/3}}{\sqrt{(e^{\delta\sigma\tau} - 1)}}sh\left(\frac{h^-}{\sqrt{3}}\right).$$

The explicit variant of the difference model for equation (111) on time grid (76) has the form:

$$\frac{\sigma u}{e^{\delta\sigma\tau} - 1}\left(\delta \ln \frac{\hat{u}}{u} - \tau\right) = -\frac{1}{18}\left(\frac{1}{th\left(\frac{h^+}{\sqrt{3}}\right)} + \frac{1}{th\left(\frac{h^-}{\sqrt{3}}\right)}\right) \times$$
$$\times \left(\frac{u_+^{-1/3} - u^{-1/3}ch\left(\frac{h^+}{\sqrt{3}}\right)}{sh\left(\frac{h^+}{\sqrt{3}}\right)} - \frac{u^{-1/3}ch\left(\frac{h^-}{\sqrt{3}}\right) - u_-^{-1/3}}{sh\left(\frac{h^-}{\sqrt{3}}\right)}\right). \tag{114}$$

We will consider this difference model on the grid, defined by conditions:

$$\varepsilon_o^+ = \frac{u^{1/3}u_+^{1/3}}{\sqrt{(e^{\delta\sigma\tau} - 1)}}sh\left(\frac{h^+}{\sqrt{3}}\right), \qquad \varepsilon_o^- = \frac{u^{1/3}u_-^{1/3}}{\sqrt{(e^{\delta\sigma\tau} - 1)}}sh\left(\frac{h^-}{\sqrt{3}}\right).$$

b.) The case $\alpha = -1$

By exchange (108) we can transfer this equation into equation (97), and by exchange (73) into equation (103). Difference model for equation (111) can be obtained with the help of the invariants

$$\left(\delta \ln \frac{\hat{u}}{u} - \tau\right), \quad \sqrt{(e^{\delta\sigma\tau} - 1)}u^{-2/3}\left(\frac{1}{tg\left(\frac{h^+}{\sqrt{3}}\right)} + \frac{1}{tg\left(\frac{h^-}{\sqrt{3}}\right)}\right), \tag{115}$$
$$\frac{u^{1/3}u_+^{1/3}}{\sqrt{(e^{\delta\sigma\tau} - 1)}}\sin\left(\frac{h^+}{\sqrt{3}}\right), \quad \frac{u^{1/3}u_-^{1/3}}{\sqrt{(e^{\delta\sigma\tau} - 1)}}\sin\left(\frac{h^-}{\sqrt{3}}\right),$$
$$\frac{\hat{u}^{1/3}\hat{u}_+^{1/3}}{\sqrt{(e^{\delta\sigma\tau} - 1)}}\sin\left(\frac{h^+}{\sqrt{3}}\right), \quad \frac{\hat{u}^{1/3}\hat{u}_-^{1/3}}{\sqrt{(e^{\delta\sigma\tau} - 1)}}\sin\left(\frac{h^-}{\sqrt{3}}\right).$$

One of possible difference models for equation (111) is

$$\frac{\sigma u}{e^{\delta\sigma\tau}-1}\left(\delta\ln\frac{\hat{u}}{u}-\tau\right)=-\frac{1}{18}\left(\frac{1}{tg\left(\frac{h^+}{\sqrt{3}}\right)}+\frac{1}{tg\left(\frac{h^-}{\sqrt{3}}\right)}\right)\times$$
$$\times\left(\frac{u_+^{-1/3}-u^{-1/3}\cos\left(\frac{h^+}{\sqrt{3}}\right)}{\sin\left(\frac{h^+}{\sqrt{3}}\right)}-\frac{u^{-1/3}\cos\left(\frac{h^-}{\sqrt{3}}\right)-u_-^{-1/3}}{\sin\left(\frac{h^-}{\sqrt{3}}\right)}\right) \tag{116}$$

on the grid defined by the following conditions:

$$\varepsilon_o^+=\frac{u^{1/3}u_+^{1/3}}{\sqrt{(e^{\delta\sigma\tau}-1)}}\sin\left(\frac{h^+}{\sqrt{3}}\right),\qquad \varepsilon_o^-=\frac{u^{1/3}u_-^{1/3}}{\sqrt{(e^{\delta\sigma\tau}-1)}}\sin\left(\frac{h^-}{\sqrt{3}}\right).$$

As a time step sequence for the model (116) the set of time intervals defined is (76) can be used. The difference models (114) and (116) are connected with model (100) by changes of variables (105) and (108) correspondingly. Variable exchange (73) transforms the difference models obtained in this point into model of the point 4.4 for corresponding values of the parameter α. This is an example of that in invariant difference modeling it is possible to get consistent models which are connected with each other by the same point transformations as their original differential counterparts.

§5. Linear heat conductivity with a source.

In this section we consider the semilinear heat transfer equation with different types of a source, namely

$$u_t=u_{xx}+Q(u). \tag{117}$$

5.1. With $Q=\pm e^u$ the equation becomes

$$u_t=u_{xx}\pm e^u, \tag{118}$$

and admits a three-dimensional algebra of infinitesimal operators, which are

$$X_1=\frac{\partial}{\partial t},\qquad X_2=\frac{\partial}{\partial x},\qquad X_3=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}-2\frac{\partial}{\partial u}. \tag{119}$$

It is easy to check that for the operators the conditions of orthogonality and uniformity conservation of a grid hold. Approximation of the equation by the invariants

$$\frac{h^2}{\tau},\quad \tau e^u,\quad (\hat{u}-u),\quad (u_+-u),\quad (u-u_-),\quad (\hat{u}_+-\hat{u}),\quad (\hat{u}-\hat{u}_-) \tag{120}$$

will give different types of difference models. An explicit one is

$$\frac{\hat{u}-u}{\tau}=\frac{1}{h}(u_{\underset{h}{x}}-u_{\underset{h}{\bar{x}}})\pm e^u. \tag{121}$$

5.2. $Q=\pm u^n$. The equation

$$u_t=u_{xx}\pm u^n \tag{122}$$

admits the following infinitesimal operators:

$$X_1=\frac{\partial}{\partial t},\quad X_2=\frac{\partial}{\partial x},\quad X_3=2(n-1)t\frac{\partial}{\partial t}+(n-1)x\frac{\partial}{\partial x}-2u\frac{\partial}{\partial u}. \tag{123}$$

The operators satisfy to conditions (27)–(29), and the finite-difference invariants

$$\frac{h^2}{\tau}, \quad \tau u^{n-1}, \quad \frac{\hat{u}}{u}, \quad \frac{u_+}{u}, \quad \frac{u_-}{u}, \quad \frac{\hat{u}_+}{\hat{u}}, \quad \frac{\hat{u}_-}{\hat{u}} \tag{124}$$

permit us the construction of the following variant of a difference model:

$$\frac{\hat{u}-u}{\tau} = \frac{1}{h}(u_{\underset{h}{x}} - u_{\underset{h}{\bar{x}}}) \pm u^n. \tag{125}$$

5.3 $Q = \delta u \ln u$, $\delta = \pm 1$.
The semilinear heat transfer equation

$$u_t = u_{xx} + \delta u \ln u, \quad \delta = \pm 1, \tag{126}$$

admits the 4-parameter Lie symmetry group of point transformations [4] corresponding to the following set of infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = 2e^{\delta t}\frac{\partial}{\partial x} - \delta e^{\delta t} x u \frac{\partial}{\partial u}, \quad X_4 = e^{\delta t} u \frac{\partial}{\partial u}. \tag{127}$$

Before constructing a difference equation and a grid that approximate (126) and inherit the whole Lie algebra (127) we should first check condition (29) for the invariance of orthogonality. The operators X_1, X_2 and X_4 conserve orthogonality, while X_3 does not: condition (29) is not true for operator X_3. Consequently an orthogonal mesh can not be used for the invariant modeling of (126). Conditions (30) are true for the complete set of operators, so it is possible to use a nonorthogonal grid with flat time layers and we will use the grid shown in Fig.2.

A possible reformulation of equation (126) by using the four differential invariants in the subspace $(t, x, u, u_x, u_{xx}, dt, dx, du)$:

$$J^1 = dt, \qquad J^2 = \left(\frac{u_x}{u}\right)^2 - \frac{u_{xx}}{u}, \qquad J^3 = 2\frac{u_x}{u} + \frac{dx}{dt}, \qquad J^4 = \frac{du}{u dt} - \delta \ln u + \frac{1}{4}\left(\frac{dx}{dt}\right)^2$$

is given by the system

$$\begin{cases} J^3 = 0; \\ J^4 = J^2; \end{cases}$$

or

$$\begin{cases} \dfrac{dx}{dt} = -2\dfrac{u_x}{u}; \\[2ex] \dfrac{du}{dt} = u_{xx} + \delta u \ln u - 2\dfrac{u_x^2}{u}. \end{cases} \tag{128}$$

So, the structure of the group (127) suggests the use of two evolution equations.

As the next step, we will find difference invariants for the set X_1–X_4 of the group (127). These invariants are necessary for the approximation of system (128). We will use the 6-point difference stencil of Fig.3 on which we will approximate system (128). The stencil defines the difference subspace $(t, \hat{t}, x, \hat{x}, h^+, h^-, \hat{h}^+, \hat{h}^-, u, u_+, u_-, \hat{u}, \hat{u}_+, \hat{u}_-)$ and the group (127) has the following difference invariants

$$I^1 = \tau, \quad I^2 = h^+, \quad I^3 = h^-, \quad I^4 = \hat{h}^+, \quad I^5 = \hat{h}^-,$$

$$I^6 = (\ln u)_x - (\ln u)_{\bar{x}}, \qquad I^7 = (\ln \hat{u})_x - (\ln \hat{u})_{\bar{x}},$$

$$I^8 = \delta \Delta x + 2(e^{\delta\tau} - 1)\left(\frac{h^-}{h^+ + h^-}(\ln u)_x + \frac{h^+}{h^+ + h^-}(\ln u)_{\bar{x}}\right),$$

$$I^9 = \delta\Delta x + 2(1-e^{-\delta\tau})\left(\frac{\hat{h}^-}{\hat{h}^+ + \hat{h}^-}(\ln \hat{u})_x + \frac{\hat{h}^+}{\hat{h}^+ + \hat{h}^-}(\ln \hat{u})_{\bar{x}}\right),$$

$$I^{10} = \delta(\Delta x)^2 + 4(1-e^{-\delta\tau})\left(\ln \hat{u} - e^{\delta\tau}\ln u\right),$$

where $\Delta x = \hat{x} - x$, $(\ln u)_x = \dfrac{\ln u_+ - \ln u}{h^+}$, $(\ln u)_{\bar{x}} = \dfrac{\ln u - \ln u_-}{h^-}$.

An explicit model is given by

$$\begin{cases} I^8 = 0; \\ I^{10} = \dfrac{8}{\delta}\dfrac{(e^{\delta I^1}-1)^2}{I^2+I^3}I^6. \end{cases}$$

or

$$\begin{cases} \delta\Delta x + 2(e^{\delta\tau}-1)\left(\dfrac{h^-}{h^++h^-}(\ln u)_x + \dfrac{h^+}{h^++h^-}(\ln u)_{\bar{x}}\right) = 0; \\ \delta(\Delta x)^2 + 4(1-e^{-\delta\tau})\left(\ln \hat{u} - e^{\delta\tau}\ln u\right) = \dfrac{8}{\delta}\dfrac{(e^{\delta\tau}-1)^2}{h^++h^-}[(\ln u)_x - (\ln u)_{\bar{x}}]. \end{cases} \tag{129}$$

As in the continuous case there is a reduction in the difference case. When we consider an invariant solution, we have the reduction of the equation-grid system to a system of ordinary difference equations. One being the difference model for the considered equation, the other for the evolution of the grid.

Let us find the solution of the difference model (129) which is invariant with respect to the operator

$$2\alpha X_2 + X_3, \quad \alpha = const. \tag{130}$$

All the invariants with respect to (130) are $u\exp\left(\dfrac{\delta e^{\delta t}}{\alpha + e^{\delta t}}\dfrac{x^2}{4}\right)$, $\left(\dfrac{\Delta x}{e^{\delta t}(e^{\delta\tau}-1)} - \dfrac{x}{\alpha+e^{\delta t}}\right)$ and t. Therefore we will seek an invariant solution of the form:

$$\begin{cases} u(x,t) = \exp\left(-\dfrac{\delta e^{\delta t}}{\alpha+e^{\delta t}}\dfrac{x^2}{4}\right)e^{f(t)}; \\ \dfrac{\Delta x}{e^{\delta t}(e^{\delta\tau}-1)} = \dfrac{x}{\alpha+e^{\delta t}} + g(t). \end{cases}$$

Substituting this form of the solution into (129) we obtain a system of ordinary difference equations to determine $f(t)$ and $g(t)$:

$$\begin{cases} \dfrac{f(t+\tau) - e^{\delta\tau}f(t)}{e^{\delta\tau}(e^{\delta\tau}-1)} = -\dfrac{1}{2}\dfrac{e^{\delta t}}{\alpha+e^{\delta t}}; \\ g(t) = 0. \end{cases}$$

The solution of this system yields the solution of equation (129):

$$u(x,t) = \exp\left(e^{\delta t}\left(f(0) - \frac{e^{\delta\tau}-1}{2}\sum_{j=1}^{n-1}\frac{e^{-\delta t_j}}{1+\alpha e^{-\delta t_j}}\right) - \frac{\delta e^{\delta t}}{\alpha+e^{\delta t}}\frac{x^2}{4}\right)$$

and the grid

$$x = x^0 \frac{e^{\delta t} + \alpha}{1+\alpha}.$$

Here $x = x_i^j = x_i(t_j)$ and $t = t_j$. For $t = 0$ the grid can be arbitrary, but if a regular grid is used, the grid will be regular on every time layer.

The obtained solution is the solution of the Cauchy problem with initial condition

$$u(x,0) = \exp\left(f(0) - \frac{\delta e^{\delta t}}{\alpha + e^{\delta t}} \frac{x^2}{4}\right).$$

5.4. A linear heat equation without a source ($Q = 0$).

1. Preliminary consideration

The linear heat transfer equation

$$u_t = u_{xx} \tag{131}$$

admits a 6-parameter Lie symmetry group of point transformations, corresponding to the following infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \tag{132}$$

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \quad X_5 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2 + 2t)u\frac{\partial}{\partial u}, \quad X_6 = u\frac{\partial}{\partial u}$$

and an infinite-dimensional symmetry obtained from the linearity of equation (131):

$$X^* = a(x,t)\ \frac{\partial}{\partial u},$$

where $a(t,x)$ in an arbitrary solution of equation (131).

We are now in a position to show that the invariant difference model for the linear heat transfer equation can not be constructed on an orthogonal grid. The model

$$\frac{\hat{u} - u}{\tau} = \frac{u_+ - 2u + u_-}{h^2} \tag{133}$$

on the regular orthogonal mesh, which is used as an invariant model in [2], (see also [1], p.363), actually doesn't admit operators X_3 and X_5.

Let us check for example the symmetry that is described by X_3. The operator X_3 generates the following transformations

$$\begin{aligned} x^* &= x + 2t\alpha, \\ t^* &= t, \\ u^* &= ue^{-x\alpha - t\alpha^2}. \end{aligned} \tag{134}$$

This transformation destroys the orthogonality of the mesh as shown in Figure 5.

The transformation (134) transforms the finite-difference equation (133) into the following:

$$\frac{\hat{u}e^{-\tau\alpha^2} - u}{\tau} = \frac{u_+ e^{-h\alpha} - 2u + u_- e^{h\alpha}}{h^2}, \tag{135}$$

which explicitly depends on a group parameter α. The zero order terms of the modified equation (135)

$$u_t = u_{xx} - 4\alpha u_x + 2u\alpha^2 + O(\tau + h)$$

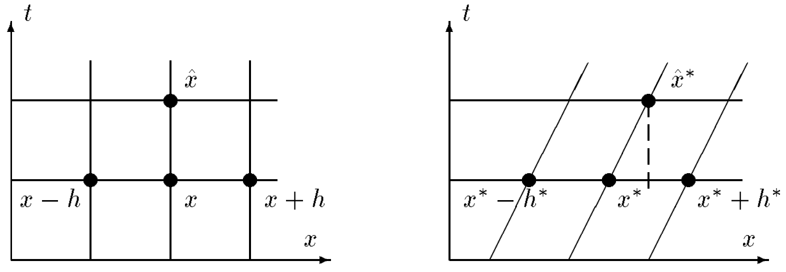

Figure 5: Deformation of the orthogonal mesh.

shows explicitly the absence of invariance for the equation (133) on an orthogonal mesh.
Consequently, we have to construct a difference model for (131) on a moving mesh.

2. Heat transfer system of equations and superposition principle.

With help of the differential invariants in the space $(t, x, u, u_x, u_{xx}, dt, dx, du)$

$$J^1 = \frac{dx + 2\frac{u_x}{u}dt}{dt^{1/2}}, \qquad J^2 = \frac{du}{u} + \frac{1}{4}\frac{dx}{dt}^2 + \left(-\frac{u_{xx}}{u} + \frac{u_x^2}{u^2}\right) dt$$

we can represent the heat equation (131) as the system

$$\begin{cases} J^1 &= 0; \\ J^2 &= 0; \end{cases}$$

or

$$\begin{cases} \dfrac{dx}{dt} &= -2\dfrac{u_x}{u}; \\ \\ \dfrac{du}{dt} &= u_{xx} - 2\dfrac{u_x^2}{u}. \end{cases} \tag{136}$$

System (136) inherits the set of operators X_1–X_6, X^*. In this case the operator X^* prolonged on the time derivatives has the form:

$$X^* = a(x,t)\,\frac{\partial}{\partial u} - 2\left(\frac{a_x}{u} - \frac{u_x}{u^2}a\right)\frac{\partial}{\partial\left(\frac{dx}{dt}\right)} + \left(a_{xx} - \frac{4u_x}{u}a_x + 2\frac{u_x^2}{u^2}a\right)\frac{\partial}{\partial\left(\frac{du}{dt}\right)},$$

where $a(x,t)$ is a solution of the system (136).

This means that system (136) possesses the entire Lie symmetry group admitted by the linear heat transfer equation (131). For system (136) the superposition principle, which has the form of summing two solutions for equation (131), acts on the trajectories on which the variable x evolves. For two arbitrary solutions of (136): $U_1(t,x)$ with the trajectory $x_1(t)$:

$$\frac{dx_1}{dt} = -\frac{2U_{1x}}{U_1},$$

and $U_2(t,x)$ with the trajectory $x_2(t)$:

$$\frac{dx_2}{dt} = -\frac{2U_{2x}}{U_2},$$

their linear combination

$$U = \alpha U_1 + \beta U_2, \quad \alpha, \beta = const, \tag{137}$$

is also the solution of (136). However, this linear combination has its own trajectory

$$\frac{dX}{dt} = \frac{\alpha U_1}{\alpha U_1 + \beta U_2}\frac{dX_1}{dt} + \frac{\beta U_2}{\alpha U_1 + \beta U_2}\frac{dX_2}{dt}. \tag{138}$$

Therefore, the superposition principle can be presented in the following form:

$$\begin{pmatrix} U(t,x) \\ \\ \dfrac{dX}{dt} \end{pmatrix} = \begin{pmatrix} \alpha & 0 \\ \\ 0 & \dfrac{\alpha U_1}{\alpha U_1 + \beta U_2} \end{pmatrix} \begin{pmatrix} U_1(t,x) \\ \\ \dfrac{dX_1}{dt} \end{pmatrix} +$$

$$+ \begin{pmatrix} \beta & 0 \\ \\ 0 & \dfrac{\beta U_2}{\alpha U_1 + \beta U_2} \end{pmatrix} \begin{pmatrix} U_2(t,x) \\ \\ \dfrac{dX_2}{dt} \end{pmatrix}.$$

Let us show the superposition principle for system (136) by an example. The solution

$$U_1 = \frac{1}{\sqrt{t+t_1}} \exp\left(-\frac{(x-a)^2}{4(t+t_1)}\right)$$

has the trajectory

$$x = a + (x_0 - a)\frac{t+t_1}{t_1},$$

while the solution

$$U_2 = \frac{1}{\sqrt{t+t_2}} \exp\left(-\frac{(x-b)^2}{4(t+t_2)}\right)$$

exists on the trajectory

$$x = b + (x_0 - b)\frac{t+t_2}{t_2}.$$

The linear combination (137) of these two solutions is also the solution of system (136). Its trajectory is

$$\frac{dX}{dt} = \frac{\frac{\alpha}{\sqrt{t+t_1}} \exp\left(-\frac{(x-a)^2}{4(t+t_1)}\right)\left(\frac{x-a}{t+t_1}\right) + \frac{\beta}{\sqrt{t+t_2}} \exp\left(-\frac{(x-b)^2}{4(t+t_2)}\right)\left(\frac{x-b}{t+t_2}\right)}{\frac{\alpha}{\sqrt{t+t_1}} \exp\left(-\frac{(x-a)^2}{4(t+t_1)}\right) + \frac{\beta}{\sqrt{t+t_2}} \exp\left(-\frac{(x-b)^2}{4(t+t_2)}\right)}.$$

Examples of evolution of grid points and corresponding solutions are shown on Figures 6-9 (for computations we used discrete model (139) which will be introduced in 5.4.3).

If we consider Lagrangian derivatives of the solution U, we could get the superposition principle in the following form:

$$\frac{dU}{dt} = \alpha\frac{dU_1}{dt} + \beta\frac{dU_2}{dt} + \frac{\alpha\beta U_1 U_2}{2(\alpha U_1 + \beta U_2)}\left(\frac{dX_1}{dt} - \frac{dX_2}{dt}\right)^2,$$

or in the form:

$$\left(\frac{dU}{dt} + \frac{U}{2}\left(\frac{dX}{dt}\right)^2\right) = \alpha\left(\frac{dU_1}{dt} + \frac{U_1}{2}\left(\frac{dX_1}{dt}\right)^2\right) + \beta\left(\frac{dU_2}{dt} + \frac{U_2}{2}\left(\frac{dX_2}{dt}\right)^2\right).$$

The expressions in brackets are partial derivatives of U with respect to time expressed in terms of total derivatives of U and X that means: $U_t = \alpha U_{1t} + \beta U_{2t}$.

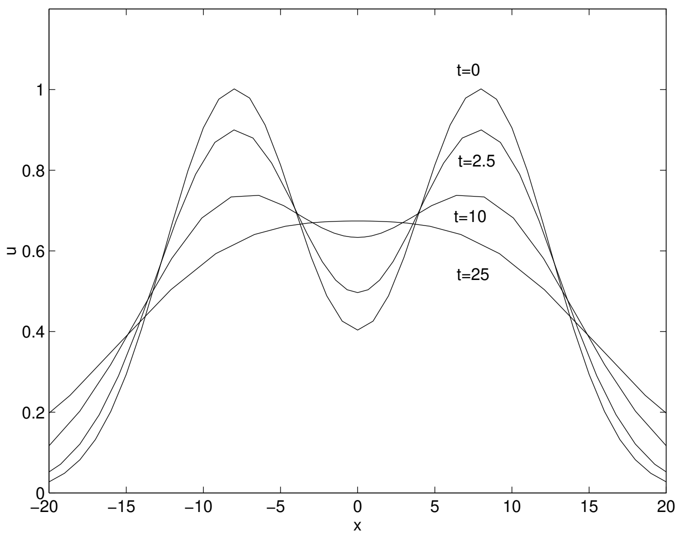

Figure 6: Evolution of solution (137): $\alpha = \beta = 1$, $t_1 = t_2 = 10$, $a = -8$, $b = 8$.

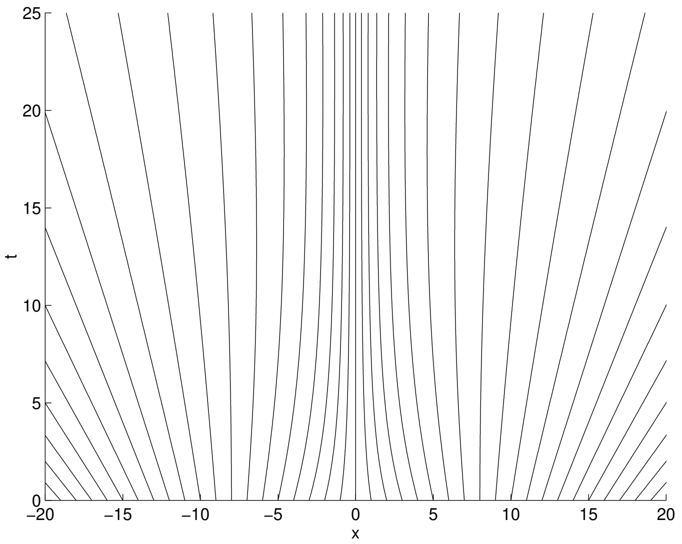

Figure 7: Mesh trajectories.

3. Invariant schemes on moving meshes.

For the difference modeling of system (136) we need the whole set of difference invariants of Lie symmetry group (132) in the difference space corresponding to the chosen stencil $(t, \hat{t}, x, \hat{x}, h^+, h^-, \hat{h}^+, \hat{h}^-, u, \hat{u}, u_+, u_-, \hat{u}_+, \hat{u}_-)$:

$$I^1 = \frac{h^+}{h^-}, \qquad I^2 = \frac{\hat{h}^+}{\hat{h}^-}, \qquad I^3 = \frac{\hat{h}^+ h^+}{\tau}, \qquad I^4 = \frac{\tau^{1/2}}{h^+}\frac{\hat{u}}{u}\exp\left(\frac{1}{4}\frac{(\Delta x)^2}{\tau}\right),$$

$$I^5 = \frac{1}{4}\frac{h^{+2}}{\tau} - \frac{h^{+2}}{h^+ + h^-}\left(\frac{1}{h^+}\ln\frac{u_+}{u} + \frac{1}{h^-}\ln\frac{u_-}{u}\right),$$

$$I^6 = \frac{1}{4}\frac{\hat{h}^{+2}}{\tau} + \frac{\hat{h}^{+2}}{\hat{h}^+ + \hat{h}^-}\left(\frac{1}{\hat{h}^+}\ln\frac{\hat{u}_+}{\hat{u}} + \frac{1}{\hat{h}^-}\ln\frac{\hat{u}_-}{\hat{u}}\right),$$

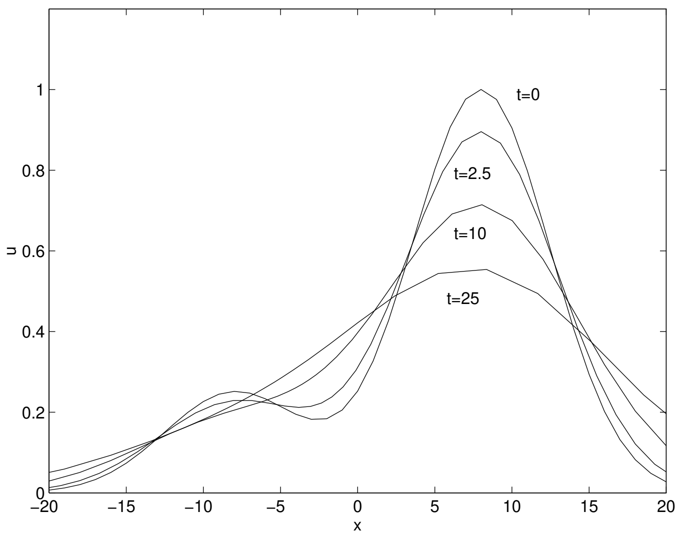

Figure 8: Evolution of solution (137): $\alpha = 0.25$, $\beta = 1$, $t_1 = t_2 = 10$, $a = -8$, $b = 8$.

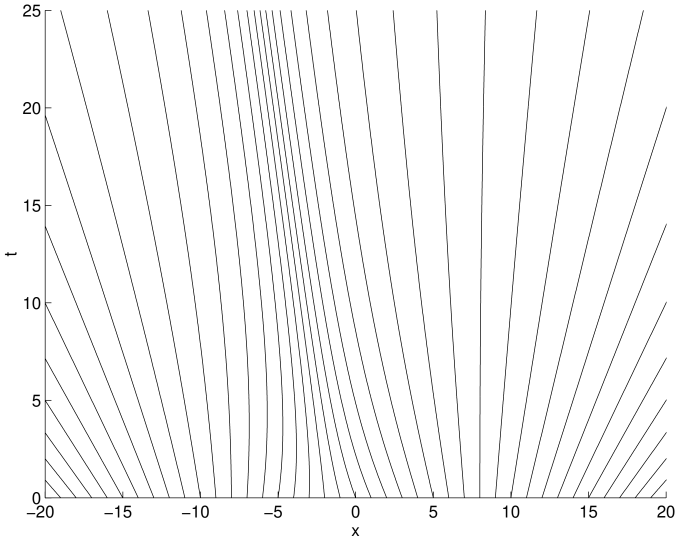

Figure 9: Mesh trajectories.

$$I^7 = \frac{\Delta x h^+}{\tau} + \frac{2h^+}{h^+ + h^-}\left(\frac{h^-}{h^+}\ln\frac{u_+}{u} - \frac{h^+}{h^-}\ln\frac{u_-}{u}\right),$$

$$I^8 = \frac{\Delta x \hat{h}^+}{\tau} + \frac{2\hat{h}^+}{\hat{h}^+ + \hat{h}^-}\left(\frac{\hat{h}^-}{\hat{h}^+}\ln\frac{\hat{u}_+}{\hat{u}} - \frac{\hat{h}^+}{\hat{h}^-}\ln\frac{\hat{u}_-}{\hat{u}}\right).$$

Approximating system (136) by these invariants as it was done for the semilinear heat equation, we obtain a system of difference evolution equations. As an example, we present here an invariant difference model which has explicit equations for the solution u and the trajectory of x:

$$\left\{\begin{array}{rcl} \Delta x & = & \dfrac{2\tau}{h^+ + h^-}\left(-\dfrac{h^-}{h^+}\ln\dfrac{u_+}{u} + \dfrac{h^+}{h^-}\ln\dfrac{u_-}{u}\right); \\[2ex] \left(\dfrac{u}{\hat{u}}\right)^2 \exp\left(-\dfrac{1}{2}\dfrac{(\Delta x)^2}{\tau}\right) & = & 1 - \dfrac{4\tau}{h^+ + h^-}\left(\dfrac{1}{h^+}\ln\dfrac{u_+}{u} + \dfrac{1}{h^-}\ln\dfrac{u_-}{u}\right). \end{array}\right. \qquad (139)$$

We also can write out implicit model

$$\begin{cases} \Delta x = \dfrac{2\tau}{\hat h^+ + \hat h^-}\left(-\dfrac{\hat h^-}{\hat h^+}\ln\dfrac{\hat u_+}{\hat u} + \dfrac{\hat h^+}{\hat h^-}\ln\dfrac{\hat u_-}{\hat u}\right); \\ \left(\dfrac{\hat u}{u}\right)^2 \exp\left(\dfrac{1}{2}\dfrac{\Delta x^2}{\tau}\right) = 1 + \dfrac{4\tau}{\hat h^+ + \hat h^-}\left(\dfrac{1}{\hat h^+}\ln\dfrac{\hat u_-}{\hat u} + \dfrac{1}{\hat h^-}\ln\dfrac{\hat u_-}{\hat u}\right) \end{cases}$$

and models which have one explicit equation and one implicit equation for mesh and heat transformation:

$$\begin{cases} \Delta x = \dfrac{2\tau}{h^+ + h^-}\left(-\dfrac{h^-}{h^+}\ln\dfrac{u_+}{u} + \dfrac{h^+}{h^-}\ln\dfrac{u_-}{u}\right); \\ \left(\dfrac{\hat u}{u}\right)^2 \exp\left(\dfrac{1}{2}\dfrac{\Delta x^2}{\tau}\right) = 1 + \dfrac{4\tau}{\hat h^+ + \hat h^-}\left(\dfrac{1}{\hat h^+}\ln\dfrac{\hat u_-}{\hat u} + \dfrac{1}{\hat h^-}\ln\dfrac{\hat u_-}{\hat u}\right) \end{cases}$$

and

$$\begin{cases} \Delta x = \dfrac{2\tau}{\hat h^+ + \hat h^-}\left(-\dfrac{\hat h^-}{\hat h^+}\ln\dfrac{\hat u_+}{\hat u} + \dfrac{\hat h^+}{\hat h^-}\ln\dfrac{\hat u_-}{\hat u}\right); \\ \left(\dfrac{u}{\hat u}\right)^2 \exp\left(-\dfrac{1}{2}\dfrac{(\Delta x)^2}{\tau}\right) = 1 - \dfrac{4\tau}{h^+ + h^-}\left(\dfrac{1}{h^+}\ln\dfrac{u_+}{u} + \dfrac{1}{h^-}\ln\dfrac{u_-}{u}\right). \end{cases}$$

There are also other ways to approximate system (136).

4. Optimal system of subalgebras and reduced systems.

For the example of difference model (139) for the linear heat transfer equation we will construct an optimal system of solutions which are invariant with respect to one-parameter groups. The optimal system of one-dimensional subalgebras of the algebra of symmetries for the linear heat equation consists of algebras corresponding to the operators (see [16]):

$$Y_1 = X_2 = \frac{\partial}{\partial x}, \qquad Y_2 = X_6 = u\frac{\partial}{\partial u}, \qquad Y_3 = X_1 + cX_6 = \frac{\partial}{\partial t} + cu\frac{\partial}{\partial u},$$

$$Y_4 = X_1 - X_3 = \frac{\partial}{\partial t} - 2t\frac{\partial}{\partial x} + xu\frac{\partial}{\partial u}, \quad Y_5 = X_4 + 2cX_6 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + 2cu\frac{\partial}{\partial u},$$

$$Y_6 = X_1 + X_5 + cX_6 = (4t^2+1)\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} + (c - x^2 - 2t)u\frac{\partial}{\partial u}.$$

1) The subalgebra corresponding to the operator Y_1 has only constant solutions.

2) The subalgebra corresponding to the operator Y_2 does not have invariant solutions (the necessary condition of existence of invariant solutions does not hold).

3) The operator Y_3 has the following invariants: $u\exp(-ct)$, τ and Δx. The time step τ is invariant, so we can consider uniform in time invariant mesh. We will seek a solution of the difference model in the form:

$$u = \exp(ct) f(x).$$

Substituting this invariant form of the solution into system (139), we get:

$$\begin{cases} \Delta x = \dfrac{2\tau}{h^+ + h^-}\left(-\dfrac{h^+}{h^-}\ln\left(\dfrac{f(x+h^+)}{f(x)}\right) + \dfrac{h^+}{h^-}\ln\left(\dfrac{f(x-h^-)}{f(x)}\right)\right); \\ \left(\dfrac{f(x)}{f(x+\Delta x)}\right)^2 \exp\left(-2c\tau - \dfrac{1}{2}\dfrac{\Delta x^2}{\tau}\right) = \\ = 1 - \dfrac{4\tau}{h^+ + h^-}\left(\dfrac{1}{h^+}\ln\left(\dfrac{f(x+h^+)}{f(x)}\right) + \dfrac{1}{h^-}\ln\left(\dfrac{f(x-h^-)}{f(x)}\right)\right). \end{cases} \tag{140}$$

System (140) will become a system of two ordinary difference equations if we project it into the invariants space. To project the system we have to impose one of the following conditions:

$$\Delta x = \begin{cases} -h^-; \\ 0; \\ h^+. \end{cases} \tag{141}$$

A solution of system (140) with one of conditions (141) is the solution of system (139) which is invariant with respect to the operator Y_3.

4) The operator Y_4 has invariants: $u \exp\left(-xt - \frac{2}{3}t^3\right)$, $x+t^2$, τ and $\frac{\Delta x}{2\tau} - t$. Let us search a solution of difference model (139) in the form:

$$u = \exp\left(tx + \frac{2}{3}t^3\right) f(x+t^2).$$

By means of variables

$$y = x + t^2, \qquad y - h_y^- = x - h^- + t^2,$$

$$y + h_y^+ = x + h^+ + t^2, \qquad y + \Delta y = x + \Delta x + (t+\tau)^2$$

we get the following system for the invariant solution of system (139):

$$\begin{cases} \Delta y - \tau^2 = \dfrac{2\tau}{h_y^+ + h_y^-}\left(-\dfrac{h_y^-}{h_y^+}\ln\left(\dfrac{f(y+h_y^+)}{f(y)}\right) + \dfrac{h_y^+}{h_y^-}\ln\left(\dfrac{f(y-h_y^-)}{f(y)}\right)\right); \\ \left(\dfrac{f(y)}{f(y+\Delta y)}\right)^2 \exp\left(-\dfrac{1}{2\tau}\Delta y^2 - \tau(2y+\Delta y) + \dfrac{1}{6}\tau^3\right) = \\ = 1 - \dfrac{4\tau}{h_y^+ + h_y^-}\left(\dfrac{1}{h_y^+}\ln\left(\dfrac{f(y+h_y^+)}{f(y)}\right) + \dfrac{1}{h_y^-}\ln\left(\dfrac{f(y-h_y^-)}{f(y)}\right)\right), \end{cases}$$

where Δy can have one of the following values

$$\Delta y = \begin{cases} -h_y^-; \\ 0; \\ h_y^+. \end{cases} \tag{142}$$

A solution of the above system with one of conditions (142) is an invariant solution for the operator Y_4.

5) Expressions $\frac{x}{\sqrt{t}}$, $t^{-c}u$, $\frac{\tau}{t}$ and $\frac{\Delta x}{x}$ are invariants of the operator Y_5. Let us search a solution of the difference model in the form:

$$u = t^c f\left(\frac{x}{\sqrt{t}}\right).$$

In variables

$$y = \frac{x}{\sqrt{t}}, \qquad y - h_y^- = \frac{x - h^-}{\sqrt{t}},$$

$$y + h_y^+ = \frac{x + h^+}{\sqrt{t}}, \qquad y + \Delta y = \frac{x + \Delta x}{\sqrt{t+\tau}}$$

we get the following system of equations:

$$\begin{cases} \sqrt{1+a}(y+\Delta y)-y \;=\; \dfrac{2a}{h_y^+ + h_y^-}\left(-\dfrac{h_y^-}{h_y^+}\ln\left(\dfrac{f(y+h_y^+)}{f(y)}\right)+\dfrac{h_y^+}{h_y^-}\ln\left(\dfrac{f(y-h_y^-)}{f(y)}\right)\right); \\ (1+a)^{-2c}\left(\dfrac{f(y)}{f(y+\Delta y)}\right)^2 \exp\left(-\dfrac{1}{2}\left((y+\Delta y)\sqrt{\dfrac{1+a}{a}}-y\dfrac{1}{\sqrt{a}}\right)^2\right) \;=\; \\ =\; 1-\dfrac{4a}{h_y^+ + h_y^-}\left(\dfrac{1}{h_y^+}\ln\left(\dfrac{f(y+h_y^+)}{f(y)}\right)+\dfrac{1}{h_y^-}\ln\left(\dfrac{f(y-h_y^-)}{f(y)}\right)\right). \end{cases}$$

Here Δy can have one of the values determined by conditions (142) and a is a constant from the condition $a=\frac{\tau}{t}$ which determines an invariant distribution of time spacing. This condition can be found if we look for a time spacing $\tau = f(t)$ which is invariant with respect to the considered operator Y_5.

6) For the operator Y_6 we have the following invariants:

$$\frac{x}{\sqrt{4t^2+1}}, \qquad (4t^2+1)^{1/4}u\exp\left(\frac{tx^2}{4t^2+1}+\frac{c}{2}\arctan(2t)\right),$$

$$\frac{4t^2+1}{\tau}+4t, \qquad \frac{\Delta x}{x}\frac{4t^2+1}{\tau}-4t.$$

We will seek a solution of the difference model in the form

$$u=(4t^2+1)^{-1/4}\exp\left(-\frac{tx^2}{4t^2+1}-\frac{c}{2}\arctan(2t)\right)f\left(\frac{x}{\sqrt{4t^2+1}}\right).$$

Involving new variables

$$y=\frac{x}{\sqrt{4t^2+1}}, \qquad y-h_y^-=\frac{x-h^-}{\sqrt{4t^2+1}},$$

$$y+h_y^+=\frac{x+h^+}{\sqrt{4t^2+1}}, \qquad y+\Delta y=\frac{x+\Delta x}{\sqrt{4(t+\tau)^2+1}},$$

the system of equations (139) can be presented in the form:

$$\begin{cases} \sqrt{b^2+1}(y+\Delta y)-by \;=\; \dfrac{1}{h_y^+ + h_y^-}\left(-\dfrac{h_y^-}{h_y^+}\ln\left(\dfrac{f(y+h_y^+)}{f(y)}\right)+\dfrac{h_y^+}{h_y^-}\ln\left(\dfrac{f(y-h_y^-)}{f(y)}\right)\right); \\ \sqrt{b^2+1}\left(\dfrac{f(y)}{f(y+\Delta y)}\right)^2 \exp\left(c\arctan\left(\dfrac{1}{b}\right)-b(y^2+(y+\Delta y)^2)+2\sqrt{b^2+1}y(y+\Delta y)\right) \;=\; \\ =\; b-\dfrac{2}{h_y^+ + h_y^-}\left(\dfrac{1}{h_y^+}\ln\left(\dfrac{f(y+h_y^+)}{f(y)}\right)+\dfrac{1}{h_y^-}\ln\left(\dfrac{f(y-h_y^-)}{f(y)}\right)\right). \end{cases}$$

where Δy has one of the values of (142), b is the constant from necessary condition of invariant grid existence

$$2b=4t+\frac{4t^2+1}{\tau}.$$

Therefore, the obtained reduced systems of equations determine the optimal system of invariant solutions for the difference model of the liner heat transfer equation. It means that each invariant solution can be found by transformation of a solution from the optimal system with the help of the corresponding element of the group.

Let us note that, while transforming a solution into other solution, the condition of invariance of a grid will take another form. This form can be got by action of the considered transformation on the initial invariance condition. For example, the transformation corresponding to the operator X_1 with the value of the parameter $-t_0$ gives shift in time $\hat{t} = t - t_0$. Since

$$Ad[\exp(-t_0 X_1)]Y_5 = X_* = Y_5 + 2t_0 X_1,$$

the action of this transformation transfers the invariant solution with respect to the operator Y_5 into the solution which is invariant with respect to the operator X_*. By this transformation the initial invariance condition $\frac{\tau}{t} = a$ will be transformed into the condition $\frac{\tau}{t+t_0} = a$.

Thus, we see that difference model (139) inherits both the group admitted by the original differential equation and the ability to be integrated on a subgroup.

Example of an exact solution.

Among all group invariant solutions for the difference model (139) there is one interesting solution which can be integrated exactly. This is the solution invariant under the action of the operator

$$2\alpha X_2 + X_3, \quad \alpha = const. \tag{143}$$

This operator has three invariants: t and the expressions $u\exp(\frac{x^2}{4(t+\alpha)})$ and $(\frac{\Delta x}{\tau} - \frac{x}{t+\alpha})$. So, we will seek the invariant solution in the form:

$$\left\{ \begin{array}{rcl} u(x,t) & = & f(t)\exp\left(-\dfrac{x^2}{4(t+\alpha)}\right); \\ \dfrac{\Delta x}{\tau} & = & g(t) + \dfrac{x}{t+\alpha}. \end{array} \right.$$

Substituting this form of the solution into system (139), we obtain ordinary difference equations for $f(t)$ and $g(t)$:

$$\left\{ \begin{array}{rcl} f(t+\tau) & = & \left(\dfrac{t+\alpha}{t+\tau+\alpha}\right)^{1/2} f(t); \\ g(t) & = & 0. \end{array} \right. \tag{144}$$

Solving this system, we find the solution of the difference equation

$$u(x,t) = f(0)\left(\frac{\alpha}{t+\alpha}\right)^{1/2} \exp\left(-\frac{x^2}{4(t+\alpha)}\right),$$

and the solution for evolution of a grid:

$$x = x^0\left(\frac{t+\alpha}{\alpha}\right).$$

The obtained solution is the solution of the Cauchy problem with the invariant initial condition

$$u(x,0) = f(0)\exp\left(-\frac{x^2}{4\alpha}\right).$$

If $\alpha = 0$, the fundamental solution of the heat equation

$$u(x,t) = C\left(\frac{1}{t}\right)^{1/2} \exp\left(-\frac{x^2}{4t}\right). \tag{145}$$

is a solution of the difference model. This solution holds on the grid:

$$\Delta x = \frac{\tau}{t}x.$$

In all cases listed above the difference mesh is arbitrary at the initial point, $t = 0$. In this case it will not be uniform on other time layers. If the grid is uniform in x–direction at $t = 0$ ($h_+ = h_- = h$), the steps of the grid in the x–direction will be equal each other on every time layer, but differ from steps on the previous time layer.

It is necessary to mention that the obtained difference invariant solution is the solution of the corresponding differential equation which is invariant with respect to operator (143). As in the differential case the above reduction procedure is applicable to every subalgebra of the algebra (132), and then one can obtain various moving meshes which are self-adaptive to corresponding solutions.

5. Numerical calculations

Here we do not discuss the questions of stability and convergence of the developed schemes. These are hard questions for nonlinear schemes but one of the ways to check them is by computing the numerical solutions to the exact solutions of the original differential equations. Below we present the results of numerical calculation of the invariant solution (145) by means of the invariant model (139). It is necessary to note that the calculations were not done for the equations (144) reduced on the subgroup, but for the nonstationary equations (139). Initial data correspond to the solution (145) with $t = 10$. In Fig.10 we present the evolution of u from invariant initial data by the invariant scheme (139).

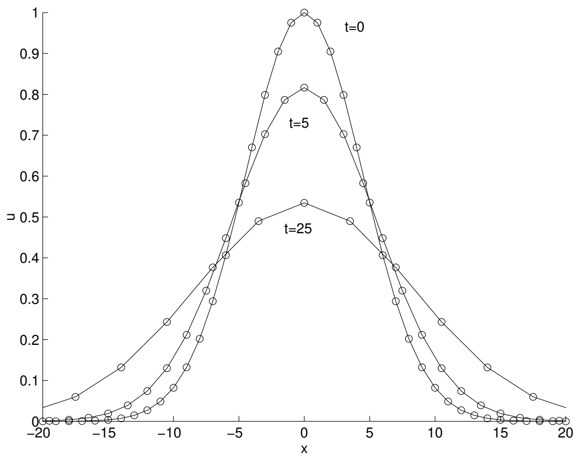

Figure 10: Solution of the invariant model.

Let us note that the difference model (139) gives us practically the exact solution for equation (131). There is only round-off error of computations.

In the Fig.11 the evolution of the grid in the plane (t, x) for the calculation of the solution (145) is shown.

The same calculations for the difference equation (133) on the orthogonal grid (as that in Fig.1) gives a similar picture (see Fig. 12).

In this case the solution does not coincide with exact solution of equation (131). The difference between the exact solution and numerical solution is shown in Fig.13.

We should note that for the calculation on the model (139) we defined the boundary values u on the moving ends of the space interval. For the difference equation (133) we defined the boundary values of u on the ends of the fixed orthogonal grid in accordance with the same solution. The comparison of the two different models shows that for the invariant model even on the decreasing number of the points of the grid on the initial space interval we have greater accuracy than for the noninvariant model.

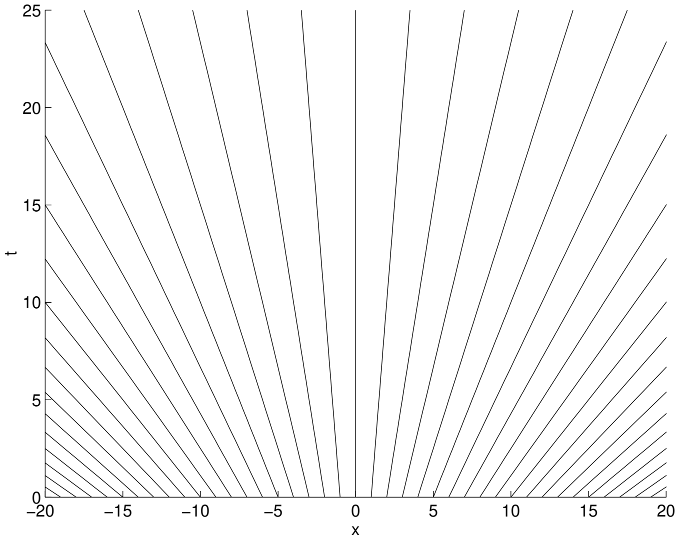

Figure 11: Evolution of the mesh.

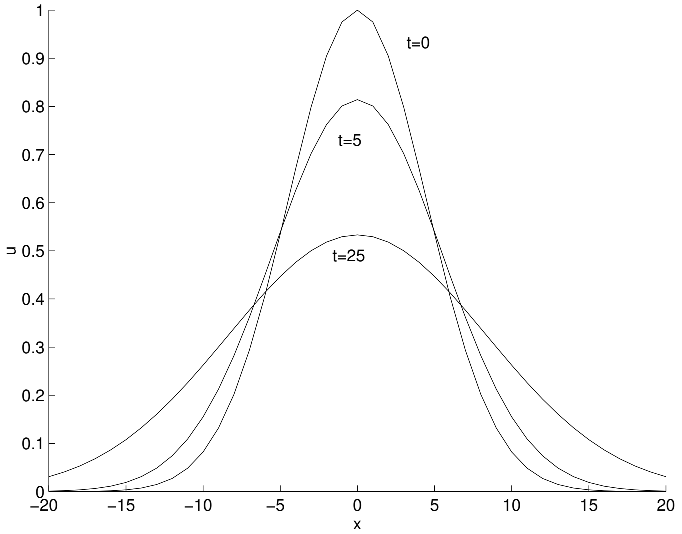

Figure 12: Solution of the noninvariant scheme.

6. The way to stop moving mesh.

The obtained difference models have appropriate adaptive nonorthogonal grids. We can find a way to stop the moving mesh, i.e., an exchange of variables which orthogonalizes the mesh. The differentiation operator of Lagrange type $\frac{d}{dt}$ can be presented in the following form:

$$\frac{d}{dt} = D_t - 2\frac{u_x}{u} D_x ,$$

where

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + \ldots,$$

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + \ldots .$$

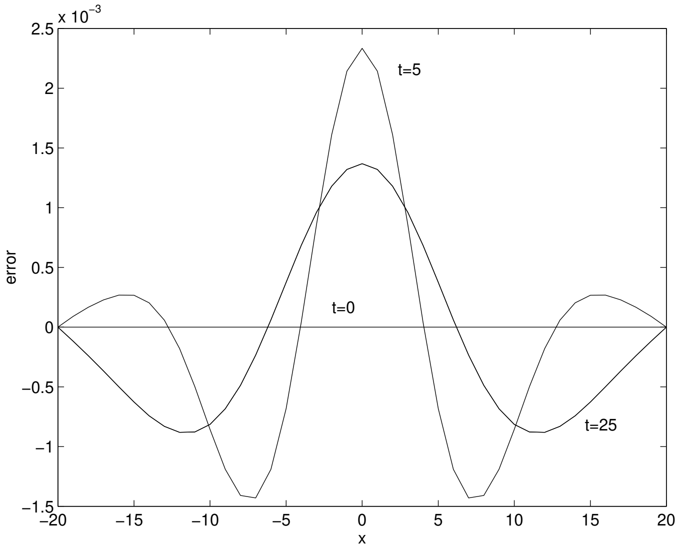

Figure 13: The error of the noninvariant scheme.

The operator $\frac{d}{dt}$ in contrast to the operators D_t and D_x does not commute with the operators of total differentiation with respect to t and x:

$$\left[\frac{d}{dt}\quad,\quad D_x\right] = 2\left(\frac{u_{xt}}{u} - \frac{u_x u_t}{u^2}\right) D_x,$$

$$\left[\frac{d}{dt}\quad,\quad D_t\right] = 2\left(\frac{u_{xx}}{u} - \frac{u_x^2}{u^2}\right) D_x.$$

It is necessary to find an operator of total differentiation with respect to a new space variable s such that

$$\left[\frac{d}{dt}\quad,\quad D_s\right] = 0. \tag{146}$$

The last commuting property is possible if we involve a new dependent variable ρ (density) [14]. The operator $D_s = \frac{1}{\rho}D_x$ satisfies (146) if ρ holds the equation

$$\rho_t - 2\rho\left(\frac{u_{xx}}{u} - \frac{u_x^2}{u^2}\right) - 2\frac{u_x}{u}\rho_x = 0.$$

The new space variable s is introduced with the help of equations

$$\begin{cases} s_x &= \rho; \\ s_t &= 2\rho\dfrac{u_x}{u}. \end{cases}$$

In the variables (t, s) the heat transfer equation will get a form of the system

$$\begin{cases} \dfrac{du}{dt} &= \rho^2\left(u_{ss} - 2\dfrac{u_s^2}{u}\right) + \rho\rho_s u_s; \\ \dfrac{d\rho}{dt} &= 2\rho^3\left(\dfrac{u_{ss}}{u} - \dfrac{u_s^2}{u^2}\right) + 2\rho^2\rho_s\dfrac{u_s}{u}; \end{cases} \tag{147}$$

or the form of conservation laws

$$\left\{\begin{array}{lcl} \frac{d}{dt}\left(\frac{1}{\rho}\right) & = & D_s\left(-2\rho\frac{u_s}{u}\right); \\ \frac{d}{dt}\left(\frac{u}{\rho}\right) & = & D_s(-\rho u_s). \end{array}\right. \tag{148}$$

In the independent variables (t,s) in addition to superposition formulas for $\frac{dU}{dt}$ and $\frac{dX}{dt}$ we have the equation for superposition of ρ:

$$\frac{1}{\rho}\frac{d\rho}{dt} = \frac{\alpha U_1}{\alpha U_1+\beta U_2}\frac{1}{\rho_1}\frac{d\rho_1}{dt} + \frac{\beta U_2}{\alpha U_1+\beta U_2}\frac{1}{\rho_2}\frac{d\rho_2}{dt} + \frac{\alpha\beta U_1U_2}{2(\alpha U_1+\beta U_2)^2}\left(\frac{dX_1}{dt}-\frac{dX_2}{dt}\right)^2,$$

where Lagrangian derivatives are taken on trajectories of the corresponding solutions. This equation yields the superposition principle for the Lagrange type derivatives of U and ρ only:

$$\left(\frac{U}{\rho}\frac{d\rho}{dt}-\frac{dU}{dt}\right) = \alpha\left(\frac{U_1}{\rho_1}\frac{d\rho_1}{dt}-\frac{dU_1}{dt}\right) + \beta\left(\frac{U_2}{\rho_2}\frac{d\rho_2}{dt}-\frac{dU_2}{dt}\right).$$

In the new coordinate system we can consider x as a potential which satisfy the system of equations

$$\left\{\begin{array}{lcl} x_t & = & -2\rho\frac{u_s}{u}; \\ x_s & = & \frac{1}{\rho}; \end{array}\right. \tag{149}$$

System (147) in the space of independent variables (t,s) and extended set of dependent variables (u,ρ,x) admits a group of point transformations determined by the following infinitesimal operators:

$$X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = 2t\frac{\partial}{\partial x} - xu\frac{\partial}{\partial u}, \tag{150}$$

$$X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + s\frac{\partial}{\partial s}, \quad X_5 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - (x^2+2t)u\frac{\partial}{\partial u} - 4t\rho\frac{\partial}{\partial \rho},$$

$$X_6 = u\frac{\partial}{\partial u}, \qquad X^* = f(s)\frac{\partial}{\partial s} + \rho f'(s)\frac{\partial}{\partial \rho},$$

where $f(s)$ is an arbitrary function of s.

In the independent variables (t,s) operators $X_1 - X_6$ are operators of Lie algebra factorized by the operator X^*. Condition (29) of grid orthogonality and condition (28) of space grid uniformity hold and it gives an opportunity to construct a difference model which is invariant with respect to operators $X_1 - X_6$ on the orthogonal grid.

Let us write system (147), (149) in the form of differential invariants. In the space of variables $(t,x,s,u,\rho,dt,dx,ds,du,d\rho,u_s,\rho_s,x_s,u_{ss})$ they are the following expressions:

$$J^1 = x_s\rho, \qquad J^2 = \frac{\rho}{ds}(dx + 2\rho\frac{u_s}{u}dt), \qquad J^3 = \frac{(ds)^2}{\rho^2 dt},$$

$$J^4 = \frac{(ds)^2}{\rho^3}\left(\frac{d\rho}{dt} - \frac{\rho_s ds}{dt} - 2\rho^3\left(\frac{u_{ss}}{u} - \frac{u_s^2}{u^2}\right) - 2\rho^2\rho_s\frac{u_s}{u}\right),$$

$$J^5 = \left(\frac{ds}{\rho}\right)^2\left(-\frac{2}{u}\frac{du}{dt} - \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + 2\rho^2\left(\frac{u_{ss}}{u} - \frac{u_s^2}{u^2}\right) + 2\rho\rho_s\frac{u_s}{u}\right).$$

In terms of these invariants we rewrite system (147), (149) as follows:

$$
\begin{cases}
\dfrac{du}{dt} & = \rho^2\left(u_{ss} - \dfrac{u_s^2}{u}\right) + \rho\rho_s u_s; \\
\dfrac{d\rho}{dt} & = 2\rho^3\left(\dfrac{u_{ss}}{u} - \dfrac{u_s^2}{u^2}\right) + 2\rho^2\rho_s\dfrac{u_s}{u}; \\
\dfrac{dx}{dt} & = -2\rho\dfrac{u_s}{u}; \\
\dfrac{\partial x}{\partial s} & = \dfrac{1}{\rho}.
\end{cases}
\tag{151}
$$

Now we can find a system of equations which approximates (151) and is invariant with respect to the set of operators (150). We can use the 6-point invariant stencil (Fig.4). We have the following invariants for the set of operators (150) where the operator X^* is changed on its difference analog

$$
X^* = f(s)\frac{\partial}{\partial s} + \rho D_{+s}(f(s))\frac{\partial}{\partial \rho}
$$

in the corresponding to the chosen stencil space $(t, \hat{t}, s, h_s^+, h_s^-, x, \hat{x}, h_x^+, h_x^-, \hat{h}_x^+, \hat{h}_x^-, u, \hat{u}, u_+, u_-, \hat{u}_+, \hat{u}_-, \rho, \hat{\rho}, \rho_+, \rho_-, \hat{\rho}_+, \hat{\rho}_-)$:

$$
I^1 = \frac{h_x^+}{h_x^-}, \qquad I^2 = \frac{\hat{h}_x^+}{\hat{h}_x^-}, \qquad I^3 = \frac{h_x^+\hat{h}_x^+}{\tau}, \qquad I^4 = \frac{\tau^{1/2}}{h_x^+}\frac{\hat{u}}{u}\exp\left(\frac{1}{4}\frac{\Delta x^2}{\tau}\right),
$$

$$
I^5 = \frac{1}{4}\frac{h_x^{+2}}{\tau} - \frac{h_x^{+2}}{h^+ + h^-}\left(\frac{1}{h_x^+}\ln\frac{u_+}{u} + \frac{1}{h_x^-}\ln\frac{u_-}{u}\right),
$$

$$
I^6 = \frac{1}{4}\frac{\hat{h}_x^{+2}}{\tau} + \frac{\hat{h}_x^{+2}}{\hat{h}_x^+ + \hat{h}_x^-}\left(\frac{1}{\hat{h}_x^+}\ln\frac{\hat{u}_+}{\hat{u}} + \frac{1}{\hat{h}_x^-}\ln\frac{\hat{u}_-}{\hat{u}}\right),
$$

$$
I^7 = \frac{\Delta x h_x^+}{\tau} + \frac{2h_x^+}{h_x^+ + h_x^-}\left(\frac{h_x^-}{h_x^+}\ln\frac{u_+}{u} - \frac{h_x^+}{h_x^-}\ln\frac{u_-}{u}\right),
$$

$$
I^8 = \frac{\Delta x \hat{h}_x^+}{\tau} + \frac{2\hat{h}_x^+}{\hat{h}_x^+ + \hat{h}_x^-}\left(\frac{\hat{h}_x^-}{\hat{h}_x^+}\ln\frac{\hat{u}_+}{\hat{u}} - \frac{\hat{h}_x^+}{\hat{h}_x^-}\ln\frac{\hat{u}_-}{\hat{u}}\right),
$$

$$
I^9 = \frac{\hat{\rho}_-}{\rho_-}, \qquad I^{10} = \frac{\hat{\rho}}{\rho}, \qquad I^{11} = \frac{\hat{\rho}_+}{\rho_+}, \qquad I^{12} = \frac{h_s^+}{\rho h_x^+}, \qquad I^{13} = \frac{h_s^-}{\rho_- h_x^-}.
$$

With the help of these invariants we can write difference model in the form of the following system of evolution difference equations (we present here only one invariant difference model which corresponds to the system (139) in variables (t, x)):

$$
\begin{cases}
\Delta x = 2\tau\dfrac{-\dfrac{h_s^-}{h_s^+}\dfrac{\rho}{\rho_-}\ln\dfrac{u_+}{u} + \dfrac{h_s^+}{h_s^-}\dfrac{\rho_-}{\rho}\ln\dfrac{u_-}{u}}{\dfrac{h_s^+}{\rho} + \dfrac{h_s^-}{\rho_-}}; \\
\left(\dfrac{u}{\hat{u}}\right)^2\exp\left(-\dfrac{1}{2}\dfrac{\Delta x^2}{\tau}\right) = 1 - 4\tau\dfrac{\dfrac{\rho}{h_s^+}\ln\dfrac{u_+}{u} + \dfrac{\rho_-}{h_s^-}\ln\dfrac{u_-}{u}}{\dfrac{h_s^+}{\rho} + \dfrac{h_s^-}{\rho_-}}; \\
\hat{\rho}\hat{h}_x^+ = \rho h_x^+ = h_s^+.
\end{cases}
$$

In the case of the uniform grid $(h_s^+ = h_s^- = h_s)$ this model could be simplified as follows:

$$\left\{\begin{array}{rcl} \Delta x & = & 2\tau\dfrac{-\rho^2 \ln \dfrac{u_+}{u} + \rho_-^2 \ln \dfrac{u_-}{u}}{h_s(\rho+\rho_-)}; \\ \left(\dfrac{u}{\hat{u}}\right)^2 \exp\left(-\dfrac{1}{2}\dfrac{\Delta x^2}{\tau}\right) & = & 1 - \dfrac{4\tau\rho\rho_-}{h_s^2(\rho+\rho_-)}\left(\rho \ln\dfrac{u_+}{u} + \rho_- \ln\dfrac{u_-}{u}\right); \\ \hat{\rho}\hat{h}_x^+ & = & \rho h_x^+ \quad = \quad h_s. \end{array}\right.$$

It is important to notice that in all cases moving in (x, t)- plane meshes could be stopped by using the Lagrange type coordinate system with one additional dependent variable (for a way to introduce Lagrange type coordinate systems see, for example, [14]).

7. Connection with Burgers' equation.

It is well know that Burgers' equation

$$v_t + v v_x = v_{xx} \tag{152}$$

is connected with the linear heat equation by Hopf transformation

$$v = -2\frac{u_x}{u}. \tag{153}$$

For Burgers' equation it is possible to introduce a potential w defined by the system

$$\left\{\begin{array}{rcl} w_x & = & v; \\ w_t & = & v_x - \dfrac{1}{2}v^2; \end{array}\right.$$

and get Burgers' equation for the potential

$$w_t + \frac{1}{2}w_x^2 = w_{xx}. \tag{154}$$

The last equation is connected with the linear heat equation by the point transformation

$$w = -2\ln u. \tag{155}$$

Here we consider connection between discrete models for Burgers' equations and the linear heat equation. Equation (152) admits the group of transformations:

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v}, \qquad X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}, \\ X_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + (x - tv)\frac{\partial}{\partial v}, \qquad X^* = \left(a\exp\left(\frac{w}{2}\right)\right)_x \frac{\partial}{\partial v}, \end{gathered} \tag{156}$$

where w is the potential for Burgers' equation and $a(t, x)$ is an arbitrary solution of heat transfer equation $a_t = a_{xx}$.

Let us write Burgers' equation in the form of differential invariants. In the space of variables $(t, x, v, v_x, v_{xx}, dt, dx, dv)$ there are three invariants:

$$J^1 = (dx - vdt)v_{xx}^{1/3}, \qquad J^2 = v_{xx}^{2/3}dt, \qquad J^3 = \frac{dv - v_x(dx - vdt)}{v_{xx}^{1/3}}.$$

We can rewrite the Burgers' equation in terms of these invariants:

$$\left\{\begin{array}{rcl} J^1 & = & 0; \\ J^2 & = & J^3; \end{array}\right.$$

or

$$\begin{cases} \dfrac{dx}{dt} &=& v; \\ \\ \dfrac{dv}{dt} &=& v_{xx}. \end{cases} \tag{157}$$

As a next step we find difference invariants of the set of operators $X_1 - X_5$. These invariants will be used for approximation of system (157). We will use the 6-point stencil of Fig.3. This stencil defines the space $(t, \hat{t}, x, \hat{x}, h^+, h^-, \hat{h}^+, \hat{h}^-, v, v_+, v_-, \hat{v}, \hat{v}_+, \hat{v}_-)$, in which group (156) has the following finite-difference invariants:

$$I^1 = \frac{h^+}{h^-}, \qquad I^2 = \frac{\hat{h}^+}{\hat{h}^-}, \qquad I^3 = \frac{\hat{h}^+ h^+}{\tau},$$

$$I^4 = h^- h^+ (v_x - v_{\bar{x}}); \qquad I^5 = \hat{h}^- \hat{h}^+ (\hat{v}_x - \hat{v}_{\bar{x}}), \qquad I^6 = h^+ \left(\frac{\Delta x}{\tau} - v \right),$$

$$I^7 = \hat{h}^+ \left(\frac{\Delta x}{\tau} - \hat{v} \right), \qquad I^8 = h^{+2} \left(\frac{1}{\tau} + v_x \right), \qquad I^9 = \hat{h}^{+2} \left(\frac{1}{\tau} + \hat{v}_x \right),$$

where $\Delta x = \hat{x} - x$, $v_x = \dfrac{v_+ - v}{h^+}$, $v_{\bar{x}} = \dfrac{v - v_-}{h^-}$.

These invariants let us write the following invariant difference models:

A.) with explicit evolution of the grid:

1. explicit model for the evolution of the solution:

$$\begin{cases} \Delta x &=& \tau v; \\ \\ \dfrac{\hat{v} - v}{\tau} \dfrac{\hat{h}^+}{h^+} \dfrac{\hat{h}^-}{h^-} &=& v_{x\bar{x}} &=& \dfrac{2}{h^+ + h^-}(v_x - v_{\bar{x}}); \end{cases}$$

2. implicit model for the evolution of the solution:

$$\begin{cases} \Delta x &=& \tau v; \\ \\ \dfrac{\hat{v} - v}{\tau} \dfrac{h^+}{\hat{h}^+} &=& \hat{v}_{x\bar{x}} &=& \dfrac{2}{\hat{h}^+ + \hat{h}^-}(\hat{v}_x - \hat{v}_{\bar{x}}); \end{cases}$$

3. implicit model for the evolution of the solution:

$$\begin{cases} \Delta x &=& \tau v; \\ \\ \dfrac{\hat{v} - v}{\tau} &=& \dfrac{2}{h^+ + h^-}(\hat{v}_x - \hat{v}_{\bar{x}}). \end{cases}$$

B.) with implicit approximation for the grid evolution:

1. explicit evolution of the solution:

$$\begin{cases} \Delta x &=& \tau \hat{v}; \\ \\ \dfrac{\hat{v} - v}{\tau} \dfrac{\hat{h}^+}{h^+} &=& v_{x\bar{x}}; \end{cases}$$

2. implicit evolution of the solution:

$$\begin{cases} \Delta x &=& \tau \hat{v}; \\ \\ \dfrac{\hat{v} - v}{\tau} \dfrac{h^+}{\hat{h}^+} \dfrac{h^-}{\hat{h}^-} &=& \hat{v}_{x\bar{x}}; \end{cases}$$

3. implicit evolution of the solution:

$$\begin{cases} \Delta x &= \tau \hat{v}; \\ \dfrac{\hat{v}-v}{\tau} &= \dfrac{2}{\hat{h}^{+}+\hat{h}^{-}}(v_x - v_{\bar{x}}). \end{cases}$$

Burgers' equation for the potential.

Here we construct a difference model for the Burgers' equation for the potential (154). This equation admits the symmetry group

$$\begin{gathered} X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial w}, \qquad X_4 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\ X_5 = t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + \left(\frac{1}{2}x^2 + t\right)\frac{\partial}{\partial w}, \qquad X_6 = \frac{\partial}{\partial w}, \qquad X^* = a\exp\left(\frac{w}{2}\right)\frac{\partial}{\partial w}, \end{gathered} \tag{158}$$

where X^* is a point symmetry operator (in contrast to the Burgers' equation (152)). In the space of variables $(t, x, w, w_x, w_{xx}, dt, dx, dw)$ there are two differential invariants:

$$J^1 = \frac{dx - w_x dt}{dt^{1/2}}, \qquad J^2 = dw - \frac{1}{2}\frac{dx^2}{dt} - w_{xx}dt.$$

By means of these invariants the potential Burgers equation can be written as a system

$$\begin{cases} J^1 &= 0; \\ J^2 &= 0; \end{cases}$$

or

$$\begin{cases} \dfrac{dx}{dt} &= w_x; \\ \dfrac{dw}{dt} &= w_{xx} + \dfrac{1}{2}(w_x)^2. \end{cases} \tag{159}$$

Let us construct invariant difference equations which approximates system (159) on the 6-point stencil (Fig.3).

The following expressions

$$\begin{gathered} I^1 = \frac{h^+}{h^-}, \quad I^2 = \frac{\hat{h}^+}{\hat{h}^-}, \quad I^3 = \frac{h^+\hat{h}^+}{\tau}, \quad I^4 = \frac{h^{+2}}{\tau}\exp\left(\hat{w} - w - \frac{\Delta x^2}{2\tau}\right), \\ I^5 = h^{+2}\left(\frac{1}{2\tau} + \frac{w_x - w_{\bar{x}}}{h^+ + h^-}\right), \qquad I^6 = \hat{h}^{+2}\left(\frac{1}{2\tau} - \frac{\hat{w}_x - \hat{w}_{\bar{x}}}{\hat{h}^+ + \hat{h}^-}\right), \\ I^7 = h^+\left(\frac{\Delta x}{\tau} - \frac{h^- w_x + h^+ w_{\bar{x}}}{h^+ + h^-}\right), \qquad I^8 = \hat{h}^+\left(\frac{\Delta x}{\tau} - \frac{\hat{h}^-\hat{w}_x + \hat{h}^+\hat{w}_{\bar{x}}}{\hat{h}^+ + \hat{h}^-}\right). \end{gathered}$$

are all the invariants for the set of the operators $X_1 - X_6$ of group (156) in the space corresponding to the chosen stencil $(t, \hat{t}, x, \Delta x, h^+, h^-, \hat{h}^+, \hat{h}^-, w, \hat{w}, w_+, w_-, \hat{w}_+, \hat{w}_-)$.

Approximating system (159) by these invariants, we can get various systems of evolution equations. For example, we present an explicit difference model for (159):

$$\begin{cases} \Delta x &= \tau\dfrac{h^- w_x + h^+ w_{\bar{x}}}{h^+ + h^-}; \\ \exp\left(\hat{w} - w - \dfrac{\Delta x^2}{2\tau}\right) &= 1 + \tau w_{x\bar{x}}; \end{cases}$$

Note that the last model is connected with model (139) for the heat transfer equation by the point transformation of the dependent variable (155) as the original differential equations (continuous limits).

5.5. If $Q = \delta u$, $\delta = \pm 1$, the equation

$$u_t = u_{xx} + \delta u \tag{160}$$

can be transformed into equation (131) by the exchange

$$\bar{u} = ue^{-\delta t}. \tag{161}$$

Reversing this exchange, one can get an invariant model for equation (160) from an invariant model for the heat transfer equation without a source (131).

5.6. $Q = \delta = const$. The equation has the form:

$$u_t = u_{xx} + \delta. \tag{162}$$

The case $Q = 0$, which was considered in point 5.4, is a partial case for $Q = \delta = const$, but the constant source can be excluded by the evident exchange:

$$\bar{u} = u - \delta t. \tag{163}$$

It means that we can get a difference model for (162) from the model for (131).

Remarks. Thus, we constructed difference models for the equation

$$u_t = (K(u)u_x)_x + Q(u). \tag{164}$$

Note that for different cases of coefficients $K(u)$ and $Q(u)$ taken in accordance with the group classification of equation (164) we have obtained different discrete models: for some cases $K(u)$ and $Q(u)$ we had to construct discrete models not for equation (164), but for the equivalent system of Lagrangian type

$$\left\{ \begin{array}{rcl} \dfrac{dx}{dt} & = & \varphi(u, u_x); \\ \\ \dfrac{du}{dt} & = & \psi(u, u_x, u_{xx}). \end{array} \right. \tag{165}$$

Consideration of equation (164) in the form of system (165) at the very beginning provides us with the classification by right parts φ and ψ. In this case there is one to one correspondence between the coordinate systems for continuous and discrete cases of the group classification. In particular, if the symmetry of equation (164) does not require the usage of Lagrangian type grid (evolving in time grid), we have $\varphi \equiv 0$. In this case the following classification of system (165) and corresponding difference equations on orthogonal grid is carried out on $\psi(u, u_x, u_{xx})$.

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