A Note on the Construction of Crouch-Grossman Methods*

Arne Marthinsen† Brynjulf Owren‡

August 26, 1998

Abstract

Numerical integration methods based on rigid frames were introduced by Crouch and Grossman. The order theory of these methods were later analysed by Owren and Marthinsen. The resulting order conditions are difficult to solve due to nonlinear relations on the weights of the methods. In this paper we propose a variant of the Crouch-Grossman method that uses modified vector fields so that the order conditions of this new method coincide with the classical order conditions for Runge-Kutta methods.

AMS Subject Classification: 65L06, 34A50

Key Words: geometric integration, numerical integration of ordinary differential equations on manifolds, rigid frame, numerical analysis, Lie groups

1 Introduction

Integration methods based on rigid frames were introduced by Crouch and Grossman [1]. These methods assume that smooth vector fields $E_1, \dots, E_d$ on a differentiable manifold $\mathcal{M}$ are available such that the differential equation can be written in the form

$$y' = F(y) = \sum_{i=1}^{d} f_i(y) E_i, \qquad y(0) = y_0 \in \mathcal{M}, \tag{1}$$

where $f : \mathbb{R} \times \mathcal{M} \to \mathbb{R}$. Let F_Y denote the vector field obtained by freezing F at a point $Y \in \mathcal{M}$, relative to the frame $E_1, \dots, E_d$:

$$F_Y = \sum_{i=1}^{d} f_i(Y) E_i, \qquad Y \in \mathcal{M},$$

and let $F^j = hF_{Y_j}$. We can now write the explicit Crouch-Grossman method that advances the solution from $y_n \approx y(t_n)$ to $y_{n+1} \approx y(t_n + h)$ as follows:

$$\begin{aligned} Y_i &= \exp(a_{i,i-1}F^{i-1}) \circ \cdots \circ \exp(a_{i,1}F^1) y_n, \qquad i = 1, \dots, s, \\ y_{n+1} &= \exp(b_s F^s) \circ \cdots \circ \exp(b_1 F^1) y_n, \end{aligned} \tag{2}$$

*This work was in part sponsored by The Norwegian Research Council under contract no. 111038/410, through the SYNODE project. WWW: `http://www.math.ntnu.no/num/synode/`

†Email: `Arne.Marthinsen@math.ntnu.no`, WWW: `http://www.math.ntnu.no/~arnema/`

‡Email: `Brynjulf.Owren@math.ntnu.no`, WWW: `http://www.math.ntnu.no/~bryn/`

where $a_{i,j}$ and b_j are integration parameters describing the particular method and $\exp(tX)p$ is the t-flow of the vector field X through $p \in \mathcal{M}$ [5].

The method defined by (2) is a one-step method, and we can write $y_{k+1} = \Psi(h, y_k)$. The exact solution of (1) through y_k can be written as $\exp(tF)y_k$. We define the order of the Crouch-Grossman method (2) to be the greatest integer p for which

$$|\psi \circ \exp(hF)y - \psi \circ \Psi(h, y)| = \mathcal{O}(h^{p+1})$$

for all smooth functions $\psi : \mathcal{M} \to \mathbb{R}$ and $y \in \mathcal{M}$. It was shown in [6] that the order conditions resulting from (2) are fairly complicated. In addition to the conditions arising from classical Runge-Kutta methods, additional ones appear due to non-commutativity of the flows. While the classical order conditions are linear in the weights, b, the new relations are highly nonlinear in these variables. Crouch and Grossman [1] constructed schemes of order three with three stages. Owren and Marthinsen [6] presented a scheme of order four with five stages and showed that there do not exist Crouch-Grossman methods of order four with four stages. In [3], the coefficients of fourth, fifth and sixth order Crouch-Grossman methods are presented.

Flow computations are normally computationally expensive, and an s-stage, explicit Crouch-Grossman method needs to compute $\frac{1}{2}s(s+1)$ flows at each step. An immediate idea, as exploited by Munthe-Kaas and Owren [5], is to use the Baker-Campbell-Hausdorff (BCH) formula to reduce this number to s. In Section 2, we modify the Crouch-Grossman method in such a way that classical Runge-Kutta coefficients yield the same order when applied in this modified setting as in the classical one. These two approaches can either be applied separately or together, the latter leading to methods of Munthe-Kaas type (see e.g. Section 2 or [4]).

1.1 Notation

Let X and Y be vector fields on the differentiable manifold $\mathcal{M}$. The Baker–Campbell–Hausdorff formula gives the vector field Z where

$$\exp(Z) = \exp\left(\mathcal{B}(X, Y)\right) = \exp(X) \circ \exp(Y).$$

The conditions on X and Y under which this formula is well-defined is discussed in [7]. We extend the notation to allow for more compositions:

$$Z = \mathcal{B}(X_1, \ldots, X_\ell) = \mathcal{B}\Big(X_1, \mathcal{B}(X_2, \ldots, \mathcal{B}(X_{\ell-1}, X_\ell) \cdots)\Big),$$

is the vector field that satisfies

$$\exp(Z) = \exp\left(\mathcal{B}(X_1, X_2, \ldots, X_\ell)\right) = \exp(X_1) \circ \exp(X_2) \circ \cdots \circ \exp(X_\ell),$$

when $X_1, \ldots, X_\ell$ are vector fields. Throughout this note we assume that the stepsize h, and hence also the vector fields, is chosen sufficiently small such that the BCH formula and its compositions are defined.

2 Modified Vector Fields

We now address the problem of constructing numerical methods of Crouch-Grossman type without having to solve the order conditions as described in [6]. In fact, we seek $\widetilde{F}^k = \widetilde{F}^k(F^1, \ldots, F^k)$, $k = 1, \ldots, s$, so that

$$\begin{aligned} Y_i &= \exp(a_{i,i-1}\widetilde{F}^{i-1}) \circ \cdots \circ \exp(a_{i,1}\widetilde{F}^1)y_n, \qquad 1 \le i \le s, \\ y_{n+1} &= \exp(b_s\widetilde{F}^s) \circ \cdots \circ \exp(b_1\widetilde{F}^1)y_n, \end{aligned} \tag{3}$$

is an approximation of order p to $y(t_{n+1})$ if the coefficients describe a classical Runge-Kutta method of order p. Note that the frozen vector fields F^j, $j = 3, \ldots, s$, differ from the vector fields in (2) in that they are frozen at different places. The new vector fields, $\widetilde{F}^k$, will be linear combinations of $F^1, \ldots, F^k$ and commutators between these vector fields.

Munthe-Kaas [4] showed that the following algorithm yields numerical approximations to the solution of (1) of at least the same order as the underlying Runge-Kutta method:

$$\left.\begin{array}{rcl} u_i & = & \sum_{j=1}^{i-1} a_{ij} \widehat{F}_j \\ Y_i & = & \exp(u_i) y_n \\ F^i & = & hF_{Y_i} \\ \widehat{F}_i & = & \operatorname{dexp}_{u_i}^{-1}(F^i) \end{array}\right\} \qquad i = 1, \ldots, s$$

$$\begin{array}{rcl} u_{s+1} & = & \sum_{j=1}^{s} b_j \widehat{F}_j \\ y_{n+1} & = & \exp(u_{s+1}) y_n . \end{array}$$

By considering the stage values, Y_i, of the two methods, it is natural to look for $\widetilde{F}^{i-1}$ such that

$$\exp(a_{i,i-1}\widetilde{F}^{i-1}) \circ \exp(a_{i,i-2}\widetilde{F}^{i-2}) \circ \cdots \circ \exp(a_{i,1}\widetilde{F}^1) = \exp(u_i).$$

After composing with $\exp(-a_{i,j}\widetilde{F}^j)$, $j = 1, \ldots, i-2$, from the right, we get

$$\exp(a_{i,i-1}\widetilde{F}^{i-1}) = \exp(u_i) \circ \exp(-a_{i,1}\widetilde{F}^1) \circ \cdots \circ \exp(-a_{i,i-2}\widetilde{F}^{i-2}).$$

This expression gives us the corrected vector field $\widetilde{F}^k$, expressed in terms of u_k and $\widetilde{F}^j$, $j = 1, \ldots, k-1$:

$$a_{i+1,i}\widetilde{F}^i = \mathcal{B}\big(u_i, -a_{i+1,1}\widetilde{F}^1, \ldots, -a_{i+1,i-1}\widetilde{F}^{i-1}\big), \qquad i = 1, \ldots, s. \tag{4}$$

Here

$$u_i = \sum_{j=1}^{i-1} a_{i+1,j} \widehat{F}_j, \qquad \widehat{F}_j = \operatorname{dexp}_{u_j}^{-1}(F^j),$$

and, to simplify notation, we define

$$a_{s+1,i} = b_i, \qquad i = 1, \ldots, s.$$

If we use the BCH formula to reduce the number of flow computations per step to s, we obtain the stages

$$\widetilde{Z}_i = \mathcal{B}\big(a_{i,i-1}\widetilde{F}^{i-1}, \ldots, a_{i,1}\widetilde{F}^1\big), \qquad i = 1, \ldots, s,$$

and

$$\widetilde{Z}_{s+1} = \mathcal{B}\big(b_s\widetilde{F}^s, \ldots, b_1\widetilde{F}^1\big),$$

which are equivalent to the stages of the Munthe-Kaas method implemented with the same Runge-Kutta coefficients.

Examples. Consider the classical Runge-Kutta method described by the Butcher tableu

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

Direct application of algorithm (4) yields:

$$
\begin{aligned}
\widetilde{F}^1 &= F^1 \\
\widetilde{F}^2 &= F^2 - \tfrac{1}{4}[F^1, F^2] + \tfrac{1}{48}[F^1, [F^1, F^2]] \\
\widetilde{F}^3 &= F^3 - \tfrac{1}{4}[F^2, F^3] + \tfrac{1}{48}[F^2, [F^2, F^3]] - \tfrac{1}{16}[F^3, [F^1, F^2]] \\
\widetilde{F}^4 &= F^4 + \tfrac{1}{6}[F^1, F^2] + \tfrac{1}{6}[F^1, F^3] + \tfrac{1}{12}[F^1, F^4] + \tfrac{1}{3}[F^2, F^3] + \tfrac{1}{6}[F^2, F^4] - \tfrac{1}{3}[F^3, F^4] - \tfrac{7}{216}[F^1, [F^1, F^2]] \\
&\quad + \tfrac{1}{108}[F^1, [F^1, F^3]] + \tfrac{1}{216}[F^1, [F^1, F^4]] + \tfrac{1}{27}[F^2, [F^1, F^2]] - \tfrac{1}{216}[F^2, [F^1, F^3]] + \tfrac{1}{54}[F^2, [F^1, F^4]] \\
&\quad - \tfrac{5}{108}[F^2, [F^2, F^3]] + \tfrac{1}{54}[F^2, [F^2, F^4]] + \tfrac{35}{216}[F^3, [F^1, F^2]] + \tfrac{1}{27}[F^3, [F^1, F^3]] - \tfrac{5}{216}[F^3, [F^1, F^4]] \\
&\quad + \tfrac{2}{27}[F^3, [F^2, F^3]] - \tfrac{5}{108}[F^3, [F^2, F^4]] + \tfrac{1}{54}[F^3, [F^3, F^4]] + \tfrac{5}{108}[F^4, [F^1, F^2]] + \tfrac{5}{108}[F^4, [F^1, F^3]] \\
&\quad + \tfrac{1}{432}[F^4, [F^1, F^4]] + \tfrac{1}{108}[F^4, [F^2, F^3]] + \tfrac{1}{216}[F^4, [F^2, F^4]] + \tfrac{1}{216}[F^4, [F^3, F^4]].
\end{aligned}
$$

The solution is advanced by computing

$$y_{n+1} = \exp\left(\tfrac{1}{6}\widetilde{F}^4\right)\exp\left(\tfrac{1}{3}\widetilde{F}^3\right)\exp\left(\tfrac{1}{3}\widetilde{F}^2\right)\exp\left(\tfrac{1}{6}\widetilde{F}^1\right) y_n.$$

The Hall basis of order 3 generated by $F^1, \ldots, F^4$, all with grade 1, has dimension 30. However, by organizing the computations as follows, only 15 commutators need to be computed. For notational convenience let $F^{i_1 i_2 \cdots i_k} = [F^{i_1}, [F^{i_2}, \cdots [F^{i_{k-1}}, F^{i_k}] \cdots]]$. We then find

$\widetilde{F}^1$: no commutators

$\widetilde{F}^2$: $F^{12} = [F^1, F^2]$ and $F^{112} = [F^1, F^{12}]$

$\widetilde{F}^3$: $F^{23} = [F^2, F^3]$, $F^{223} = [F^2, F^{23}]$ and $F^{312} = [F^3, F^{12}]$

$\widetilde{F}^4$: $F^{13} = [F^1, F^3]$, $F^{14} = [F^1, F^4]$, $F^{24} = [F^2, F^4]$, $F^{34} = [F^3, F^4]$ as well as

$$
\begin{array}{ll}
\left[\tfrac{1}{27}F^2 + \tfrac{5}{108}F^4, F^{12}\right] & \left[\tfrac{1}{108}F^1 - \tfrac{1}{216}F^2 + \tfrac{1}{27}F^3 + \tfrac{5}{108}F^4, F^{13}\right] \\
\left[\tfrac{1}{216}F^1 + \tfrac{1}{54}F^2 - \tfrac{5}{216}F^3 + \tfrac{1}{432}F^4, F^{14}\right] & \left[\tfrac{2}{27}F^3 + \tfrac{1}{108}F^4, F^{23}\right] \\
\left[\tfrac{1}{54}F^2 - \tfrac{5}{108}F^3 + \tfrac{1}{216}F^4, F^{24}\right] & \left[\tfrac{1}{54}F^3 + \tfrac{1}{216}F^4, F^{34}\right].
\end{array}
$$

Since $F^j = hF_{Y_j} = hF_{y_n} + \mathcal{O}(h^2)$, we have that $[F^k, F^\ell] = \mathcal{O}(h^3)$ when $k \neq \ell$, and so the commutator $F^{i_1 i_2 \cdots i_k} = \mathcal{O}(h^{k+1})$. Hence it suffices to consider a Hall basis of order $q-1$ when the underlying Runge-Kutta method has order q and all the generators have grade 1.

Based on graded free Lie algebras, Munthe-Kaas and Owren [5] showed how to introduce a new basis $Q_1, \ldots, Q_s$ with $Q_i = \mathcal{O}(h^{q_i})$ and $q_i > 0$ as large as possible. Using the Q_i instead of the F^i, the total number of commutators in the $\widetilde{F}^\ell$ is significantly reduced. Only the terms of order less or equal to the method's order are interesting. One choice of variables is

$$
\begin{array}{ll}
Q_1 = F^1 = \mathcal{O}(h) & Q_3 = F^3 - F^2 = \mathcal{O}(h^3) \\
Q_2 = F^2 - F^1 = \mathcal{O}(h^2) & Q_4 = F^4 - 2F^2 + F^1 = \mathcal{O}(h^3).
\end{array}
$$

In terms of these new variables, the vector fields read

$$
\begin{aligned}
\widetilde{F}^1 &= Q_1 \\
\widetilde{F}^2 &= Q_1 + Q_2 - \tfrac{1}{4}[Q_1, Q_2] + \tfrac{1}{48}[Q_1, [Q_1, Q_2]] \\
\widetilde{F}^3 &= Q_1 + Q_2 + Q_3 - \tfrac{1}{4}[Q_1, Q_3] - \tfrac{1}{16}[Q_1, [Q_1, Q_2]] \\
\widetilde{F}^4 &= Q_1 + 2Q_2 + Q_4 + \tfrac{1}{3}[Q_1, Q_2] + \tfrac{5}{6}[Q_1, Q_3] - \tfrac{1}{12}[Q_1, Q_4] + \tfrac{11}{36}[Q_1, [Q_1, Q_2]].
\end{aligned}
$$

We now need to compute only 4 commutators.

Note that if we advance the solution using the modified Crouch-Grossman method (3), all the modified vector fields $\widetilde{F}^i$, $i = 1, \ldots, s$, must be of order p. However, if we use the BCH formula to compute one single vector field, $\widetilde{Z}_i$, at each stage, the $s-1$ first vector fields need only be computed to order $p-1$. The reason is that they are not explicitly used to advance the solution. Only $\widetilde{Z}_s$ must be of order p. In this case, the algorithm in the above example reduces to the RKMK algorithm based on RK4 in [5].

Another example is given by `dopri5(4)`; a 7 stage, 5th order method by Dormand and Prince (see description in [2, pp. 178–179]). To find a new basis that suits our purpose, we look for linear combinations of the vector fields F^i such that

$$Q_i = \sum_{j=1}^{i} r_j F^j = \mathcal{O}(h^{q_i}), \qquad i = 1, \ldots, s,$$

with $q_i > 0$ as large as possible. This change of basis depends only on the vector fields $F^i = \mathcal{O}(h)$ and the method coefficients, and we can use the same algorithm as specified in [5] for the Munthe-Kaas methods. As described in [5], a new basis is given by

$$\begin{aligned}
Q_1 &= F^1 = \mathcal{O}(h) \\
Q_2 &= F^2 - F^1 = \mathcal{O}(h^2) \\
Q_3 &= F^3 - \tfrac{3}{2}F^2 + \tfrac{1}{2}F^1 = \mathcal{O}(h^3) \\
Q_4 &= F^4 - 6F^3 + 5F^2 = \mathcal{O}(h^3) \\
Q_5 &= F^5 - \tfrac{106}{81}F^4 + \tfrac{128}{243}F^3 - \tfrac{53}{243}F^1 = \mathcal{O}(h^4) \\
Q_6 &= F^6 - \tfrac{567}{212}F^5 + \tfrac{7}{4}F^4 - \tfrac{4}{53}F^3 = \mathcal{O}(h^4) \\
Q_7 &= F^7 - \tfrac{176}{105}F^6 + \tfrac{17253}{8480}F^5 - \tfrac{71}{48}F^4 + \tfrac{568}{3339}F^3 - \tfrac{71}{1440}F^1 = \mathcal{O}(h^5).
\end{aligned}$$

Equation (4) now gives

$$\begin{aligned}
\widetilde{F}^1 &= Q_1 \\
\widetilde{F}^2 &= Q_1 + Q_2 - \tfrac{1}{16}[Q_1,Q_2] + \tfrac{37}{19200}[Q_1,[Q_1,Q_2]] + \tfrac{9}{6400}[Q_2,[Q_1,Q_2]] - \tfrac{1}{25600}[Q_1,[Q_1,[Q_1,Q_2]]] \\
\widetilde{F}^3 &= Q_1 + \tfrac{3}{2}Q_2 + Q_3 - \tfrac{151}{192}[Q_1,Q_2] - \tfrac{55}{36}[Q_1,Q_3] - \tfrac{95}{48}[Q_2,Q_3] + \tfrac{1123}{4129}[Q_1,[Q_1,Q_2]] \\
&\quad + \tfrac{449}{677}[Q_1,[Q_1,Q_3]] - \tfrac{215}{3079}[Q_2,[Q_1,Q_2]] + \tfrac{191}{1663}[Q_1,[Q_1,[Q_1,Q_2]]] \\
\widetilde{F}^4 &= Q_1 + 4Q_2 + 6Q_3 + Q_4 + \tfrac{5714}{99}[Q_1,Q_2] + \tfrac{3313}{34}[Q_1,Q_3] + \tfrac{692}{3645}[Q_1,Q_4] + \tfrac{13361}{107}[Q_2,Q_3] \\
&\quad + \tfrac{1407}{1829}[Q_2,Q_4] + \tfrac{16527}{113}[Q_1,[Q_1,Q_2]] + \tfrac{35953}{123}[Q_1,[Q_1,Q_3]] + \tfrac{59}{2843}[Q_1,[Q_1,Q_4]] \\
&\quad + \tfrac{21093}{62}[Q_2,[Q_1,Q_2]] + \tfrac{11449}{38}[Q_1,[Q_1,[Q_1,Q_2]]] \\
\widetilde{F}^5 &= Q_1 + \tfrac{40}{9}Q_2 + \tfrac{1780}{243}Q_3 + \tfrac{106}{81}Q_4 + Q_5 + \tfrac{14667}{134}[Q_1,Q_2] + \tfrac{11495}{62}[Q_1,Q_3] + \tfrac{209}{3426}[Q_1,Q_4] \\
&\quad + \tfrac{546}{2839}[Q_1,Q_5] + \tfrac{16257}{65}[Q_2,Q_3] + \tfrac{863}{5876}[Q_2,Q_4] + \tfrac{5665}{21}[Q_1,[Q_1,Q_2]] + \tfrac{19893}{37}[Q_1,[Q_1,Q_3]] \\
&\quad - \tfrac{33}{142576}[Q_1,[Q_1,Q_4]] + \tfrac{33577}{51}[Q_2,[Q_1,Q_2]] + \tfrac{25179}{47}[Q_1,[Q_1,[Q_1,Q_2]]] \\
\widetilde{F}^6 &= Q_1 + 5Q_2 + \tfrac{55}{6}Q_3 + \tfrac{7}{4}Q_4 + \tfrac{567}{212}Q_5 + Q_6 - \tfrac{8539}{559}[Q_1,Q_2] - \tfrac{3944}{165}[Q_1,Q_3] - \tfrac{1699}{1640}[Q_1,Q_4] \\
&\quad - \tfrac{161}{3244}[Q_1,Q_5] - \tfrac{11}{168}[Q_1,Q_6] - \tfrac{1164}{4247}[Q_2,Q_3] - \tfrac{1164}{287}[Q_2,Q_4] - \tfrac{21579}{200}[Q_1,[Q_1,Q_2]] \\
&\quad - \tfrac{14197}{69}[Q_1,[Q_1,Q_3]] - \tfrac{193}{921}[Q_1,[Q_1,Q_4]] - \tfrac{17255}{76}[Q_2,[Q_1,Q_2]] - \tfrac{35557}{120}[Q_1,[Q_1,[Q_1,Q_2]]] \\
\widetilde{F}^7 &= Q_1 + 5Q_2 + \tfrac{55}{6}Q_3 + \tfrac{7}{4}Q_4 + \tfrac{879}{359}Q_5 + \tfrac{176}{105}Q_6 + Q_7 - \tfrac{6015}{37}[Q_1,Q_2] - \tfrac{14426}{53}[Q_1,Q_3] - \tfrac{863}{610}[Q_1,Q_4] \\
&\quad + \tfrac{853}{3278}[Q_1,Q_5] + \tfrac{339}{11260}[Q_1,Q_6] - \tfrac{38644}{119}[Q_2,Q_3] - \tfrac{2869}{536}[Q_2,Q_4] - \tfrac{9621}{19}[Q_1,[Q_1,Q_2]] \\
&\quad - \tfrac{38771}{39}[Q_1,[Q_1,Q_3]] - \tfrac{500}{1103}[Q_1,[Q_1,Q_4]] - \tfrac{41389}{34}[Q_2,[Q_1,Q_2]] - \tfrac{44104}{39}[Q_1,[Q_1,[Q_1,Q_2]]].
\end{aligned}$$

Although these expressions are large, only 12 commutators as well as linear combinations of these need being computed. The dimension of the Hall basis of order 5 generated by $Q_1, \ldots, Q_7$ with grades as above is 19.

If we use the BCH formula to reduce number of flow computations per step to 7, we obtain exactly the stages given by Munthe-Kaas and Owren [5, Equation (39)].

3 Concluding Remarks

In this note we have analysed the Crouch-Grossman methods from a numerical point of view only. We have optimized with respect to order, but we have not considered the geometrical aspects of the method. It is possible that the terms introduced into the modified vector fields destroy some advantageous properties of the methods, and it is also not clear if the terms excluded by the graded free Lie algebra have geometrical significance. On the other hand, we have proposed a modification of the original method such that the process of constructing new schemes is substantially simplified. The minimal number of stages needed to obtain order p is the same as in the classical case. This is not the case for the original Crouch-Grossman methods: for instance, there exists no Crouch-Grossman method of order four with four stages.

In its original form, the Crouch-Grossman method integrates by using flows of frozen vector fields. The frozen vector fields are expressed in terms of a d-dimensional frame of vector fields on the underlying manifold, but no brackets between vector fields are used. This can be advantageous when the dimension of the Lie algebra generated by $E_1, \ldots, E_d$ is much larger than the dimension of the manifold. The approach proposed in this paper, however, introduces commutators between frozen vector fields in the modified vector fields, $\widetilde{F}^k$.

Acknowledgements. The authors are grateful to Department of Mathematics at Arizona State University and in particular Zdzislaw Jackiewicz and Bruno D. Welfert for their hospitality during the authors' visit in Spring of 1998.

References

[1] P. E. Crouch and R. Grossman. Numerical Integration of Ordinary Differential Equations on Manifolds. *J. Nonlinear Sci.*, 3:1–33, 1993.

[2] E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I, Nonstiff Problems.* Springer-Verlag, Second revised edition, 1993.

[3] Z. Jackiewicz, A. Marthinsen, and B. Owren. Construction of Runge-Kutta methods of Crouch-Grossman type of high order. Manuscript, 1998.

[4] H. Munthe-Kaas. High Order Runge-Kutta Methods on Manifolds. Technical Report 1997/NA14, Department of Applied Mathematics and Theoretical Physics, University of Cambridge, England, 1997. To appear in Journal of Applied Numerical Mathematics.

[5] H. Munthe-Kaas and B. Owren. Computations in a Free Lie Algebra. Technical Report No. 148, Department of Informatics, University of Bergen, Norway, 1998. To appear in Philosophical Transactions of the Royal Society.

[6] B. Owren and A. Marthinsen. Integration Methods Based on Rigid Frames. Technical Report Numerics No. 1/1997, Department of Mathematical Sciences, The Norwegian University of Science and Technology, 1997. To appear in BIT.

[7] V. S. Varadarajan. *Lie Groups, Lie Algebras, and Their Representations.* GTM 102. Springer-Verlag, 1984.