Second-order ordinary difference equations admitting Lie-point transformations. [1]

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Abstract

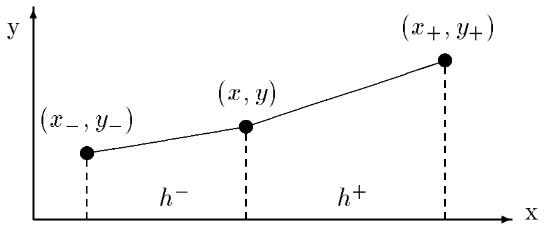
Classification of second order ordinary differential equations admitting groups of point transformations is known from S.Lie [11]. In the present paper we consider the whole set of invariant discrete models, i.e. invariant difference equations and meshes, which correspond to Lie's classification of ODEs. Applications of the discrete version of the Noether theorem are illustrated by examples.

Introduction

In the paper [11] S.Lie considered the group classification of second–order ODEs admitting point transformations. This classification is based upon the enumeration of all possible Lie algebras of operators acting on the plane (x, y). The classification was done up to similarity of the subgroups, i.e. up to suitable changes of variables. The dimension of the admitted group can be 0, 1, 2, 3 or 8. If the equation admits a 8–dimensional group, then it can be linearized by an appropriate change of variables. Furthermore, in this case the equation is equivalent to the simplest second–order equation $y'' = 0$. Group classification of second–order ODEs is presented in Table 1 of the Appendix.

In this letter we consider the whole set of invariant discrete models (i.e. invariant difference equations and meshes) that correspond to Lie's classification of ODEs.

Let x be an independent variable and y a dependent variable. To consider second-order difference equations and lattices in the x-direction we need at least 3-point stencils corresponding to the subspace $(x, x_-, x_+, y, y_-, y_+)$:



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The discrete model under consideration can be presented in terms of two difference equations:

$$\left\{\begin{array}{l} F(x, x_-, x_+, y, y_-, y_+) = 0, \\ \Omega(x, x_-, x_+, y, y_-, y_+) = 0, \end{array}\right. \tag{1}$$

The first equation presents a second–order difference equation. In the continuous limit it should become a second–order ordinary differential equation at some point of the finite–dimensional space of continuous variables (x, y, y', y''). The second equation gives a lattice (spacing) on which the first equation is considered. It is not necessary for the second equation to be a second–order difference equation in the continuous limit. For example, we can consider the uniform lattice equation $h_+ = h_-$ which "disappears" in the continuous limit. The difference stencil is a nonlocal object with its own geometrical structure, since we have to include a lattice equation into the formulation of the discrete model. It is the main difference between continuous and finite–difference models in our approach.

In some cases we will prefer the following form of the difference model

$$\left\{\begin{array}{l} x_+ = f(x, x_-, y, y_-), \\ y_+ = g(x, x_-, y, y_-), \end{array}\right. \tag{2}$$

or

$$\left\{\begin{array}{l} y_x = f(x, h_-, y, y_{\bar{x}}), \\ h_+ = g(x, h_-, y, y_{\bar{x}}), \end{array}\right. \tag{3}$$

where $h_+ = x_+ - x$, $h_- = x - x_-$, $y_x = \frac{y_+ - y}{h_+}$ and $y_{\bar{x}} = \frac{y - y_-}{h_-}$. We denote the continuous derivatives by y' and y'' to distinguish them from the discrete derivatives. We will consider models in more complicated forms than (3) only when these forms are prescribed by complicated structure of the admitted Lie groups.

It has been known for a long time that conservation laws are connected with symmetry properties. E.Noether [13] was the first to combine the methods of variational calculus with the theory of Lie groups for the Euler–Lagrange equations. This connection is exhibited by the well known Noether's theorem. Some new details were added in [4], [10] and some other papers. We recall here the continuous and difference versions of Noether's theorem of the forms in which they are applied to second–order ordinary equations (see for the general case [1], [9], [14], [15]).

It is well known that the functional

$$\mathbf{L} = \int_{\Omega} L(x, y, y')dx, \quad \Omega \subset R^1$$

achieves its extremal values on the Euler–Lagrange equation

$$\frac{\delta L}{\delta y} = \frac{\partial L}{\partial y} - D\left(\frac{\partial L}{\partial y'}\right) = 0.$$

Let us consider a Lie point symmetry generated by the operator

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \ldots.$$

Then the first Noether's theorem establishes the invariance of a variational functional. The functional $\mathbf{L}$ is invariant under the action of a group G of point transformations with infinitesimal operator X if and only if

$$X(L) + LD(\xi) = 0, \qquad \text{where} \quad D = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + \ldots. \tag{4}$$

The invariance of a functional is connected with the Euler–Lagrange equations and the Noether operator

$$N = \xi + (\eta - \xi y')\frac{\partial}{\partial y'}$$

by means of the Noether identity.

The Noether identity:

$$X + D(\xi) \equiv (\eta - \xi y')\frac{\delta}{\delta y} + DN.$$

This operators identity (actually stated by N.Ibragimov [9]) makes transparent the content and proof of the Noether theorem.

Noether's Theorem . For a given functional **L** the value $N(L)$ is a first integral for the Euler–Lagrange equation $\frac{\delta L}{\delta y} = 0$ if and only if the functional is invariant on any solution of the Euler–Lagrange equation.

Remark.

It directly follows from the Noether identity that if instead of the invariance condition (4) the divergence condition [4]

$$X(L) + LD(\xi) = D(B),$$

where $B = B(x, y, y')$ is some function, holds, then there is a first integral $(N(L) - B)$ on the Euler–Lagrange equation.

The variational formulation of discrete equations and a discrete analog of the Noether theorem were developed in [6] and [7]. Here we overview briefly the main results.

Let a finite difference functional

$$\mathbf{L_h} = \sum_{\underset{h}{\Omega}} \mathcal{L}(x, x_+, y, y_+)h_+ \tag{5}$$

be defined on some one dimensional mesh $\underset{h}{\Omega}$ with the step h_+ explicitly dependent on the solution:

$$h_+ = \varphi(x, y). \tag{6}$$

We will use discrete Lagrangians in the form $\mathcal{L}(x, x_+, y, y_+)$, which is equivalent the form $\mathcal{L}(x, y, y_x)$.

Let us have in $(x, y, \ldots, h_+, h_-)$ a one-parameter Lie group of point transformations G_1 with the generator

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \ldots + h_+ \underset{+h}{D}(\xi)\frac{\partial}{\partial h_+} + h_- \underset{-h}{D}(\xi)\frac{\partial}{\partial h_-}, \tag{7}$$

where $\underset{+h}{D} = \dfrac{\underset{+h}{S} - 1}{h_+}$ and $\underset{-h}{D} = \dfrac{1 - \underset{-h}{S}}{h_-}$ are right and left difference operators, defined by means of right and left shift operators $\underset{+h}{S}$ and $\underset{-h}{S}$: $f^+ = \underset{+h}{S}(f(x, y)) = f(x_+, y_+)$, $f^- = \underset{-h}{S}(f(x, y)) = f(x_-, y_-)$.

Then the invariance condition of (5) on the mesh (6) is given by two equations

$$\xi\frac{\partial \mathcal{L}}{\partial x} + \xi^+\frac{\partial \mathcal{L}}{\partial x_+} + \eta\frac{\partial \mathcal{L}}{\partial y} + \eta^+\frac{\partial \mathcal{L}}{\partial y_+} + \mathcal{L}\underset{+h}{D}(\xi) = 0,$$

$$\underset{+h}{S}(\xi) - \xi(1 + \varphi_x) - \eta\varphi_y = 0.$$

In contrast to the continuous case a stationary value of the difference functional is not reached on difference Euler's equation, but on new difference equations, called quasi extremal equations, which are different for different subgroups. Let us consider the variations of the functional along the orbit of a one–parameter subgroup with the operator coordinates ξ_α and η_α, then the stationary value of the functional is given by the quasi extremal equation

$$\xi_\alpha \left(h_+ \frac{\partial \mathcal{L}}{\partial x} + h_- \frac{\partial \mathcal{L}^-}{\partial x} + \mathcal{L}^- - \mathcal{L} \right) + \eta_\alpha \left(h_+ \frac{\partial \mathcal{L}}{\partial y} + h_- \frac{\partial \mathcal{L}^-}{\partial y} \right) = 0, \qquad \mathcal{L}^- = \underset{-h}{S}(\mathcal{L}),$$

which can be rewritten as

$$\xi_\alpha \frac{\delta \mathcal{L}}{\delta x} + \eta_\alpha \frac{\delta \mathcal{L}}{\delta y} = 0, \tag{8}$$

where $\alpha = 1, 2, 3, \ldots$ is the number of a subgroup of the admitted Lie groups G_r . There are the following notations used in (8):

$$\frac{\delta \mathcal{L}}{\delta x} = h_+ \frac{\partial \mathcal{L}}{\partial x} + h_- \frac{\partial \mathcal{L}^-}{\partial x} + \mathcal{L}^- - \mathcal{L}, \tag{9}$$

$$\frac{\delta \mathcal{L}}{\delta y} = h_+ \frac{\partial \mathcal{L}}{\partial y} + h_- \frac{\partial \mathcal{L}^-}{\partial y} . \tag{10}$$

The quasi extremal equations (8) have the following intersection:

$$\begin{cases} \dfrac{\delta \mathcal{L}}{\delta x} = 0, \\[2ex] \dfrac{\delta \mathcal{L}}{\delta y} = 0. \end{cases} \tag{11}$$

It was shown [8] that every quasi extremal equation is invariant with respect to its "own subgroup". Next proposition shows that the intersection of quasi extremals admits every subgroup G_1, and thus the whole symmetry group G_r.

Proposition

Let difference functional (5) be invariant with respect to a group G_1 with the operator

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \xi^+ \frac{\partial}{\partial x_+} + \eta^+ \frac{\partial}{\partial y_+}. \tag{12}$$

Then system (11) admits the same symmetry group G_1 (see [2], [8]).

The discrete formulation of Noether's theorem is based on the following operators identity (see [5], [7], [8] for many–dimensional cases as well):

$$\begin{aligned} \xi \frac{\partial \mathcal{L}}{\partial x} + \xi^+ \frac{\partial \mathcal{L}}{\partial x_+} + \eta \frac{\partial \mathcal{L}}{\partial y} + \eta^+ \frac{\partial \mathcal{L}}{\partial y_+} + \mathcal{L} \underset{+h}{D}(\xi) = \xi \left(\frac{\partial \mathcal{L}}{\partial x} + \frac{h_-}{h_+} \frac{\partial \mathcal{L}^-}{\partial x} - \underset{+h}{D}(\mathcal{L}^-) \right) + \\ + \eta \left(\frac{\partial \mathcal{L}}{\partial y} + \frac{h_-}{h_+} \frac{\partial \mathcal{L}^-}{\partial y} \right) + \underset{+h}{D} \left(h_- \eta \frac{\partial \mathcal{L}^-}{\partial y} + h_- \xi \frac{\partial \mathcal{L}^-}{\partial x} + \xi \mathcal{L}^- \right) \end{aligned} \tag{13}$$

This identity yields the following conservative property of quasi extremal equations.

Theorem

Let the difference equation

$$F(z) = 0, \tag{14}$$

be given on a mesh

$$\underset{h}{\Omega}(z, h) = 0. \tag{15}$$

Let the solution of (14) on mesh (15) be a quasi extremal of functional (5) that yield the stationary value while the functional is being transformed by a group G_1. Let ξ and η be coordinate of the operator corresponding to the group G_1. Then the invariance of (5) on equations (14)–(15) is the necessary and sufficient condition for existence of the conservation law

$$\underset{+h}{D}\left(h_-\eta\frac{\partial\mathcal{L}^-}{\partial y}+h_-\xi\frac{\partial\mathcal{L}^-}{\partial x}+\xi\mathcal{L}^-\right)=0 \tag{16}$$

on equations (14)–(15).

The theorem has the following corollary, which can be viewed as a difference analog of Noether's theorem.

Corollary

Let the functional (5) admit an r-parameter group G_r and let the equations

$$\xi^\alpha\frac{\delta\mathcal{L}}{\delta x}+\eta^\alpha\frac{\delta\mathcal{L}}{\delta y}=0, \quad \alpha=1,2,\dots,r. \tag{17}$$

be r quasi extremal equations. Then the intersection system

$$\left\{\begin{array}{rcl}\dfrac{\delta\mathcal{L}}{\delta x} & = & 0,\\[2ex] \dfrac{\delta\mathcal{L}}{\delta y} & = & 0,\end{array}\right. \tag{18}$$

possesses r conservation laws of type (16) (see [2], [8]).

These results will be illustrated by forthcoming examples.

Invariant difference models

Here we begin the consideration of different groups, which can be admitted by second–order ODEs. For this purpose we will have to compute invariants of groups in spaces (x,y,y',y'') and (x,x_-,x_+,y,y_-,y_+). The procedure of computation of invariants can be found, for example, in [14]. We demonstrate the computational method in the differential case. Given a group defined by operators X_α, $\alpha=1,...,n$:

$$X_\alpha=\xi_\alpha(x,y)\frac{\partial}{\partial x}+\eta_\alpha(x,y)\frac{\partial}{\partial y},$$

it is necessary to prolong all operators on the variables of the considered space. With the help of formulas

$$\zeta^1_\alpha=D(\eta_\alpha(x,y))-y'D(\xi_\alpha(x,y)),$$
$$\zeta^2_\alpha=D(\zeta^1_\alpha)-y''D(\xi_\alpha(x,y)).$$

we obtain prolonged operators

$$\hat{X}_\alpha=X_\alpha+\zeta^1_\alpha\frac{\partial}{\partial y'}+\zeta^2_\alpha\frac{\partial}{\partial y''}.$$

Now we can find invariants $I_\beta(x,y,y',y'')$, $\beta=1,...,k$ as all functionally independent solutions of the system of equations

$$\hat{X}_\alpha(I)=0, \qquad \alpha=1,...,n.$$

If the system is not singular, one can expect that $n+k=4$ because 4 is the dimension of the space. Thus, we expect to find 3 invariants for the 1–parameter group, 2 invariants for 2–parameter groups, 1 invariant for 3–parameter groups and no invariants for the 8–parameter group.

In case of discrete space the procedure to find invariants is quite the same except for the prolongation formula for the operators:

$$\hat{X}_\alpha = X_\alpha + \underset{-h}{S}(\xi_\alpha(x,y))\frac{\partial}{\partial x_-} + \underset{+h}{S}(\xi_\alpha(x,y))\frac{\partial}{\partial x_+} + \underset{-h}{S}(\eta_\alpha(x,y))\frac{\partial}{\partial y_-} + \underset{+h}{S}(\eta_\alpha(x,y))\frac{\partial}{\partial y_+} =$$

$$= X_\alpha + \xi_\alpha(x_-,y_-)\frac{\partial}{\partial x_-} + \xi_\alpha(x_+,y_+)\frac{\partial}{\partial x_+} + \eta_\alpha(x_-,y_-)\frac{\partial}{\partial y_-} + \eta_\alpha(x_+,y_+)\frac{\partial}{\partial y_+}.$$

It is easy to see that there should be 2 more discrete invariants than differential invariants for the same group.

One–parameter group G_1

We start with the simplest case of the admitted group (see [11]) — the one–parameter group with the translation operator

$$X_1 = \frac{\partial}{\partial x}. \tag{19}$$

In the continuous case an invariant equation can be presented in the form (see [9]):

$$y'' = F(y, y'), \tag{20}$$

where F is an arbitrary function. For the discrete formulation of an invariant model we need difference invariants of the operator (19):

$$h_+, \quad h_-, \quad y, \quad y_x, \quad y_{\bar{x}}.$$

The general form of the invariant with respect to operator (19) discrete model is

$$\begin{cases} y_x = f\,(h_-, h_+, y, y_{\bar{x}}); \\ h_+ = g\,(h_-, y, y_x, y_{\bar{x}}), \end{cases} \tag{21}$$

where f and g are arbitrary functions of their arguments.

As an invariant approximation of (20) we can choose the equation

$$\frac{2}{h_+ + h_-}(y_x - y_{\bar{x}}) = F\left(y, \frac{1}{2}(y_x + y_{\bar{x}})\right)$$

on the regular lattice $h_- = h_+ = h = const$.

Remark: a regular lattice is not necessary; one can choose any nonregular lattice of type (21).

Two–parameter groups G_2

There are two substantially different two–parameter groups [11].

a) The group with infinitesimal operators

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \tag{22}$$

corresponds to the invariant equation

$$y'' = F(y'), \tag{23}$$

where F is an arbitrary function.

The expressions

$$h_+, \quad h_-, \quad y_x, \quad y_{\bar{x}}.$$

form the whole set of difference invariants of operators (22). The general form of the invariant model can be written as

$$\begin{cases} y_x = f\,(h_-, y_{\bar{x}}); \\ \\ h_+ = g\,(h_-, y_{\bar{x}}). \end{cases} \tag{24}$$

As an invariant approximation of equation (23) we can choose

$$\frac{2}{h_+ + h_-}(y_x - y_{\bar{x}}) = F\left(\frac{1}{2}(y_x + y_{\bar{x}})\right)$$

on uniform grid $h_- = h_+ = h$.

b) The group with operators

$$X_1 = \frac{\partial}{\partial y}; \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \tag{25}$$

yields the following invariant equation

$$y'' = \frac{1}{x}F(y'), \tag{26}$$

for an arbitrary function F.

Finite–difference invariants of the operators (25) are

$$\frac{h_+}{h_-}, \quad \frac{x}{h_-}, \quad y_x, \quad y_{\bar{x}}$$

and the invariant model can be presented as

$$\begin{cases} y_x = y_{\bar{x}} + f\left(\dfrac{x}{h_-}, y_{\bar{x}}\right); \\ \\ h_+ = h_- g\left(\dfrac{x}{h_-}, y_{\bar{x}}\right). \end{cases} \tag{27}$$

As an invariant approximation of (26) we can take

$$\frac{2}{h_+ + h_-}(y_x - y_{\bar{x}}) = \frac{1}{x}F\left(\frac{1}{2}(y_x + y_{\bar{x}})\right)$$

on the uniform grid $h_- = h_+ = h$.

Three–parameter groups G_3

There are five different three-dimensional groups which have invariant second–order differential equations.

a) The group with operators

$$X_1 = \frac{\partial}{\partial x} + \frac{\partial}{\partial y}; \qquad X_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \qquad X_3 = x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}; \tag{28}$$

has

$$y'' + 2\frac{y' + Cy'\sqrt{y'} + y'^2}{x - y} = 0 \tag{29}$$

as the general form of invariant equation. This equation can be obtained by variation of the Lagrangian function

$$L = \frac{Cy' + 4\sqrt{y'} + C}{2(x - y)}.$$

It is easy to check that the operators (28) represent variational symmetries of the Lagrangian:

$$X_1L + LD(\xi_1) = 0;$$

$$X_2L + LD(\xi_2) = 0;$$

$$X_3L + LD(\xi_3) = \frac{C}{2}(1 - y') = D\left(\frac{C}{2}(x - y)\right).$$

Note that X_3 is divergently variational symmetry. Applying the Noether theorem (see [5,9]), we get three first integrals from the Lagrangian :

$$J_1 = \frac{y' + C\sqrt{y'} + 1}{(x - y)\sqrt{y'}} = A^0, \quad J_2 = \frac{xy' + C\frac{x+y}{2}\sqrt{y'} + y}{(x - y)\sqrt{y'}} = B^0, \quad J_3 = \frac{x^2y' + Cxy\sqrt{y'} + y^2}{(x - y)\sqrt{y'}} = C^0.$$

According to the structure of the adjoint actions

$$X_1(X_3) = 2X_2, \quad X_1(X_2) = X_1$$

integrals J_1 and J_2 can be obtained from the basic integral J_3 (see [9]):

$$X_1(J_3) = 2J_2, \qquad X_1(J_2) = J_1.$$

The following difference invariants for the set of operators (28) exist:

$$I_1 = \frac{h_+^2 y_x}{(x - y_+)(x_+ - y)}, \qquad I_2 = \frac{h_-^2 y_{\bar{x}}}{(x - y_-)(x_- - y)}, \qquad I_3 = \frac{x_+ - y}{x - y}\frac{h_-}{h_- + h_+}.$$

With help of these invariants we get the following form of a discrete model which is invariant for the whole set of operators:

$$\left\{\begin{array}{l} y_x = \dfrac{(x - y)f\left(\dfrac{h_-^2 y_{\bar{x}}}{(x - y_-)(x_- - y)}\right)}{\dfrac{h_+^2}{x + h_+ - y} + h_+ f\left(\dfrac{h_-^2 y_{\bar{x}}}{(x - y_-)(x_- - y)}\right)}; \\ \\ h_+ = \dfrac{h_-\left(1 - g\left(\dfrac{h_-^2 y_{\bar{x}}}{(x - y_-)(x_- - y)}\right)\right)}{-\dfrac{h_-}{x - y} + g\left(\dfrac{h_-^2 y_{\bar{x}}}{(x - y_-)(x_- - y)}\right)} \end{array}\right. \tag{30}$$

As an invariant discrete model we can consider the equation

$$\left(\sqrt{I_1}\frac{1}{1 - I_3} - \sqrt{I_2}\frac{1}{I_3}\right) = \frac{C}{2}\left(\sqrt{I_1} + \sqrt{I_2}\right)^2$$

or

$$\left(\sqrt{\frac{h_+^2 y_x}{(x - y_+)(x_+ - y)}}\,\frac{x - y}{x_- - y}\,\frac{h_+ + h_-}{h_+} - \sqrt{\frac{h_-^2 y_{\bar{x}}}{(x - y_-)(x_- - y)}}\,\frac{y - x}{y - x_+}\,\frac{h_- + h_+}{h_-}\right) =$$

$$= \frac{C}{2}\left(\sqrt{\frac{h_+^2 y_x}{(x - y_+)(x_+ - y)}} + \sqrt{\frac{h_-^2 y_{\bar{x}}}{(x - y_-)(x_- - y)}}\right)^2$$

on the invariant grid

$$\frac{h_+^2 y_x}{(x - y_+)(x_+ - y)} = \frac{h_-^2 y_{\bar{x}}}{(x - y_-)(x_- - y)} = \varepsilon^2, \qquad 0 < \varepsilon \ll 1.$$

To consider a variational model we choose the Lagrangian

$$\mathcal{L} = \frac{Cy_x + 4\sqrt{y_x} + C}{2\sqrt{(x-y)(x_+ - y_+)}}.$$

This Lagrangian is invariant with respect to the operators X_1 and X_2, but is not invariant with respect to the operator X_3:

$$X_1\mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_1) = 0;$$

$$X_2\mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_2) = 0;$$

$$X_3\mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_3) = \frac{C}{4\sqrt{(x-y)(x_+ - y_+)}} \underset{+h}{D}\left((y-x)^2\right).$$

By means of Norther's theorem we get the following quasi extremal equations for the Lagrangian

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta y}: \quad & -\frac{C + 2(y_x)^{-1/2}}{\sqrt{(x-y)(x_+ - y_+)}} + \frac{C + 2(y_{\bar{x}})^{-1/2}}{\sqrt{(x_- - y_-)(x-y)}} + \\
& + \frac{1}{2(x-y)}\left(\frac{Cy_x + 4\sqrt{y_x} + C}{\sqrt{(x-y)(x_+ - y_+)}}h_+ + \frac{Cy_{\bar{x}} + 4\sqrt{y_{\bar{x}}} + C}{\sqrt{(x_- - y_-)(x-y)}}h_-\right) = 0; \\
\frac{\delta\mathcal{L}}{\delta x}: \quad & -\frac{C + 2\sqrt{y_x}}{\sqrt{(x-y)(x_+ - y_+)}} + \frac{C + 2\sqrt{y_{\bar{x}}}}{\sqrt{(x_- - y_-)(x-y)}} + \\
& - \frac{1}{2(x-y)}\left(\frac{Cy_x + 4\sqrt{y_x} + C}{\sqrt{(x-y)(x_+ - y_+)}}h_+ + \frac{Cy_{\bar{x}} + 4\sqrt{y_{\bar{x}}} + C}{\sqrt{(x_- - y_-)(x-y)}}h_-\right) = 0;
\end{aligned} \tag{31}$$

There are only too independent difference first integrals which hold on the quasi extremal equations (31)

$$I_1 = \frac{y_x + C\sqrt{y_x} + 1}{\sqrt{(x-y)(x_+ - y_+)}\sqrt{y_x}} = A^h,$$

$$I_2 = \frac{\frac{x + x_+}{2}y_x + C\frac{x_+ + x + y + y_+}{4}\sqrt{y_x} + \frac{y + y_+}{2}}{\sqrt{(x-y)(x_+ - y_+)}\sqrt{y_x}} = B^h.$$

Note that $X_1(I_2) = I_1$. The complicated form of the first integrals I_1 and I_2 does not allow us to express the general from of the solution for the discrete model specified by the quasi extremal equations (31). In the differential case the first integrals yield the continuous solution

$$\left(B^0 - A^0 y - \frac{C}{2}\right)\left(A^0 x - B^0 - \frac{C}{2}\right) = 1,$$

or

$$y = \frac{1}{A^0(B^0 + C/2 - A^0 x)} + \frac{2B^0 - C}{2A^0}.$$

Remark This function is a solution of (29) if a square root of the first derivative is chosen as

$$\sqrt{y'} = -\frac{1}{B^0 + C/2 - A^0 x}$$

b) For the operators

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}; \qquad X_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}; \tag{32}$$

there is an invariant equation

$$y'' = y^{-3}. \tag{33}$$

This equation can be obtained from the Lagrangian function (see [3]):

$$L = y'^2 - \frac{1}{y^2}, \tag{34}$$

which admits all operators as variational symmetries:

$$X_1 L + LD(\xi_1) = 0,$$

$$X_2 L + LD(\xi_2) = 0,$$

$$X_3 L + LD(\xi_3) = 2y'y = D(y^2).$$

There are the following first integrals

$$J_1 = y'^2 + \frac{1}{y^2} = A^0, \qquad J_2 = 2\frac{x}{y^2} - 2(y - y'x)y' = 2B^0, \qquad J_3 = \frac{x^2}{y^2} + (y - xy')^2 = C^0.$$

The structure of action of the adjoint Lie algebra

$$X_1(X_3) = X_2; \qquad X_1(X_2) = 2X_1$$

allows us to find all integrals from the basic integral J_3:

$$X_1(J_3) = J_2; \qquad X_1(J_2) = 2J_1.$$

The finite–difference invariants of (32) are

$$\frac{h_+}{yy_+}, \qquad \frac{h_-}{yy_-}, \qquad y^2\left(\frac{1}{h_+} + \frac{1}{h_-}\right).$$

With the help of these invariants we get the following form of discrete model which is invariant for the whole set of operators:

$$\left\{ \begin{array}{l} y_+ = \dfrac{h_+}{y} f\left(\dfrac{h_-}{yy_-}\right); \\ \\ h_+ = \dfrac{h_- y^2}{-y^2 + h_- g\left(\dfrac{h_-}{yy_-}\right)} \end{array} \right. \tag{35}$$

Below we construct discrete analogs of the Lagrangian and all integrals. Let us consider the discrete Lagrangian function

$$\mathcal{L} = \left(y_x^2 - \frac{1}{y_+ y}\right) \tag{36}$$

which admits all operators (32):

$$X_1\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0,$$

$$X_2\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_2) = 0,$$

$$X_3\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_3) = \underset{+h}{D}(y^2).$$

The Noether theorem gives us the following Euler equations for this Lagrangian

$$
\begin{aligned}
\frac{\delta \mathcal{L}}{\delta y}: &\qquad 2(y_x - y_{\bar{x}}) = \frac{h_+}{y^2 y_+} + \frac{h_-}{y^2 y_-},\\
\frac{\delta \mathcal{L}}{\delta x}: &\quad (y_x)^2 + \frac{1}{y_+ y} - (y_{\bar{x}})^2 - \frac{1}{y y_-} = 0.
\end{aligned}
\tag{37}
$$

On these equations we have conservation of first integrals

$$
I_1 = {y_x}^2 + \frac{1}{y y_+} = A^h,
$$

$$
I_2 = \frac{x + x_+}{y y_+} - (y + y_+) y_x + (x + x_+) {y_x}^2 = 2B^h,
$$

$$
I_3 = \frac{x x_+}{y y_+} + \left(\frac{y + y_+}{2} - \frac{x + x_+}{2} y_x \right)^2 = C^h.
$$

As in the continuous case the discrete first integrals obey relations

$$
X_1(I_3) = I_2; \qquad X_1(I_2) = 2 I_1.
$$

If we denote

$$
\frac{h_-}{y y_-} = \varepsilon,
$$

then it is easy to find the general solution of (37) by means of integrals:

$$
A^h y^2 = (A^h x - B^h)^2 + 1 - \frac{\varepsilon^2}{2}. \tag{38}
$$

This solution differs from the exact solution of the original equation (33)

$$
A^0 y^2 = (A^0 x - B^0)^2 + 1,
$$

by $\frac{\varepsilon^2}{2}$. Notice that the estimation is homogeneous.

The second equations of system (37) can be solved for h_+:

$$
\frac{h_+}{y_+ y} = \varepsilon,
$$

that gives us as a discrete model for (33) equation

$$
2(y_x - y_{\bar{x}}) = \frac{h_+}{y^2 y_+} + \frac{h_-}{y^2 y_-} \tag{39}
$$

on the invariant nonregular lattice

$$
\frac{h_-}{y y_-} = \frac{h_+}{y y_+} = \varepsilon, \qquad 0 < \varepsilon \ll 1. \tag{40}
$$

In this case the first equation of system (37) can be considered independently in the form:

$$
y_+ y_- (2 - \varepsilon^2) = y (y_+ + y_-),
$$

which is a one–dimensional mapping.

It is interesting to note, that the nonregular lattice (40) can be straightened, i.e. there exists a change of variables, [1] which transforms this lattice into uniform one. This transformation is nonlocal:

[1] This exchange was found by M.Bakirova

$$\hbar = \frac{h_+}{yy_+} = \frac{h_-}{yy_-}, \quad v(t) = y(x), \tag{41}$$

where $\hbar = t_+ - t = t - t_-$ is a regular step along the new independent variable t. The transform (41) change the equation (39) into the following difference equation

$$\left(\frac{v_t}{vv_+}\right)_{\bar{t}} = \frac{1}{v}$$

on a regular mesh for t.

In the continuous limit the transform (41) has the following non-point counterpart

$$dx = y^2 dt, \quad v(t) = y(x),$$

which transforms (33) into

$$\left(\frac{v_t}{v^2}\right)_t = \frac{1}{v}.$$

c) For the operators

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}; \tag{42}$$

there is an invariant equation

$$y'' = C\exp(-y'). \tag{43}$$

One can find a Lagrangian

$$L = \exp(y') + Cy, \tag{44}$$

which admits X_1 and X_2 as variational symmetries:

$$\begin{aligned} &X_1 L + LD(\xi_1) = 0; \\ &X_2 L + LD(\xi_2) = C = D(Cx); \\ &X_3 L + LD(\xi_3) = 2\exp(y') + C(2y + x) \neq 0. \end{aligned}$$

It is possible to show that there is no Lagrangian which admits all operators as variational symmetries. Two different variational symmetries will give two functionally independent first integrals, that is enough to integrate a second-order ODE.

Lemma There is no Lagrangian function $L(x, y, y')$ which gives (43) as the extremal equation and is divergently invariant with respect to the whole set of symmetries (42) .

Proof If it is not so, then there must exist a Lagrangian $L(x, y, y')$ which is divergently invariant for set (42):

$$X(L) + LD_x(\xi) = D_x(B(x, y)) = B_x(x, y) + B_y(x, y)y'$$

with some functions $B(x, y)$ which are different for different symmetries. Thus we have

$$\begin{cases} \dfrac{\partial}{\partial x}L = B_x^1(x, y) + B_y^1(x, y)y'; \\[2ex] \dfrac{\partial}{\partial y}L = B_x^2(x, y) + B_y^2(x, y)y'; \\[2ex] x\dfrac{\partial}{\partial x} + (x+y)\dfrac{\partial}{\partial y}L + \dfrac{\partial}{\partial y'}L + L = B_x^3(x, y) + B_y^3(x, y)y'. \end{cases}$$

Note that the right hand sides of the equations are linear with respect to y'. It gives us

$$\begin{cases} \dfrac{\partial^3 L}{\partial^2 y' \partial x} = 0; \\ \dfrac{\partial^3 L}{\partial^2 y' \partial y} = 0; \\ x\dfrac{\partial^3 L}{\partial^2 y' \partial x} + (x+y)\dfrac{\partial^3 L}{\partial^2 y' \partial y} + \dfrac{\partial^3 L}{\partial^3 y'} + \dfrac{\partial^2 L}{\partial^2 y'} = 0. \end{cases}$$

We found that the solution of this system has the form

$$L(x,y,y') = A(x,y)\exp(-y') + B(x,y)y' + C(x,y)$$

It is trivial to show that Lagrangian of this form can not give equation (43) as the extremal equation. □

With the help of Noether's theorem we derive the following first integrals from the Lagrangian (44):

$$J_1 = \exp(y')(1-y') + Cy = A^0;$$

$$J_2 = \exp(y') - Cx = B^0.$$

We have the following difference invariants for the operators (42)

$$\frac{h_+}{h_-}, \qquad y_x - y_{\bar{x}}, \qquad \frac{1}{h_-}\exp(y_{\bar{x}}).$$

The general form of the invariant model can be written as

$$\begin{cases} y_x = y_{\bar{x}} + f\left(\dfrac{1}{h_-}\exp(y_{\bar{x}})\right); \\ h_+ = h_- g\left(\dfrac{1}{h_-}\exp(y_{\bar{x}})\right); \end{cases} \tag{45}$$

As an invariant difference approximation of (43) we can consider equation

$$\frac{2}{h_- + h_+}\left(\exp(y_x) - \exp(y_{\bar{x}})\right) = C$$

on some invariant lattice, for example $h_+ = h_-$.

Let us choose a difference Lagrangian in the following form:

$$\mathcal{L} = \left(\exp(y_x) + C\frac{y_+ + y}{2}\right),$$

for which

$$X_1\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0,$$

$$X_2\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_2) = C = \underset{+h}{D}(Cx).$$

Then the variations of $\mathcal{L}$ yield:

$$\begin{aligned} \frac{\delta\mathcal{L}}{\delta y}: &\quad \frac{2}{h_- + h_+}\left(\exp(y_x) - \exp(y_{\bar{x}})\right) = C, \\ \frac{\delta\mathcal{L}}{\delta x}: &\quad \exp(y_x)(y_x - 1) - C\frac{y_+ + y}{2} + \exp(y_{\bar{x}})(1 - y_{\bar{x}}) + C\frac{y + y_-}{2} = 0. \end{aligned} \tag{46}$$

Because the Lagrangian is invariant with respect to the operators X_1 and X_2, Noether's theorem gives the first integrals

$$I_1 = \exp(y_x)(1 - y_x) + C\frac{y_+ + y}{2} = A^h,$$

$$I_2 = \exp(y_x) - C\frac{x_+ + x}{2} = B^h.$$

Let us denote

$$h_- \exp(-y_{\bar{x}}) = \varepsilon, \quad \varepsilon = const,$$

for initial points, then the quasi extremal equations (46) can be integrated with the help of the first integrals

$$\left(\frac{B^h + Cx}{1 - C\varepsilon}\right)\left(1 - \ln\left(\frac{B^h + Cx}{1 - C\varepsilon}\right)\right) + C\left(y + \varepsilon\left(\frac{B^h + Cx}{1 - C\varepsilon}\right)\ln\left(\frac{B^h + Cx}{1 - C\varepsilon}\right)\right) = A^h \qquad (47)$$

that corresponds to the solution of the continuous equation

$$(B^0 + Cx)(1 - \ln(B^0 + Cx)) + Cy = A^0.$$

or

$$y = (B^0/C + x)[\ln(B^0 + Cx) - 1] + A^0/C,$$

where A^0 and B^0 are arbitrary constants.

The solution (47) exists on the spacing given by the system of quasi extremal equations (46). This system gives space points x_i and values of the function y_i in these points. These values will obey (47).

d) For the set of operators

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}, \qquad k \neq 0, \frac{1}{2}, 1, 2 \qquad (48)$$

there is an invariant equation of the form

$$y'' = Cy'^{\frac{k-2}{k-1}}. \qquad (49)$$

This equation can be obtained from the Lagrangian

$$L = \frac{(k-1)^2}{k}(y')^{\frac{k}{k-1}} + Cy \qquad (50)$$

which admits X_1 and X_2 as variational symmetries:

$$X_1 L + LD(\xi_1) = 0;$$

$$X_2 L + LD(\xi_2) = C = D(Cx);$$

$$X_3 L + LD(\xi_3) = (k+1)L \neq 0.$$

As in the previous point it is possible to show that there is no Lagrangian which is divergently invariant with respect to all three symmetries.

With the help of Noether's theorem, first integrals

$$J_1 = \frac{(1-k)}{k}(y')^{\frac{k}{k-1}} + Cy = A^0, \qquad J_2 = (k-1)(y')^{\frac{1}{k-1}} - Cx = B^0;$$

can be obtained.

The finite–difference invariants of (48)

$$\frac{h_+}{h_-}, \qquad y_x {h_-}^{(1-k)}, \qquad y_{\bar{x}} {h_-}^{(1-k)}$$

gives us the general form of the invariant model for the operators (48):

$$\begin{cases} y_x = {h_-}^{(k-1)} f\left(y_{\bar{x}} {h_-}^{(1-k)} \right); \\ \\ h_+ = h_- g\left(y_{\bar{x}} {h_-}^{(1-k)} \right); \end{cases} \tag{51}$$

As a discrete approximation for (49) we can take the equation

$$\frac{2(k-1)}{h_- + h_+} \left((y_x)^{\frac{1}{k-1}} - (y_{\bar{x}})^{\frac{1}{k-1}} \right) = C$$

on the invariant lattice $h_+ = h_-$.

We can also construct difference analogs of Lagrangian (50) and first integrals. We choose

$$\mathcal{L} = \left(\frac{(1-k)^2}{k} (y_x)^{\frac{k}{k-1}} + C \frac{y_+ + y}{2} \right)$$

as a discrete Lagrangian:

$$X_1 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_1) = 0,$$

$$X_2 \mathcal{L} + \mathcal{L} \underset{+h}{D}(\xi_2) = C = \underset{+h}{D}(Cx).$$

Applying Noether's theorem, we get quasi extremal equations

$$\begin{aligned} \frac{\delta \mathcal{L}}{\delta y} : \qquad & \frac{2(k-1)}{h_- + h_+} \left((y_x)^{\frac{1}{k-1}} - (y_{\bar{x}})^{\frac{1}{k-1}} \right) = C, \\ \frac{\delta \mathcal{L}}{\delta x} : \quad & \frac{(k-1)}{k} \left((y_x)^{\frac{k}{k-1}} - (y_{\bar{x}})^{\frac{k}{k-1}} \right) - C \frac{y_+ + y}{2} + C \frac{y + y_-}{2} = 0, \end{aligned} \tag{52}$$

on which there are the following first integrals

$$I_1 = \frac{(1-k)}{k} (y_x)^{\frac{k}{k-1}} + C \frac{y_+ + y}{2} = A^h,$$

$$I_2 = (k-1)(y_x)^{\frac{1}{k-1}} - C \frac{x_+ + x}{2} = B^h.$$

Let us denote

$$(y_{\bar{x}})^{\frac{1}{1-k}} h_- = 2(k-1)\varepsilon$$

for the starting points of computations. In this spacing we can find the solution of the quasi extremal equations

$$Cy - \frac{1}{k} \left(\frac{1}{k-1} \right)^{(k-1)} \frac{(B^h + Cx)^k}{(1 - C\varepsilon)^k} + C(k+1)\varepsilon \frac{(k-1)^k (1 - C\varepsilon)^k}{(B^h + Cx)^k} = A^h, \tag{53}$$

corresponding to the continuous solution of (49)

$$Cy - \frac{1}{k} \left(\frac{1}{k-1} \right)^{(k-1)} (B^0 + Cx)^k = A^0.$$

The solution (53) exists on the grid defined by system (52).

e) The last case of three–parameter groups is defined by the operators

$$X_1 = \frac{\partial}{\partial x}; \qquad X_2 = \frac{\partial}{\partial y}; \qquad X_3 = (kx+y)\frac{\partial}{\partial x} + (ky-x)\frac{\partial}{\partial y}. \tag{54}$$

The corresponding invariant equation is:

$$y'' = C(1+(y')^2)^{3/2}\exp(k\arctan(y')). \tag{55}$$

There are two first integrals

$$I_1 = Cy + \frac{\exp(-k\arctan(y'))}{\sqrt{1+(y')^2}}\frac{1+ky'}{1+k^2} = A^0, \quad I_2 = \frac{\exp(-k\arctan(y'))}{\sqrt{1+(y')^2}}\frac{y'-k}{1+k^2} - Cx = B^0,$$

which correspond operators X_1 and X_2.

There are the following finite–difference invariants for the operators (54):

$$h_+\sqrt{1+y_x^2}\exp\left(k\arctan y_x\right), \qquad h_-\sqrt{1+y_{\bar{x}}^2}\exp\left(k\arctan y_{\bar{x}}\right), \qquad (\arctan(y_x) - \arctan(y_{\bar{x}})).$$

The general form of the invariant model is

$$\left\{ \begin{array}{l} \arctan(y_x) = \arctan(y_{\bar{x}}) + f\left(h_-\sqrt{1+y_{\bar{x}}^2}\exp\left(k\arctan(y_{\bar{x}})\right)\right); \\ \\ h_+ = \sqrt{1+y_x^2}\exp\left(-k\arctan(y_x)\right)g\left(h_-\sqrt{1+y_{\bar{x}}^2}\exp\left(k\arctan(y_{\bar{x}})\right)\right); \end{array} \right. \tag{56}$$

As an invariant discrete model for (55) we can choose the discrete equation

$$\frac{2\left(y_x - y_{\bar{x}}\right)}{h_+ + h_-} = \frac{C}{h_+ + h_-}\left(1 + y_x y_{\bar{x}}\right)\left(h_+\sqrt{1+y_x^2}\exp\left(k\arctan(y_x)\right) + h_-\sqrt{1+y_{\bar{x}}^2}\exp\left(k\arctan(y_{\bar{x}})\right)\right)$$

on the grid

$$h_+ = \frac{\varepsilon}{\sqrt{1+y_x^2}}\exp\left(-k\arctan(y_x)\right), \qquad h_- = \frac{\varepsilon}{\sqrt{1+y_{\bar{x}}^2}}\exp\left(-k\arctan(y_{\bar{x}})\right), \qquad 0 < \varepsilon \ll 1.$$

The solution in the continuous case can be presented in an implicit way

$$\exp\left(k\arctan\left(\frac{(A^0 - Cy)k + (B^0 + Cx)}{(A^0 - Cy) - (B^0 + Cx)k}\right)\right)\sqrt{1+k^2}\sqrt{(A^0 - Cy)^2 + (B^0 + Cx)^2} = 1 \tag{57}$$

Consideration of the groups of this case with arbitrary parameter k is difficult. We consider the Lagrangian formulation only for the value $k = 0$.

For this value of the parameter we have the group

$$Y_1 = \frac{\partial}{\partial x}; \qquad Y_2 = \frac{\partial}{\partial y}; \qquad Y_3 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}. \tag{58}$$

and the corresponding invariant equation

$$y'' = C(1+(y')^2)^{3/2}. \tag{59}$$

can be obtained from the Lagrangian

$$L = \sqrt{1+(y')^2} + Cy \tag{60}$$

which respects all symmetries:

$$Y_1 L + LD(\xi_1) = 0;$$

$$Y_2 L + LD(\xi_2) = C = D_x(Cx);$$

$$Y_3 L + LD(\xi_3) = C(yy' - x) = D_x\left(C\frac{y^2 - x^2}{2}\right).$$

With the help of Noether's theorem we get first integrals

$$J_1 = \frac{1}{\sqrt{1+(y')^2}} + Cy, \qquad J_2 = \frac{y'}{\sqrt{1+(y')^2}} - Cx, \qquad J_3 = \frac{y - xy'}{\sqrt{1+(y')^2}} + C\frac{x^2+y^2}{2}.$$

According to the structure of adjoint actions

$$X_1(X_3) = -X_2, \quad X_2(X_3) = X_1$$

integrals J_1 and J_2 can be obtained from the basic integral J_3:

$$X_1(J_3) = -J_2, \qquad X_2(J_3) = J_1.$$

In the discrete case we can choose Lagrangian

$$\mathcal{L} = \left(\sqrt{1+y_x^2} + C\frac{y_+ + y}{2}\right) \tag{61}$$

which admits all operators (58):

$$X_1\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_1) = 0,$$

$$X_2\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_2) = C = \underset{+h}{D}(Cx),$$

$$X_3\mathcal{L} + \mathcal{L}\underset{+h}{D}(\xi_3) = \underset{+h}{D}\left(C\frac{y^2 - x^2}{2}\right).$$

The Noether theorem gives us the following Euler equations for this Lagrangian

$$\begin{aligned}
\frac{\delta\mathcal{L}}{\delta y}: &\qquad \frac{2}{h_+ + h_-}\left(\frac{y_x}{\sqrt{1+y_x^2}} - \frac{y_{\bar{x}}}{\sqrt{1+y_{\bar{x}}^2}}\right) = C, \\
\frac{\delta\mathcal{L}}{\delta x}: &\quad -\frac{1}{\sqrt{1+y_x^2}} - C\frac{y_+ + y}{2} + \frac{1}{\sqrt{1+y_{\bar{x}}^2}} + C\frac{y + y_-}{2} = 0.
\end{aligned} \tag{62}$$

On these equations we have conservation of the first integrals

$$I_1 = \frac{1}{\sqrt{1+y_x^2}} + C\frac{y_+ + y}{2} = A_h \qquad I_2 = \frac{y_x}{\sqrt{1+y_x^2}} - C\frac{x_+ + x}{2} = B_h$$

$$I_3 = \frac{1}{\sqrt{1+y_x^2}}\left(\frac{y_+ + y}{2} - \frac{x_+ + x}{2}y_x\right) + C\left(\frac{y_+ y}{2} + \frac{x_+ x}{2}\right) = C_h$$

For discrete first integrals we have the same relations

$$X_1(I_3) = -I_2, \qquad X_2(I_3) = I_1$$

as for continuous ones.

If we use the notation $h_+ = \frac{\varepsilon}{\sqrt{1+y_x^2}}$, then the system (62) can be integrated by means of the first integrals. We get the solution

$$(A_h - Cy)^2 + (B_h + Cx)^2 = 1 + \left(\frac{C\varepsilon}{2}\right)^2$$

on the grid defined by (62). If we compare this solution with the solution of the original differential equation corresponding to $k = 0$ (given by (57)), we note that the difference is equal to $\left(\frac{C\varepsilon}{2}\right)^2$.

Eight–parameter groups G_8

Up to equivalence transformations there exists one eight–parameter group with operators

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = \frac{\partial}{\partial y}, \qquad X_3 = x\frac{\partial}{\partial y}, \qquad X_4 = x\frac{\partial}{\partial x}, \qquad X_5 = y\frac{\partial}{\partial x},$$
$$X_6 = y\frac{\partial}{\partial y}, \qquad X_7 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, \qquad X_8 = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} \tag{63}$$

which has an invariant equation

$$y'' = 0. \tag{64}$$

It is important to note that equation (64) is only an invariant manifold for the operators (63) and it can not be expressed in terms of invariants (there are no invariants in the space (x, y, y', y'') for the operators (63)).

In the finite-difference subspace $(x, h_+, h_-, y, y_+, y_-)$ there are no invariants as well. There is only one invariant manifold

$$y_x - y_{\bar{x}} = 0. \tag{65}$$

The invariant mesh can be expressed by a duplicated equation but in other variables:

$$\frac{h_-}{y - y_-} = \frac{h_+}{y - y_+}. \tag{66}$$

Concluding remarks

We have presented the whole set of invariant models among second–order difference equations, that corresponds to Lie's classification of second–order ODEs [11]. We believe that existence of other invariant difference models which have no continuous counterparts is very unlikely, although theoretically possible.

Incorporation of a mesh equation into the discrete model is an essential part of our approach. Not considering both equations (one equation represents a difference analog of the continuous equation from Lie's classification and the other yields a mesh) it would not be possible to construct discrete counterparts of Lie's equations. Indeed, without the lattice equation and transformations of independent variables it is possible to consider only one–variable groups. The only finite–dimensional groups which can be realized in this case up to an arbitrary change of variables are the group generated by the operators (see [12])

$$X_1 = \frac{\partial}{\partial y}, \qquad X_2 = y\frac{\partial}{\partial y}, \qquad X_3 = y^2\frac{\partial}{\partial y}$$

and its subgroups, since there exist groups G_1, G_2, G_3 only. This set of groups definitely does not correspond to Lie's group classification of second–order ODEs.

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Appendix

Table 1. Group classification of second–order equations [2]

Group		Equation	Mesh
G_1		$X_1 = \dfrac{\partial}{\partial x}$	$y'' = F(y, y')$
G_2	a)	$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = \dfrac{\partial}{\partial y}$	$y'' = F(y')$
	b)	$X_1 = \dfrac{\partial}{\partial y}, \quad X_2 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}$	$y'' = \dfrac{1}{x}F(y')$
G_3	a)	$X_1 = \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial y}, \quad X_2 = x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}, \quad X_3 = x^2\dfrac{\partial}{\partial x} + y^2\dfrac{\partial}{\partial y}$	$y'' + 2\dfrac{y' + Cy'\sqrt{y'} + y'^2}{x - y} = 0$
	b)	$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = 2x\dfrac{\partial}{\partial x} + y\dfrac{\partial}{\partial y}, \quad X_3 = x^2\dfrac{\partial}{\partial x} + xy\dfrac{\partial}{\partial y}$	$y'' = y^{-3}$
	c)	$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = \dfrac{\partial}{\partial y}, \quad X_3 = x\dfrac{\partial}{\partial x} + (x + y)\dfrac{\partial}{\partial y}$	$y'' = C\exp(-y')$
	d)	$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = \dfrac{\partial}{\partial y}, \quad X_3 = x\dfrac{\partial}{\partial x} + ky\dfrac{\partial}{\partial y}, \quad k \neq 0, \dfrac{1}{2}, 1, 2$	$y'' = Cy'^{\frac{k-2}{k-1}}$
	e)	$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = \dfrac{\partial}{\partial y}, \quad X_3 = (kx + y)\dfrac{\partial}{\partial x} + (ky - x)\dfrac{\partial}{\partial y}$	$y'' = C(1 + (y')^2)^{3/2}\exp(k\arctan(y'))$
G_8		$X_1 = \dfrac{\partial}{\partial x}, \quad X_2 = \dfrac{\partial}{\partial y}, \quad X_3 = x\dfrac{\partial}{\partial y}, \quad X_4 = x\dfrac{\partial}{\partial x}, \quad X_5 = y\dfrac{\partial}{\partial x},$ $X_6 = y\dfrac{\partial}{\partial y}, \quad X_7 = x^2\dfrac{\partial}{\partial x} + xy\dfrac{\partial}{\partial y}, \quad X_8 = xy\dfrac{\partial}{\partial x} + y^2\dfrac{\partial}{\partial y}$	$y'' = 0$

[2] S.Lie carried out the classification in the complex space, using complex substititions and complex bases of algebras as needed. Therefore his classification does not contain the point G_3 e).

Table 2. Group classification of second–order difference equations and meshes.

Group		Equation	Mesh
G_1		$\frac{2}{h_+ + h_-}(y_x - y_{\bar{x}}) = F\left(y, \frac{1}{2}(y_x + y_{\bar{x}})\right)$	$h_- = h_+$
G_2	a)	$\frac{2}{h_+ + h_-}(y_x - y_{\bar{x}}) = F\left(\frac{1}{2}(y_x + y_{\bar{x}})\right)$	$h_- = h_+$
	b)	$\frac{2}{h_+ + h_-}(y_x - y_{\bar{x}}) = \frac{1}{x}F\left(\frac{1}{2}(y_x + y_{\bar{x}})\right)$	$h_- = h_+$
G_3	a)	$\left(\sqrt{\frac{h_+^2 y_x}{(x-y_+)(x_+-y)}}\,\frac{x-y}{x_--y}\,\frac{h_++h_-}{h_+} - \sqrt{\frac{h_-^2 y_{\bar{x}}}{(x-y_-)(x_--y)}}\,\frac{y-x}{y-x_+}\,\frac{h_-+h_+}{h_-}\right) =$ $= \frac{C}{2}\left(\sqrt{\frac{h_+^2 y_x}{(x-y_+)(x_+-y)}} + \sqrt{\frac{h_-^2 y_{\bar{x}}}{(x-y_-)(x_--y)}}\right)^2$	$\frac{h_+^2 y_x}{(x-y_+)(x_+-y)} = \varepsilon^2,$ $\frac{h_-^2 y_{\bar{x}}}{(x-y_-)(x_--y)} = \varepsilon^2$
	b)	$2(y_x - y_{\bar{x}}) = \frac{h_+}{y^2 y_+} + \frac{h_-}{y^2 y_-}$	$\frac{h^-}{yy_-} = \frac{h^+}{yy_+} = \varepsilon$
	c)	$\frac{2}{h_- + h_+}\left(\exp(y_x) - \exp(y_{\bar{x}})\right) = C$	$h_+ = h_-$
	d)	$\frac{2(k-1)}{h_- + h_+}\left((y_x)^{\frac{1}{k-1}} - (y_{\bar{x}})^{\frac{1}{k-1}}\right) = C$	$h_+ = h_-$
	e)	$\frac{2\,(y_x - y_{\bar{x}})}{h_+ + h_-} = \frac{C}{h_+ + h_-}\,(1 + y_x y_{\bar{x}}) \times$ $\times\left(h_+\sqrt{1+y_x^2}\exp\left(k\arctan(y_x)\right) + h_-\sqrt{1+y_{\bar{x}}^2}\exp\left(k\arctan(y_{\bar{x}})\right)\right)$	$h_+ = \frac{\varepsilon}{\sqrt{1+y_x^2}}\exp\left(-k\arctan(y_x)\right),$ $h_- = \frac{\varepsilon}{\sqrt{1+y_{\bar{x}}^2}}\exp\left(-k\arctan(y_{\bar{x}})\right)$
G_8		$y_x - y_{\bar{x}} = 0$	$\frac{h_-}{y-y_-} = \frac{h_+}{y-y_+}$