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A POSTERIORI ERROR ESTIMATION FOR FINITE ELEMENT METHODS, AN OVERVIEW

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Abstract.

This is an overview of the most common a posteriori error estimation techniques for finite element methods. It covers methods based on gradient recovery, explicit and implicit methods, goal oriented techniques, and some applications.

Key words: A posteriori error estimators, finite element methods.

Contents

1 Introduction

When using finite element methods in applications, such as physics or engineering, one typically finds that in certain areas of the domain the numerical approximation of the solution is rather poor. In addition to removing information about the actual solution in these areas, such errors may influence the solution in the rest of the domain, making the global solution inaccurate. The natural way to handle this problem is to refine the discretisation in areas where the error contributions are large, but then the following question arises: How should such areas be identified, and what rate of refinement should be applied in order to balance the errors in the different regions and control the global error?

One answer to the above question is to develop a local error estimate for each element to indicate the magnitude of the local error. A priori error estimates known from standard finite element analysis are usually not practical as they only show the asymptotic behaviour of the error. However, it is possible to derive quite reliable *a posteriori* error estimates that are based on the solution of the discrete problem.

Such a posteriori error estimates have become powerful tools in finite element analysis since they provide control of the global error at the same time as they give information about the error distribution. A clear picture of the error distribution is important when running an adaptive procedure, while quantifying the total error provides the analyst with information about how good the overall approximation is.

An important issue is that a posteriori estimates should be cheap to compute compared to computing the discrete solution of the entire problem. This suggests that the estimators should be computed either by explicit use of the data and the finite element solution or by solving only local problems. Both these strategies will be pursued in this report, and in addition to traditional error estimators we will also present more recently developed techniques for goal oriented error estimation. These techniques are aimed at quantifying each element's contribution to the error in an a priori chosen quantity.

This report gives an overview of the most commonly used a posteriori error estimation techniques for finite element methods, and it is mainly based on the survey article *A posteriori error estimation in finite element analysis* by M.Ainsworth and J.T.Oden [4], and the book *A review of a posteriori error estimation and adaptive mesh-refinement techniques* by R.Verfürth [24].

2 Preliminaries

2.1 Notations and model problem

We will use standard notations for functional spaces. Let Ω be an open, bounded domain in $\mathbb{R}^n$, $n = 1, 2, 3$ with boundary $\partial\Omega$. For any open subset ω of Ω with Lipschitz boundary γ we denote by $W^{m,p}(\omega)$, $m \in \mathbb{Z}^+$, $1 \leq p \leq \infty$ the standard Sobolev spaces and by $L^m(\gamma)$ the standard Lebesgue spaces equipped with the norms

$$\begin{aligned} ||\phi||_{W^{m,p}(\omega)} &= \left\{ \sum_{|\alpha| \leq m} \int_{\omega} |D^\alpha \phi|^p \, dx \right\}^{1/p} \\ ||\phi||_{L^p(\gamma)} &= \left\{ \int_{\gamma} |\phi|^p \, ds \right\}^{1/p} , \end{aligned}$$

and the seminorm

$$|\phi|_{W^{m,p}(\omega)} = \left\{ \sum_{|\alpha| = m} \int_{\omega} |D^\alpha \phi|^p \, dx \right\}^{1/p} .$$

We will often be using the convenient abbreviations $L^2(\omega) = W^{0,2}(\omega)$, $H^m(\omega) = W^{m,2}(\omega)$, and $H_0^m(\omega) = W_0^{m,2}(\omega)$, in which the subscript '0' means that the space only contains functions that vanish on the boundary. Note that $|| \cdot ||_{L^\infty(\omega)}$ is used to denote the infinity norm. We will also be applying the L^2-norm to vectors, which should be interpreted as taking the inner-product of the vector with itself before integrating as usual.

The symbol $\mathcal{T}_i$ (or just $\mathcal{T}$) will be used to denote the finite element partition of Ω, and we will assume that $\mathcal{T}$ is a *proper partition*, which means it is a collection of elements satisfying:

1. $\bar{\Omega} = \cup_{T \in \mathcal{T}} \bar{T}$
2. Each element is a non-empty open subset of Ω.
3. The intersection of each distinct pair of elements T, K is empty.
4. Each element is a triangle or a convex quadrilateral.
5. Each side of an element is either a subset of the boundary $\partial\Omega$ or a full side of another element in the partition.

The set of all element sides will denoted by $\partial\mathcal{T}$ and the set of nodes will be denoted by $\mathcal{N}$. Different subdomains, or *patches*, that will be referred to can be seen in Figure 2.1. The partition $\mathcal{T}_i$ belongs to a family of partitions $\mathcal{F}$ which is assumed to be *locally quasi-uniform*. This means that there exists a constant C (note that the letter C throughout the report will be used to denote generic constants that may take on different values from place to place), such that for

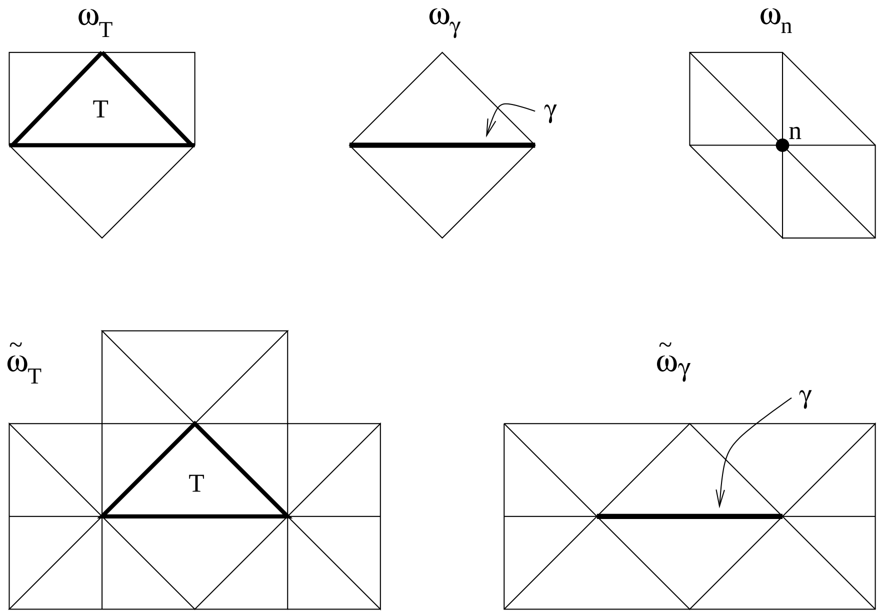

Figure 2.1: Domains ω_T, ω_γ, ω_n, $\tilde{\omega}_T$, and $\tilde{\omega}_\gamma$.

all patches $\tilde{\omega}_T$, $T \in \mathcal{T}_i$, the following inequalities hold for each element $K \in \tilde{\omega}_T$,

$$\frac{1}{C} \leq \frac{h_T}{h_K} \leq C.$$

Here h_T is the diameter of T. For any element T the ratio between h_T and the radius of the largest ball contained in T is assumed to be uniformly bounded by a constant κ, i.e. the grid is *regular*. The constant κ expresses the shape regularity of the grid, and C in the above inequalities depends on this constant. Note that this condition rules out highly stretched meshes.

As a model problem we consider the Poisson equation defined on a domain $\Omega \subset \mathbb{R}^2$ with partly Dirichlet and partly Neumann boundary conditions,

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma_D \\ \tfrac{\partial u}{\partial n} &= g && \text{on } \Gamma_N \ , \end{aligned}$$

with $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$, $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, and $\Gamma_D \cap \Gamma_N = \emptyset$. The outward unit normal vector on $\partial\Omega$ is denoted by $\mathbf{n}$, and $\mathbf{n} \cdot \nabla u$ is denoted by $\partial u/\partial n$.

Multiplying by a test function, integrating over the domain, and applying Green's formula leads us to the variational form of problem (2.1), which reads: Find $u \in V$

such that

$$a(u,v) = L(v) \quad \forall v \in V, \tag{2.1}$$

where

$$V = \{v \in H^1(\Omega) : \; v = 0 \text{ on } \Gamma_D\} \tag{2.2}$$

and

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \cdot \nabla v \, dx \\ L(v) &= \int_\Omega fv \, dx + \int_{\Gamma_N} gv \, ds \equiv F(v) + \int_{\Gamma_N} gv \, ds \, . \end{aligned}$$

Both $a(\cdot,\cdot)$, $L(\cdot)$, and $F(\cdot)$ may be used as local quantities with a subscript indicating the integral domain, e.g.

$$\begin{aligned} a_T(u,v) &= \int_T \nabla u \cdot \nabla v \, dx \\ L_T(v) &= \int_T fv \, dx + \int_{\Gamma_N \cap \partial T} gv \, ds \equiv F_T(v) + \int_{\Gamma_N \cap \partial T} gv \, ds \, . \end{aligned}$$

Also V has a local counterpart,

$$V(T) = \{v \in H^1(T) : \; v = 0 \text{ on } \partial T \cap \Gamma_D\}.$$

Suppose that $V_h \subset V$ is a finite element subspace corresponding to $\mathcal{T}$. The finite element approximation to problem (2.1) is to find $u_h \in V_h$ such that

$$a(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h \, . \tag{2.3}$$

Define the polynomial space of degree $p \in \mathbb{N}$ on triangular elements as

$$\mathbb{P}_p = span\{x^i y^j \; : \; 0 \leq i + j \leq p\}, \tag{2.4}$$

and on quadrilateral elements as

$$\mathbb{Q}_p = span\{x^i y^j \; : \; 0 \leq i, \, j \leq p\} \, . \tag{2.5}$$

If V_h is the space of continuous piecewise polynomials of order p, the finite element subspace is said to be of order p. The notation $V_h(T)$ will be used in the sense

$$V_h(T) = \{v_h|_T \, , \; v_h \in V_h\} \, .$$

The error $e = u - u_h$ belongs to the space V and is implicitly given as

$$a(e,v) = a(u,v) - a(u_h,v) = L(v) - a(u_h,v) \quad \forall v \in V \, . \tag{2.6}$$

Moreover,

$$a(e, v_h) = L(v_h) - a(u_h, v_h) = L(v_h) - L(v_h) = 0 \quad \forall v_h \in V_h \,. \tag{2.7}$$

The above equation expresses orthogonality (Galerkin orthogonality) between the error and the space V_h. We further define the *energy norm* as

$$|||\phi|||^2 = a(\phi, \phi)\,, \tag{2.8}$$

which in this simple test case is nothing but the seminorm

$$|\phi|_{H^1(\Omega)} \equiv ||\nabla\phi||_{L^2(\Omega)} = \int_\Omega \nabla\phi \cdot \nabla\phi \, dx \,.$$

Also energy norms will be used as local quantities, for instance $|||\phi|||_T^2 \equiv a_T(\phi, \phi)$. Note that the energy norm may also be expressed as

$$|||\phi||| = \sup_{v \in V} \frac{|a(\phi, v)|}{|||v|||} \,. \tag{2.9}$$

What we will aim at in this report is to obtain local error estimators η_T on each element T. The local error estimators then usually define the global error estimator as

$$\eta = \left(\sum_{T \in \mathcal{T}} \eta_T^2 \right)^{1/2} \,. \tag{2.10}$$

We may then define the local and the global effectivity indices as

$$\frac{\eta_T}{|||e|||_T} \quad \text{and} \quad \frac{\eta}{|||e|||} \,.$$

If $lim_{h \to 0} \frac{\eta}{|||e|||} = 1$ the estimator is said to be asymptotically exact.

2.2 Mesh refinement algorithm

Assuming that local error estimates can be computed, an adaptive mesh refinement process may have the following structure.

1. Construct an initial coarse mesh, $\mathcal{T}_0$, put $k := 0$.
2. Solve the discrete problem on the present mesh, $\mathcal{T}_k$.
3. For each element T in $\mathcal{T}_k$ compute the a posteriori error estimate.
4. If the estimated global error is sufficiently small, then stop. Else decide which elements need to be refined and construct a refined mesh $\mathcal{T}_{k+1}$. Put $k := k + 1$ and return to step 2.

Instead of refining elements in step 4, one could increase the order of the interpolation functions on the elements that should have been refined. The rate of refinement may depend on the magnitude of the local error estimator.

3 Gradient recovery

A simple approach for getting an a posteriori error estimate, is first to post process the finite element solution to obtain a smoother approximation for the gradient than what is found directly from the discrete solution. The difference between the two gradient fields can then be viewed as an expression for the error. Such estimators are quite easy to implement, and they can often give good results.

3.1 A theoretical framework

To obtain a gradient recovery estimator one needs to to approximate ∇u in the expression

$$|||e|||^2 = \int_\Omega |\nabla u - \nabla u_h|^2 \, dx \; . \tag{3.1}$$

Let I_{V_h} be the nodal interpolation operator onto the finite element space, and let the finite element space be of order p. Denoting by $G(u_h)$ the approximation of the true gradient, we first identify some conditions, originally suggested in [1], assuring that $G(u_h)$ is a good approximation to ∇u.

(C1) *Consistency condition.* If $u \in \mathbb{P}_{p+1}(\tilde{\omega}_T)$ then

$$G(I_{V_h} u) \equiv I_{V_h} \nabla u \qquad \text{on } T. \tag{3.2}$$

For the definition of $\tilde{\omega}_T$ see Figure 2.1, and note that this condition does not uniquely define the recovery operator.

(C2) *Localisation condition.* Let $T \in \mathcal{T}$. If $x^* \in T$ then the value of $G(\phi)(x^*)$ depends only on values of $\nabla\phi$ sampled on $\tilde{\omega}_T$.

Condition (C2) makes the computation of G a local and cheap operation. The important point here is to avoid global, expensive computations. If the recovery procedure is too expensive, it would be better to improve the gradient by resolving the original problem on a finer grid.

(C3) *Boundedness and linearity condition.* $G : V_h \to V_h \times V_h$ is a linear operator and there exists a constant C, independent of h, such that

$$||G(v_h)||_{L^\infty(T)} \le C||\nabla v_h||_{L^\infty(\tilde{\omega}_T)} \qquad \forall T \in \mathcal{T} \quad \forall v_h \in V_h \; . \tag{3.3}$$

Together, (C1)-(C3) assures that when u is smooth, $G(I_{V_h} u)$ is a good and locally based approximation to ∇u, and the following result [4, Lemma 2.3] states in exact terms how good the approximation will be.

Lemma 3.1 *If G satisfies (C1)-(C3) and $u \in H^{p+2}(\tilde{\omega}_T)$ then*

$$||\nabla u - G(I_{V_h} u)||_{L^2(T)} \le C h^{p+1} |u|_{H^{p+2}(\tilde{\omega}_T)} \, . \tag{3.4}$$

□

One remaining problem that arises when approximating ∇u using G is that one has to use the finite element solution u_h instead of $I_{V_h} u$. This means that even if the operator G is good we may still get inaccurate results. The difference between u_h and $I_{V_h} u$ is sometimes referred to as the *pollution* error because it is a globally transported error which has nothing to do with the local approximation properties. To make sure that this error is small, we will in some cases be assuming the finite element solution to be *superconvergent* according to the definition below.

(SC) *Superconvergence condition.* There exist constants $C(u)$ and $\tau \in [0, 1)$, independent of h, such that

$$||\nabla(u_h - I_{V_h} u)||_{L^2(\Omega)} \leq C(u)h^{p+\tau} \ . \tag{3.5}$$

This states that when superconvergence occurs, ∇u_h converges faster towards $\nabla I_{V_h} u$ than it does towards ∇u itself, provided h is small enough. Superconvergence only occurs under very special circumstances, and a survey of superconvergence results is found in [16].

Lemma 3.2 *Assume that $u \in H^{p+2}(\Omega)$, G satisfies (C1)-(C3), and that (SC) holds. Then*

$$||\nabla u - G(u_h)||_{L^2(\Omega)} \leq C(u)h^{p+\tau} \ . \tag{3.6}$$

PROOF. Using the linearity of G and the triangle inequality

$$||\nabla u - G(u_h)||_{L^2(T)} \leq ||\nabla u - G(I_{V_h} u)||_{L^2(T)} + ||G(I_{V_h} u - u_h)||_{L^2(T)} \ . \tag{3.7}$$

The property (C3) and an inverse estimate shows that

$$||G(I_{V_h} u - u_h)||_{L^2(T)} \quad \leq \quad C||\nabla(I_{V_h} u - u_h)||_{L^2(\tilde{\omega}_T)}. \tag{3.8}$$

Summing over all elements, using (3.7), (3.4), (3.8), and (SC), yields

$$\begin{aligned} ||\nabla u - G(u_h)||_{L^2(\Omega)} &\leq \sum_{T \in \mathcal{T}} \Big\{ Ch^{p+1}|u|_{H^{p+2}(\tilde{\omega}_T)} + C||\nabla(I_{V_h} u - u_h)||_{L^2(\omega_T)} \Big\} \\ &\leq C\Big\{ h^{p+1}|u|_{H^{p+2}(\Omega)} + ||\nabla(I_{V_h} u - u_h)||_{L^2(\Omega)} \Big\} \\ &\leq C(u)h^{p+\tau} \, . \end{aligned} \tag{3.9}$$

□

Based on (3.1) we now define the error estimator on element T as

$$\eta_T = ||G(u_h) - \nabla u_h||_{L^2(T)} \ , \tag{3.10}$$

where the global error estimator is obtained from summing all the element contributions, see (2.10). Note that for the h-version of the finite element method

$$|||e||| \geq C(u)h^p \ , \tag{3.11}$$

for non-trivial cases. Before we state the main result, also note that

$$\Big| \, ||v|| - ||w|| \, \Big| \leq ||v - w||.$$

This is easily seen from the triangle inequality:

$$\begin{array}{lll} \text{If } ||w|| \leq ||v|| \ : & \Big| ||v|| - ||w|| \Big| = ||v|| - ||w|| \leq ||v - w|| \ . \\ \text{If } ||w|| > ||v|| \ : & \Big| ||v|| - ||w|| \Big| = ||w|| - ||v|| \leq ||w - v|| = ||v - w|| \ . \end{array}$$

Theorem 3.3 *Let η be the global a posteriori error estimator defined by (3.10) and (2.10). Assume that (C1)-(C3), (SC), and (3.11) hold. Let p be the order of the finite element space and assume that $u \in H^{p+2}(\Omega)$. Then*

$$\lim_{h \to 0} \frac{\eta}{|||e|||} = 1 \ . \tag{3.12}$$

PROOF. We prove the theorem by first bounding the expression $\Big| \eta - |||e||| \Big|$. Using the triangle inequality and Lemma 3.2 yields,

$$\begin{aligned} \Big| \eta - |||e||| \Big| &= \big| \, ||G(u_h) - \nabla u_h||_{L^2(\Omega)} - ||\nabla e||_{L^2(\Omega)} \big| \\ &\leq ||G(u_h) - \nabla u_h - \nabla e||_{L^2(\Omega)} \\ &= ||G(u_h) - \nabla u||_{L^2(\Omega)} \\ &\leq C(u) h^{p+\tau} \ . \end{aligned}$$

Therefore, from (3.11),

$$\left| \frac{\eta}{|||e|||} - 1 \right| \leq C(u) \frac{h^{p+\tau}}{|||e|||} \leq C(u) h^{\tau} \ .$$

□

Theorem 3.3 states that the estimator is asymptotically exact.

3.2 Two recovery based estimators

Consider a piecewise linear finite element solution $u_h \in V_h$ in one dimension. Then a superconvergence result is known, see [27]. If $u \in H^3(\Omega)$ and the mesh is quasi-uniform, then

$$||\nabla(u_h - I_{V_h} u)||_{L^2(\Omega)} \leq C h^2 |u|_{H^3(\Omega)} \ , \tag{3.13}$$

where $I_{V_h} u$ is the nodal interpolant of u. Figure 3.1 shows how an approximation $G(u_h)$ can be computed, and it is clear that the locality condition (C2) is satisfied. In Figure 3.2 it is shown that G satisfies the consistency condition (C1). As we can see, the approximated gradient exactly reproduces the true gradient when u is a parabola on each patch. Moreover,

$$||G(v_h)||_{L^\infty(T)} \leq 3|\nabla v_h|_{L^\infty(\tilde{\omega}_T)} \ , \tag{3.14}$$

which means that (C1)-(C3) and (SC) are all valid. The estimator is defined in (3.10), and Theorem 3.3 states that it is asymptotically exact. This estimator was first suggested by Babuška and Rheinboldt [8].

The most famous estimator based on this concept is probably the Zienkiewicz-Zhu patch recovery technique. It was proposed in [25, 26], and it fitted perfectly into the framework presented here, which was already established by Ainsworth and Craig [1].

For linear triangular elements the Zienkiewicz-Zhu estimator samples the gradient of the finite element solution in the centroids of the elements in each patch ω_n, see Figure 2.1. Then, for each component k in the gradient, a function $v_k \in \mathbb{P}_1$ is fitted to these datapoints in a least squares sense. Finally, the approximated

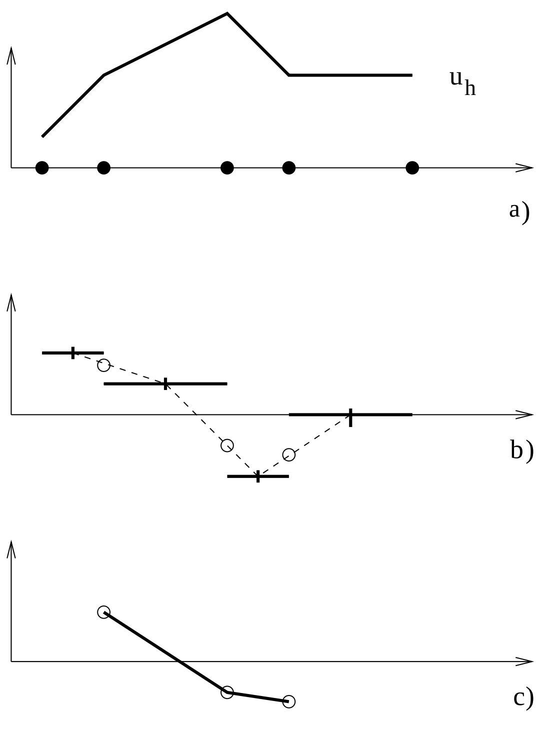

Figure 3.1: The figure shows how $G(u_h)$ is computed. a): The finite element solution. b): The gradient (solid line) and the postprocessed gradient (dashed line) of the finite element solution. c): The recovered gradient.

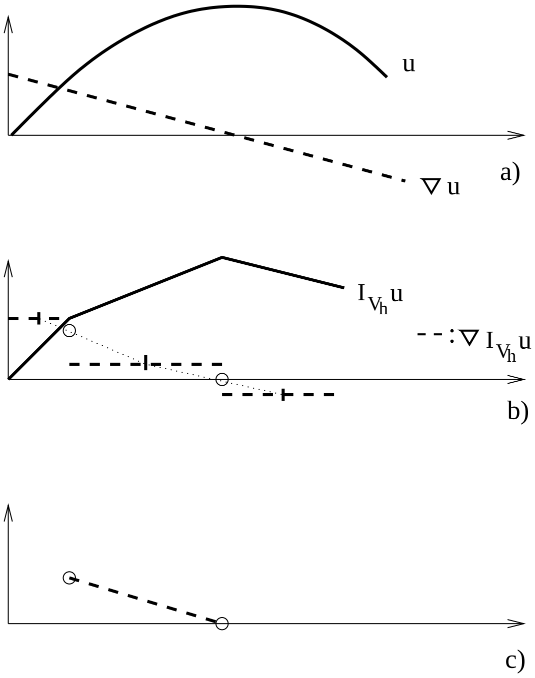

Figure 3.2: The figure shows that the recovered gradient in c) is exact as long as u is a second order polynomial. The gradient recovery procedure therefore satisfies the consistency condition.

gradient, evaluated in the central node n, is taken to be

$$g_k(u_h) = v_k(x_n),$$

where $g_k(u_h)$ is the k'th component of $G_n(u_h)$, the value of the recovered gradient in node n. Now the recovered gradient is defined as

$$G(u_h) = \sum_{n \in \mathcal{N}} G_n(u_h)\, \theta_n,$$

in which θ_n is the Lagrange basis function associated with node n.

For other types of elements the sample points are chosen differently and v_k is taken from a different polynomial space, but except for that the procedure is the same. This method clearly satisfies (C2) and (C3), and for certain grids (C1) and (SC) are also satisfied. Although the superconvergence property is generally lost in practical cases, this estimator has proved to be surprisingly accurate.

4 Explicit error estimators

In this chapter we will develop *explicit* error estimators, which means the estimators are computed by direct use of the finite element solution without post-processing the solution or solving local problems. The estimators will be based on various kinds of residuals that are directly computable. The main part of this section is dedicated to showing that one such estimator is an *equivalent* measure of the error, in the sense that it bounds the norm of the error from above and below within a multiplicative constant. Explicit estimators were first introduced by Babuška and Rheinboldt [7].

4.1 An explicit energy norm estimator

Again we consider the model problem and we start by decomposing the equation for the error (2.6) into separate contributions from each element,

$$\begin{aligned} a(e,v) &= L(v) - a(u_h, v) \\ &= \sum_{T\in\mathcal{T}} \left\{ \int_T fv\,dx + \int_{\partial T\cap\Gamma_N} gv\,ds - \int_T \nabla u_h \cdot \nabla v\,dx \right\} \end{aligned} \tag{4.1}$$

for all $v \in V$. Using Green's formula on element T shows that

$$\int_T \nabla u_h \cdot \nabla v\,dx = \int_{\partial T} \frac{\partial u_h}{\partial n_T} v\,ds - \int_T \Delta u_h v\,dx\ .$$

As usual $\mathbf{n}_T$ is the outward unit normal vector on ∂T, and $\mathbf{n}_T \cdot \nabla u$ is denoted by $\partial u/\partial n_T$. Substituting this into (4.1) gives

$$a(e,v) = \sum_{T\in\mathcal{T}} \left\{ \int_T rv\,dx + \int_{\partial T\cap\Gamma_N} Rv\,ds - \int_{\partial T\setminus\partial\Omega} \frac{\partial u_h}{\partial n_T} v\,ds \right\}, \tag{4.2}$$

where

$$r = f + \Delta u_h \quad \text{in } T \tag{4.3}$$

is called the *interior residual*, and where

$$R = g - \frac{\partial u_h}{\partial n_T} \quad \text{on } \partial T \cap \Gamma_N \tag{4.4}$$

is called the *boundary residual*. Both r and R are well defined since u_h is smooth on the interior of each element.

Define the *jump* in the flux across an edge γ as

$$\left[\frac{\partial u_h}{\partial n_\gamma}\right] = \mathbf{n}_T \cdot (\nabla u_h)_T + \mathbf{n}_K \cdot (\nabla u_h)_K\ , \tag{4.5}$$

where γ separates element T and K. Then equation (4.2) can be written as

$$a(e,v) = \sum_{T\in\mathcal{T}} \left\{ \int_T rv\,dx + \int_{\partial T\cap\Gamma_N} Rv\,ds \right\} - \sum_{\gamma\in\partial\mathcal{T}\setminus\partial\Omega} \int_\gamma \left[\frac{\partial u_h}{\partial n_\gamma}\right] v\,ds\ .$$

By extending the definition of R to

$$R = \begin{cases} g - \frac{\partial u_h}{\partial n_T} & \text{on } \partial T\cap\Gamma_N \\ 0 & \text{on } \Gamma_D \\ -\left[\frac{\partial u_h}{\partial n_\gamma}\right] & \text{on } \gamma\in\partial\mathcal{T}\setminus\partial\Omega, \end{cases}$$

we obtain

$$a(e,v) = \sum_{T\in\mathcal{T}} \int_T rv\,dx + \sum_{\gamma\in\partial\mathcal{T}} \int_\gamma Rv\,ds \quad \forall v\in V\ . \tag{4.6}$$

For $v\in V$ let $I_{V_h}v$ be an interpolant of v from the subspace V_h. Combining (2.7) and (4.6) then gives

$$0 = \sum_{T\in\mathcal{T}} \int_T r\, I_{V_h}v\,dx + \sum_{\gamma\in\partial\mathcal{T}} \int_\gamma R\, I_{V_h}v\,ds\ ,$$

and therefore

$$a(e,v) = \sum_{T\in\mathcal{T}} \int_T r(v - I_{V_h}v)\,dx + \sum_{\gamma\in\partial\mathcal{T}} \int_\gamma R\,(v - I_{V_h}v)\,ds \quad \forall v\in V\ . \tag{4.7}$$

It is possible [14] to choose the interpolant $I_{V_h}v$ such that

$$||v - I_{V_h}v||_{L^2(T)} \le Ch_T||\nabla v||_{L^2(\tilde\omega_T)}$$

and

$$||v - I_{V_h}v||_{L^2(\gamma)} \le Ch_\gamma^{1/2}||\nabla v||_{L^2(\tilde\omega_\gamma)}\ ,$$

where h_γ is the length of edge γ. Applying the Cauchy-Schwarz inequality to (4.7) and thereafter using the above results, yields

$$\begin{aligned} a(e,v) &\le \sum_{T\in\mathcal{T}} ||r||_{L^2(T)}||v - I_{V_h}v||_{L^2(T)} + \sum_{\gamma\in\partial\mathcal{T}} ||R||_{L^2(\gamma)}||v - I_{V_h}v||_{L^2(\gamma)} \\ &\le C\left\{ \sum_{T\in\mathcal{T}} ||\nabla v||_{L^2(\tilde\omega_T)}h_T||r||_{L^2(T)} + \sum_{\gamma\in\partial\mathcal{T}} ||\nabla v||_{L^2(\tilde\omega_\gamma)}h_\gamma^{1/2}||R||_{L^2(\gamma)} \right\}. \end{aligned} \tag{4.8}$$

The two sums can be rewritten as one large sum,

$$\sum_i \alpha_i \beta_i,$$

where α's denote the entries $||\nabla v||_{L^2(\tilde{\omega}_T)}$ and $||\nabla v||_{L^2(\tilde{\omega}_\gamma)}$, and where β's denote $h_T||r||_{L^2(T)}$ and $h_\gamma^{1/2}||R||_{L^2(\gamma)}$. Once again thanks to the Cauchy-Schwarz inequality, the following is obtained,

$$\begin{aligned} a(e,v) &\leq C\sum \alpha_i\beta_i \\ &\leq C\Big(\sum \alpha_i^2\Big)^{1/2}\Big(\sum \beta_i^2\Big)^{1/2} \\ &\leq C||\nabla v||_{L^2(\Omega)}\left\{\sum_{T\in\mathcal{T}} h_T^2||r||^2_{L^2(T)} + \sum_{\gamma\in\partial\mathcal{T}} h_\gamma||R||^2_{L^2(\gamma)}\right\}^{1/2}. \end{aligned} \tag{4.9}$$

Finally, we choose $v = e$ such that the left hand side in the above inequality becomes $a(e,e) = |||e|||^2 \equiv ||\nabla e||^2_{L^2(\Omega)}$, and thereby

$$|||e||| \leq C\left\{\sum_{T\in\mathcal{T}} h_T^2||r||^2_{L^2(T)} + \sum_{\gamma\in\partial\mathcal{T}} h_\gamma||R||^2_{L^2(\gamma)}\right\}^{1/2}.$$

Since the partition is assumed to be shape regular and locally quasi-uniform, h_T and h_γ (with $\gamma\subset\partial T$) differ only by a multiplicative constant, and it is usual to rewrite the expression as

$$|||e||| \leq C\sum_{T\in\mathcal{T}}\left\{h_T^2||r||^2_{L^2(T)} + \frac{1}{2}h_T||R||^2_{L^2(\partial T)}\right\}^{1/2} \tag{4.10}$$

to identify the local error estimator η_T,

$$\eta_T^2 = h_T^2||r||^2_{L^2(T)} + \frac{1}{2}h_T||R||^2_{L^2(\partial T)}. \tag{4.11}$$

The factor one half is present because now every interior edge is accounted for twice. The edges on the Neumann part of the boundary are accounted for only once, and they could therefore have been given a unity weight, but this only changes the constant C in (4.10). All terms in the expression for η_T can be computed directly from the finite element solution.

4.1.1 Equivalence of estimator

The error estimator obtained by summing η_T defined in (4.11) as suggested in (2.10), clearly satisfies

$$|||e||| \leq C\eta\ , \tag{4.12}$$

but it is desirable that the estimator is an *equivalent measure of the error*, meaning it also needs to satisfy

$$\eta \le C|||e||| \; . \tag{4.13}$$

We will now show that this is indeed the case if f and g can be well approximated. In order to prove such a result we will follow the strategy of Verfürth [23], which requires the use of so-called *bubble functions*. For this purpose two kinds of bubble functions are needed. An *interior* bubble function $b_T \, : \, T \to \mathbb{R}$ is the lowest order polynomial which vanishes on the boundary ∂T and is nonzero on the interior of T. An *edge* bubble function $b_\gamma \, : \, \omega_\gamma \to \mathbb{R}$ is, for each $K \in \omega_\gamma$, the lowest order polynomial which vanishes on $\partial K \setminus \gamma$, and which is nonzero on the interior of K. On the reference triangle

$$\hat{T} = \{(\hat{x}, \hat{y}) \, : \, 0 \le \hat{x} \le 1 \, ; \, 0 \le \hat{y} \le 1 - \hat{x}\} \; ,$$

the interior bubble function is expressed as

$$b_{\hat{T}} = 27 \hat{\lambda}_1 \hat{\lambda}_2 \hat{\lambda}_3 \; ,$$

where $\hat{\lambda}_i$ denotes a barycentric (area) coordinate on $\hat{T}$. $\hat{\lambda}_i$ is linear and it takes on the value 1 in corner number i and vanishes on the opposite side. Similarly the first edge bubble function is defined as

$$b_{\hat{\gamma}_1} = 4 \hat{\lambda}_2 \hat{\lambda}_3 \; .$$

The coefficients in front makes the value of the bubble functions equal to unity in the middle of the element and in the middle of the edge, respectively. Unity is the maximum value obtained by bubble functions. For the reference quadrilateral element

$$\hat{T} = \{(\hat{x}, \hat{y}) \, : \, -1 \le \hat{x}, \, \hat{y} \le 1\},$$

the bubble functions are

$$\begin{aligned} b_{\hat{T}} &= (1 - \hat{x}^2)(1 - \hat{y}^2) \; , \\ b_{\hat{\gamma}_1} &= \frac{1}{2}(1 - \hat{x}^2)(1 - \hat{y}) \; . \end{aligned}$$

The two following theorems concerning properties of bubble functions will prove useful. The proofs are based on scaling arguments and the equivalence of norms on finite dimensional spaces, see [4].

Theorem 4.1 *Let $\hat{P} \subset H^1(\hat{T})$ be a finite dimensional subspace of functions defined on the reference element, and let $F_T \, : \, \hat{T} \to T$ be an invertible mapping. Define $P = \{\hat{v} \circ F_T^{-1} \, : \, \hat{v} \in \hat{P}\}$. Then there exists a constant C such that for all $v \in P$*

$$C^{-1} ||v||^2_{L^2(T)} \le \int\limits_T b_T v^2 \, dx \le C ||v||^2_{L^2(T)}$$

and

$$C^{-1}||v||_{L^2(T)} \leq ||b_T v||_{L^2(T)} + h_T |b_T v|_{H^1(T)} \leq C||v||_{L^2(T)} \,.$$

□

Theorem 4.2 *Let $\gamma \subset \partial T$ be and edge and b_γ the corresponding bubble function. Let P be the finite dimensional space of functions defined on γ obtained by mapping $\hat{P} \subset H^1(\hat{T})$ onto γ. Then there exists a constant C such that for all $v \in P$*

$$C^{-1}||v||^2_{L^2(\gamma)} \leq \int_\gamma b_\gamma v^2 \, ds \leq C||v||^2_{L^2(\gamma)} \,.$$

Moreover, there exists an extension of v to T (again denoted by v) such that

$$h_T^{-1/2}||b_\gamma v||_{L^2(T)} + h_T^{1/2}|b_\gamma v|_{H^1(T)} \leq C||v||_{L^2(\gamma)} \,.$$

□

Bounding the interior and boundary residuals

We will now find upper bounds for the norm of the interior and the boundary residual which are present in the expression for η_T, equation (4.10). Starting with r, let $\bar{r}_T \in C^0(T)$ be an approximation, from some finite dimensional space, to the interior residual r on element T. Theorem 4.1 gives

$$||\bar{r}_T||^2_{L^2(T)} \leq C \int_T b_T \bar{r}_T^2 \, dx \,. \tag{4.14}$$

As $b_T \bar{r}_T$ vanishes on ∂T it may be extended, with the value zero, to the whole of Ω giving a function in the space V. With this choice for v in (4.6),

$$a(e, b_T \bar{r}_T) = \int_T b_T r \bar{r}_T \, dx \,.$$

This implies the identity

$$\int_T b_T \bar{r}_T^2 \, dx = \int_T b_T \bar{r}_T (\bar{r}_T - r) \, dx + a(e, b_T \bar{r}_T) \,, \tag{4.15}$$

which will be our point of departure when bounding r. From Theorem 4.1 we have

$$||b_T v||_{L^2(T)} \leq C||v||_{L^2(T)} \,,$$

so by using this and the Cauchy-Schwarz inequality, the first term on the right hand side of (4.15) can be bounded,

$$\begin{aligned}\int_T b_T\bar{r}_T(\bar{r}_T - r)\,dx &\leq ||b_T\bar{r}_T||_{L^2(T)}\,||\bar{r}_T - r||_{L^2(T)} \\ &\leq C||\bar{r}_T||_{L^2(T)}\,||\bar{r}_T - r||_{L^2(T)}\ . \end{aligned}\tag{4.16}$$

Turning now to the second term,

$$\begin{aligned} a(e, b_T\bar{r}_T) &= \int_T \nabla e\cdot\nabla(b_T\bar{r}_T)\,dx \\ &\leq ||\nabla e||_{L^2(T)}||\nabla(b_T\bar{r}_T)||_{L^2(T)} \\ &= |||e|||_T\,|b_T\bar{r}_T|_{H^1(T)}\ . \end{aligned}$$

Further, from Theorem 4.1 we have

$$h_T|b_T\bar{r}_T|_{H^1(T)} \leq C||\bar{r}_T||_{L^2(T)}\ ,$$

so

$$a(e, b_T\bar{r}_T) \leq Ch_T^{-1}|||e|||_T\,||\bar{r}_T||_{L^2(T)}\ .$$

Summarising these results,

$$||\bar{r}_T||^2_{L^2(T)} \leq C\Big\{||\bar{r}_T||_{L^2(T)}||\bar{r}_T - r||_{L^2(T)} + h_T^{-1}|||e|||_T\ ||\bar{r}_T||_{L^2(T)}\Big\}\ ,$$

and hence

$$||\bar{r}_T||_{L^2(T)} \leq C\Big\{h_T^{-1}|||e|||_T + ||\bar{r}_T - r||_{L^2(T)}\Big\}\ .$$

By the triangle inequality we now obtain the bound

$$\begin{aligned} ||r||_{L^2(T)} &\leq ||\bar{r}_T||_{L^2(T)} + ||\bar{r}_T - r||_{L^2(T)} \\ &\leq C\Big\{h_T^{-1}|||e|||_T + ||\bar{r}_T - r||_{L^2(T)}\Big\}\ . \end{aligned}\tag{4.17}$$

Next $||R||_{L^2(\gamma)}$ needs to be bounded, and the argument is very much the same as for $||r||_{L^2(T)}$. Let $\bar{R}_\gamma$ be a continuous approximation, from some finite dimensional space, to the boundary residual on γ. From Theorem 4.2

$$||\bar{R}_\gamma||^2_{L^2(\gamma)} \leq C\int_\gamma b_\gamma\bar{R}_\gamma^2\,ds\ . \tag{4.18}$$

The function $v = b_\gamma\bar{R}_\gamma$ is continuous and vanishes on the boundary of ω_γ, so it can be extended with the value zero to the whole of Ω, giving a function $v\in V$. Now equation (4.6) gives

$$a(e, b_\gamma\bar{R}_\gamma) = \int_{\omega_\gamma} b_\gamma r\bar{R}_\gamma\,dx + \int_\gamma b_\gamma R\bar{R}_\gamma\,ds\ ,$$

so the right hand side of (4.18) satisfies the identity

$$\int_\gamma b_\gamma \bar{R}_\gamma^2 \, ds \;=\; \int_\gamma b_\gamma \bar{R}_\gamma (\bar{R}_\gamma - R) \, ds \;+\; a(e, b_\gamma \bar{R}_\gamma) \;-\; \int_{\omega_\gamma} b_\gamma r \bar{R}_\gamma \, dx \, . \tag{4.19}$$

Applying the Cauchy-Schwarz inequality and utilising the trivial fact that $||b_\gamma v||_{L^2(\gamma)} \le ||v||_{L^2(\gamma)}$, the first term on the right hand side is bounded by

$$\int_\gamma b_\gamma \bar{R}_\gamma (\bar{R}_\gamma - R) \le ||\bar{R}_\gamma||_{L^2(\gamma)} ||\bar{R}_\gamma - R||_{L^2(\gamma)} \, . \tag{4.20}$$

Consider an edge $\gamma \subset \partial T$. From Theorem 4.2 we get

$$|b_\gamma \bar{R}_\gamma|_{H^1(T)} \le C h_T^{-1/2} ||\bar{R}_\gamma||_{L^2(\gamma)} \, . \tag{4.21}$$

When bounding $a(e, b_\gamma \bar{R}_\gamma)$ we will need to bound $||\nabla(b_\gamma \bar{R}_\gamma)||_{L^2(\omega_\gamma)}$, which equals $|b_\gamma \bar{R}_\gamma|_{H^1(T)} \; + \; |b_\gamma \bar{R}_\gamma|_{H^1(K)}$, where $\gamma = \bar{T} \cap \bar{K}$. First using the Cauchy-Schwarz inequality and then (4.21) on T and K separately, gives

$$\begin{aligned} a(e, b_\gamma \bar{R}_\gamma) \;&=\; \int_{\omega_\gamma} \nabla e \cdot \nabla(b_\gamma \bar{R}_\gamma) \, dx \\ &\le\; ||\nabla e||_{L^2(\omega_\gamma)} ||\nabla(b_\gamma \bar{R}_\gamma)||_{L^2(\omega_\gamma)} \\ &\le\; C|||e|||_{\omega_\gamma} \; h_T^{-1/2} ||\bar{R}_\gamma||_{L^2(\gamma)} \, . \end{aligned} \tag{4.22}$$

Turning now to the last of the three terms in (4.19), we have from Theorem 4.2 that

$$||b_\gamma \bar{R}_\gamma||_{L^2(T)} \le C h_T^{1/2} ||\bar{R}_\gamma||_{L^2(\gamma)} \, .$$

By the Cauchy-Schwarz inequality and the same argument as above, one therefore obtains

$$\int_{\omega_\gamma} b_\gamma r \bar{R}_\gamma \, dx \;\le\; ||r||_{L^2(\omega_\gamma)} ||b_\gamma \bar{R}_\gamma||_{L^2(\omega_\gamma)} \;\le\; C||r||_{L^2(\omega_\gamma)} h_T^{1/2} ||\bar{R}_\gamma||_{L^2(\gamma)} \, . \tag{4.23}$$

Combining (4.18), (4.19), (4.20), (4.22), and (4.23) yields

$$||\bar{R}_\gamma||^2_{L^2(\gamma)} \;\le\; C||\bar{R}_\gamma||_{L^2(\gamma)} \Big\{ h_T^{-1/2} |||e|||_{\omega_\gamma} \;+\; h_T^{1/2} ||r||_{L^2(\omega_\gamma)} \;+\; ||\bar{R}_\gamma - R||_{L^2(\gamma)} \Big\} \, . \tag{4.24}$$

The term $||r||_{L^2(\omega_\gamma)}$ in (4.24) will now be replaced using (4.17). Note that it has to be used twice to cover the whole of ω_γ, which in the following equation is assumed to be comprised of the two elements T and K.

$$
\begin{aligned}
\|\bar{R}_\gamma\|_{L^2(\gamma)} \;&\leq\; C\Big\{ h_T^{-1/2} |||e|||_{\omega_\gamma} + \|\bar{R}_\gamma - R\|_{L^2(\gamma)} \\
&\qquad +\; h_T^{1/2}\Big[h_T^{-1}|||e|||_T + h_K^{-1}|||e|||_K \\
&\qquad\qquad +\; \|\bar{r}_T - r\|_{L^2(T)} + \|\bar{r}_K - r\|_{L^2(K)} \Big] \Big\} \\
&\leq\; C\Big\{ h_T^{-1/2} |||e|||_{\omega_\gamma} + h_T^{1/2} \sum_{T_* \subset \omega_\gamma} \|\bar{r}_{T_*} - r\|_{L^2(T_*)} + \|\bar{R}_\gamma - R\|_{L^2(\gamma)} \Big\} .
\end{aligned}
\tag{4.25}
$$

Finally, the triangle inequality is used exactly as in (4.17), and the following lemma is hereby proved:

Lemma 4.3 *Let r and R denote the interior and boundary residuals associated with the finite element approximation constructed from the subspace V_h. Suppose that $\bar{r}_T$ and $\bar{R}_\gamma$ are continuous finite dimensional approximations to the interior residual on element T and the boundary residual on edge $\gamma \subset \partial T$, respectively. Then*

$$\|r\|_{L^2(T)} \leq C\Big\{ h_T^{-1}|||e|||_T + \|\bar{r}_T - r\|_{L^2(T)} \Big\}$$

and

$$\|R\|_{L^2(\gamma)} \leq C\Big\{ h_T^{-1/2} |||e|||_{\omega_\gamma} + h_T^{1/2} \sum_{T_* \subset \omega_\gamma} \|\bar{r}_{T_*} - r\|_{L^2(T_*)} + \|\bar{R}_\gamma - R\|_{L^2(\gamma)} \Big\} .$$

□

Equivalence

Choose $\bar{r}_T$ such that $\bar{r}_T \in V_h(T)$ and note that $\bar{r}_T$ is allowed to be nonzero on ∂T. In the same manner, let $\bar{R}_\gamma$ be a polynomial of order p, the order of the finite element space, on edge γ. Lemma 4.3 now applies, and the approximations may be chosen such that they are identical to the actual residuals except for the terms f and g which may be impossible to represent exactly by functions from the finite element space.

The boundary residual R is the jump in the normal derivative of the finite element solution across an edge. The above choice of $\bar{R}_\gamma$ therefore implies $\bar{R}_\gamma = R$ on internal edges. On global Neumann boundaries $R - \bar{R}_\gamma \equiv g - \Pi_p g$ where $\Pi_p g$ is the order p approximation of g on edge γ. On global Dirichlet boundaries $\bar{R}_\gamma = R = 0$. On each element T the term $r - \bar{r}_T \equiv f - \Pi_p f$ according to equation (4.3). We then have

$$\|r\|_{L^2(T)} \leq C\Big\{ h_T^{-1}|||e|||_T + \|f - \Pi_p f\|_{L^2(T)} \Big\}$$

and

$$||R||_{L^2(\gamma)} \leq C\left\{ h_T^{-1/2}|||e|||_{\omega_\gamma} + h_T^{1/2} \sum_{T_* \subset \omega_\gamma} ||f - \Pi_p f||_{L^2(T_*)} + ||g - \Pi_p g||_{L^2(\gamma \cap \Gamma_N)} \right\} .$$

Therefore, the local error estimator η_T defined in (4.11) can be bounded by

$$\eta_T^2 \;\leq\; C\left\{ |||e|||_{\omega_T}^2 + h_T^2 \sum_{T_* \in \omega_T} ||f - \Pi_p f||_{L^2(T_*)}^2 + h_T \sum_{\gamma \in \partial T \cap \Gamma_N} ||g - \Pi_p g||_{L^2(\gamma)}^2 \right\} , \tag{4.26}$$

where first the inequality $(\sum \alpha_i)^2 \leq C \sum \alpha_i^2$ has been used, which is valid with a constant C that depends on the number of terms in the sum. However, the maximum number of elements in ω_T can be bounded, so the constant in (4.26) only depends on the shape regularity of the elements. Note that the expression bounding η_T is local in the sense that it only involves T and its nearest neighbours. The following theorem is hereby proved:

Theorem 4.4 *Let η_T denote the local error estimator defined in (4.11), where r and R are the interior and boundary residuals. Then there exists a constant C, depending only on the shape regularity of the elements, such that*

$$\begin{aligned} \frac{1}{C}|||e|||^2 \;&\leq\; \sum_{T \in \mathcal{T}} \eta_T^2 \\ \sum_{T \in \mathcal{T}} \eta_T^2 \;&\leq\; C\left\{ |||e|||^2 + \sum_{T \in \mathcal{T}} h_T^2 ||f - \Pi_p f||_{L^2(T)}^2 + \sum_{\gamma \subset \Gamma_N} h_\gamma ||g - \Pi_p g||_{L^2(\gamma)}^2 \right\} . \end{aligned}$$

Moreover,

$$\eta_T^2 \leq C\left\{ |||e|||_{\omega_T}^2 + h_T^2 \sum_{T_* \in \omega_T} ||f - \Pi_p f||_{L^2(T_*)}^2 + h_T \sum_{\gamma \in \partial T \cap \Gamma_N} ||g - \Pi_p g||_{L^2(\gamma)}^2 \right\} .$$

PROOF. The first inequality in the first part of the theorem is due to equation (4.10). The second inequality in the first part of the theorem is due to summing the expression for η_T in (4.26) over all elements. The second part of the theorem is exactly equation (4.26). $\square$

Although in practice η_T will not be computed exactly, but rather by numerical quadrature, one can prove, see [4], that under very mild assumptions the same bounds will be valid also for the numerical approximation to the estimator.

4.2 L^p-estimators, $1 < p < \infty$

In the previous section an error estimator for the energy norm was derived and found to be equivalent to the energy norm of the error. However, it can sometimes be desirable to get an error estimate for instance in the L^2-norm instead of in the energy norm, and such estimators can be derived in a very similar fashion, see e.g. [?].

Let L^p be the space in which we want to estimate the error, and let q be the conjugate exponent of p, i.e.

$$\frac{1}{p} + \frac{1}{q} = 1 \ .$$

To obtain an L^p-estimate we will use the so-called Aubin-Nitsche trick, which means we will consider the adjoint, or the *dual*, of the model problem: Find $\phi_F \in V$ such that

$$a(v, \phi_F) = (F, v) \quad \forall v \in V \ , \tag{4.27}$$

where $F \in L^q(\Omega)$ is given and

$$(v, w) = \int_\Omega vw \, dx \ .$$

We will need to assume that $\phi_F \in W^{2,q}(\Omega) \cap V$ and that the solution operator is continuous, i.e.

$$||\phi_F||_{W^{2,q}(\Omega)} \leq C||F||_{L^q(\Omega)} \ . \tag{4.28}$$

This condition is satisfied if Ω is convex [?]. Note that for L^2-estimates these assumptions become: $\phi_F \in H^2(\Omega) \cap V$ and

$$||\phi_F||_{H^2(\Omega)} \leq C||F||_{L^2(\Omega)} \ . \tag{4.29}$$

Also note that F and ϕ_F are quantities that only take part in the derivation of the error estimate, and as they are not part of the final estimator we do not need to know or compute them in practice. The problem (4.27) should be viewed as a way of rewriting the inner product (F, v) such that it is possible to get a useful upper bound for it.

Now, choose $v = e$ so that

$$(e, F) = a(e, \phi_F) \ ,$$

and since $\phi_F \in V$, we can use (4.7) to obtain

$$a(e, \phi_F) = \sum_{T \in \mathcal{T}} \int_T r(\phi_F - I_{V_h}\phi_F) \, dx + \sum_{\gamma \in \partial\mathcal{T}} \int_\gamma R(\phi_F - I_{V_h}\phi_F) \, ds \ .$$

If the bilinear form was nonsymmetric, using the conjugate form would be essential for obtaining an expression corresponding to the one above. Application of Hölder's inequality now shows that

$$(e, F) \ \leq \ \sum_{T \in \mathcal{T}} ||r||_{L^p(T)} ||\phi_F - I_{V_h}\phi_F||_{L^q(T)} + \sum_{\gamma \in \partial\mathcal{T}} ||R||_{L^p(\gamma)} ||\phi_F - I_{V_h}\phi_F||_{L^q(\gamma)} \ . \tag{4.30}$$

Further, we need the following approximation results [14],

$$\begin{aligned} ||v - I_{V_h} v||_{L^q(T)} &\leq Ch_T^2 |v|_{W^{2,q}(\tilde{\omega}_T)} \\ ||v - I_{V_h} v||_{L^q(\gamma)} &\leq Ch_T^{2-1/q} \, |v|_{W^{2,q}(\tilde{\omega}_\gamma)} \, . \end{aligned}$$

Substituting this into (4.30), gives

$$\begin{aligned} (e, F) &\leq C \Big\{ \sum_{T \in \mathcal{T}} ||r||_{L^p(T)} h_T^2 |\phi_F|_{W^{2,q}(\tilde{\omega}_T)} \\ &\qquad + \sum_{\gamma \in \partial \mathcal{T}} ||R||_{L^p(\gamma)} h_\gamma^{2-(1-1/p)} \, |\phi_F|_{W^{2,q}(\tilde{\omega}_\gamma)} \Big\} \\ &= C \Big\{ \sum_{T \in \mathcal{T}} ||r||_{L^p(T)} h_T^2 |\phi_F|_{W^{2,q}(\tilde{\omega}_T)} \\ &\qquad + \sum_{\gamma \in \partial \mathcal{T}} ||R||_{L^p(\gamma)} h_\gamma^{\frac{p+1}{p}} \, |\phi_F|_{W^{2,q}(\tilde{\omega}_\gamma)} \Big\}. \end{aligned} \tag{4.31}$$

Once again using Hölder's inequality in the same fashion as the Cauchy-Schwarz inequality was used prior to equation (4.9),

$$(e, F) \leq C |\phi_F|_{W^{2,q}(\Omega)} \Big\{ \sum_{T \in \mathcal{T}} h_T^{2p} ||r||_{L^p(T)}^p + \sum_{\gamma \in \partial \mathcal{T}} h_\gamma^{p+1} ||R||_{L^p(\gamma)}^p \Big\}^{1/p} ,$$

and now (4.28) implies that

$$(e, F) \leq C ||F||_{L^q(\Omega)} \Big\{ \sum_{T \in \mathcal{T}} h_T^{2p} ||r||_{L^p(T)}^p + \sum_{\gamma \in \partial \mathcal{T}} h_\gamma^{p+1} ||R||_{L^p(\gamma)}^p \Big\}^{1/p} .$$

Finally, the identity

$$||e||_{L^p(\Omega)} = \sup_{F \in L^q(\Omega)} \frac{(e, F)}{||F||_{L^q(\Omega)}} ,$$

can be applied to obtain

$$||e||_{L^p(\Omega)} \leq C \Big\{ \sum_{T \in \mathcal{T}} h_T^{2p} ||r||_{L^p(T)}^p + \frac{1}{2} h_T^{p+1} ||R||_{L^p(\partial T)}^p \Big\}^{1/p} .$$

Here the sums have been rearranged exactly as was done in Section 4.1. We identify the local error estimator $\eta_{L^p(T)}$ as

$$\eta_{L^p(T)}^p = h_T^{2p} ||r||_{L^p(T)}^p + \frac{1}{2} h_T^{p+1} ||R||_{L^p(\partial T)}^p ,$$

and for Ω convex we have hereby proved that

$$||e||_{L^p(\Omega)} \leq C \Big\{ \sum_{T \in \mathcal{T}} \eta_{L^p(T)}^p \Big\}^{1/p} .$$

5 Implicit error estimators

So far we have only looked at estimators that are computed explicitly from the numerical solution or by post processing it to recover gradients, and especially the explicit estimators display several undesirable features. One difficulty is that they contain generic constants that are unknown, which makes the bounds uncertain. In addition, of course the correct value of the constants come from the worst case scenario, so approximating it and including it in the estimator would make the estimate very pessimistic. Moreover, there could be cancellations between interior and boundary residuals that are not accounted for in explicit estimates. How to weight the two types of residuals is also an unclear issue.

In this section so-called *implicit* estimators will be presented, and these estimators avoid some of the problems mentioned above, in particular this is true for the equilibrated residual method. In this context the word *implicit* means that we will use the already known finite element approximation to construct local boundary value problems for the error. The local error estimator is taken to be the norm of the local error approximation, which is given as the solution of the local problem.

5.1 The subdomain residual method

The main issue in this chapter is to approximate the residual problem which describes the true error. The relevant equation was formulated in Section 2.1, and it reads

$$a(e,v) = a(u-u_h,v) = L(v) - a(u_h,v) \quad \forall v \in V \ . \tag{5.1}$$

Solving the discretised counterpart of this global problem would involve solving a problem even larger than the problem defining u_h, because due to Galerkin orthogonality one now needs a larger (or at least different) global solution space. This could for instance be accomplished by refining the grid. In the *subdomain residual method* the idea is the following: Instead of solving the global problem, we will split it up and solve it on local patches, associated with each node, and equip each local problem with homogeneous Dirichlet boundary conditions. This is a lot cheaper than solving the refined global problem, and in addition it is perfectly suited for parallel processing. The boundary conditions chosen mean that we will assume the error to be zero on the boundary of each patch. This is of course not fully satisfied, but despite this fact the resulting estimator can be shown to be effective. A *patch* associated with a node n is what was defined as ω_n in Figure 2.1, Section 2.1. For node n the local problem now reads: Find $\phi_n \in H_0^1(\omega_n)$ such that

$$a_{\omega_n}(\phi_n,v) = L_{\omega_n}(v) - a_{\omega_n}(u_h,v) \quad \forall v \in H_0^1(\omega_n) \ . \tag{5.2}$$

The error estimator η_n associated with a subdomain ω_n is then taken to be

$$\eta_n^2 = |||\phi_n|||^2_{\omega_n} = a_{\omega_n}(\phi_n,\phi_n) \ , \tag{5.3}$$

and the global estimator is defined by

$$\eta = \Big\{ \sum_{n \in \mathcal{N}} \eta_n^2 \Big\}^{1/2} , \tag{5.4}$$

in which $\mathcal{N}$ is the set of all nodes. In practice one uses a finite dimensional subspace of $H_0^1(\omega_n)$ to find an approximate solution. Note that this is an estimator associated with nodes (and domains ω_n) instead of elements.

5.1.1 Equivalence of estimator

Similar to what was done in Section 4.1.1, we will now prove equivalence of the estimator defined above. First some definitions are needed. Let θ_n be the continuous piecewise (bi)linear function defined on Ω such that

$$\theta_n(x_n) = 1 \quad \text{and} \quad \theta_n = 0 \ \text{ on } \ \Omega \setminus \omega_n \ . \tag{5.5}$$

Let $\mathcal{N}_1$, $\mathcal{N}_2$, ... be subsets of $\mathcal{N}$ with the nodes in $\mathcal{N}_i$ chosen such that:

$$\text{If} \quad m, n \in \mathcal{N}_i \quad \text{and } m \neq n \quad \text{then } \omega_m \cap \omega_n = \emptyset \ .$$

The smallest number of subsets ($\mathcal{N}_i$) needed to cover all of $\mathcal{N}$ is denoted by ρ. Further, the following result concerning how multiplication by θ_n affects an interpolant is known [4, Lemma 4.1]:

Lemma 5.1 *For any $v \in H^1(\Omega)$ there exists a constant C and an interpolant $I_{V_h} v$ from the finite element space, such that*

$$\sum_{n \in \mathcal{N}} ||\theta_n(v - I_{V_h} v)||^2_{H^1(\Omega)} \leq C |v|^2_{H^1(\Omega)} \ .$$

□

The following theorem, stating the equivalence of the estimator obtained from the subdomain residual method, can now be proved.

Theorem 5.2 *Let η be the estimator defined by (5.4). Then there exists a constant C, depending only on the regularity of the elements, such that*

$$\frac{\eta}{\sqrt{\rho}} \leq |||e||| \leq C\eta.$$

PROOF. First we will show that $|||e|||$ is bounded from above by $C\eta$. From (2.7) we get

$$a(e, v) = a(e, v - I_{V_h} v) \ ,$$

and because $\sum_{n \in \mathcal{N}} \theta_n \equiv 1$,

$$a(e, v) = a(e, v - I_{V_h} v) \sum_{n \in \mathcal{N}} \theta_n = \sum_{n \in \mathcal{N}} a\Big(e, \ \theta_n(v - I_{V_h} v)\Big) \ .$$

From (5.1) and (5.2)

$$a_{\omega_n}(e,v) = a_{\omega_n}(\phi_n, v) \quad \forall v \in H_0^1(\omega_n). \tag{5.6}$$

Therefore, since $\theta_n(v - I_{V_h}v) \in H_0^1(\omega_n)$*,*

$$a(e,v) = \sum_{n\in\mathcal{N}} a_{\omega_n}\Big(\phi_n,\ \theta_n(v - I_{V_h}v)\Big),$$

and hence, by the Cauchy-Schwarz inequality and the definition of η_n*,*

$$|a(e,v)| \le \sum_{n\in\mathcal{N}} \eta_n |||\theta_n(v - I_{V_h}v)||| \ .$$

The last sum is nothing but a discrete inner product, and the Cauchy-Schwarz inequality and Lemma 5.1 immediately give

$$\begin{aligned} a(e,v) &\le \Big\{\sum_{n\in\mathcal{N}} \eta_n^2\Big\}^{1/2} \Big\{\sum_{n\in\mathcal{N}} ||\theta_n(v - I_{V_h}v)||^2_{H^1(\Omega)}\Big\}^{1/2} \\ &\le C\eta|||v||| \ . \end{aligned} \tag{5.7}$$

Finally, choosing $v = e$ *yields*

$$|||e||| \le C\eta \ . \tag{5.8}$$

To obtain the lower bound for $|||e|||$*, note that according to (5.6) the test function* v *can be chosen as* ϕ_n *to obtain*

$$a_{\omega_n}(e,\phi_n) = a_{\omega_n}(\phi_n,\phi_n) \ .$$

By extending ϕ_n *with zero to all of* Ω*,*

$$\begin{aligned} \eta^2 &= \sum_{n\in\mathcal{N}} \eta_n^2 = \sum_{n\in\mathcal{N}} a_{\omega_n}(\phi_n,\phi_n) = \sum_{n\in\mathcal{N}} a_{\omega_n}(e,\phi_n) = \sum_{n\in\mathcal{N}} a(e,\phi_n) \\ &= a\big(e, \sum_{n\in\mathcal{N}} \phi_n\big) \ . \end{aligned}$$

Further, the Cauchy-Schwarz inequality gives

$$\eta^2 \le |||e|||\ |||\sum_{n\in\mathcal{N}} \phi_n||| \ . \tag{5.9}$$

Note that by the Cauchy-Schwarz inequality a vector $\alpha \in \mathbb{R}^k$ *satisfies*

$$\Big(\sum_{i=1}^k \alpha_i\Big)^2 = \Big(\sum_{i=1}^k 1\cdot\alpha_i\Big)^2 \le \Big(\sum_{i=1}^k 1^2\Big)\Big(\sum_{i=1}^k \alpha_i^2\Big) = k\sum_{i=1}^k \alpha_i^2,$$

such that if

$$\Big|\Big|\sum_{i=1}^k v_i\Big|\Big|^2 = \int \Big(\sum_{i=1}^k v_i\Big)^2 dx$$

and

$$\sum_{i=1}^{k} ||v_i||^2 = \int \sum_{i=1}^{k} v_i^2 dx \, ,$$

then

$$\Big|\Big| \sum_{i=1}^{k} v_i \Big|\Big|^2 \leq k \sum_{i=1}^{k} ||v_i||^2.$$

In order to bound the right hand side of (5.9), we partition the set of nodes, $\mathcal{N}$, as previously described. Then, from the above result

$$||| \sum_{n\in\mathcal{N}} \phi_n |||^2 = ||| \sum_{\mathcal{N}_i} \sum_{n\in\mathcal{N}_i} \phi_n |||^2 \leq \rho \sum_{\mathcal{N}_i} ||| \sum_{n\in\mathcal{N}_i} \phi_n |||^2 = \rho \sum_{\mathcal{N}_i} \sum_{n\in\mathcal{N}_i} |||\phi_n|||^2 . \quad (5.10)$$

The last equality holds due to the fact that the functions ϕ_n, corresponding to nodes in $\mathcal{N}_i$, are non-overlapping. From (5.10)

$$||| \sum_{n\in\mathcal{N}} \phi_n |||^2 \leq \rho \eta^2 \, ,$$

and substituting this into (5.9) finally yields

$$\eta^2 \leq |||e||| \sqrt{\rho} \eta \, ,$$

which proves the result. $\square$

It can be shown that the estimator described in this section is equivalent to the explicit energy norm estimator, and as the explicit estimator is easier to compute, it is in many cases more favourable. In fact, it is rather expensive to compute good finite dimensional approximations to the infinite dimensional problem (5.2), due to the typical irregularity of the patches. The method seems not to be much used, but it was one of the earliest established a posteriori error estimators, first suggested by Babuška and Rheinboldt [6].

5.2 The element residual method

Also the element residual method is based on solving a residual-like equation describing the error, but this time the local problems will be equipped with Neumann type boundary conditions, and they will be posed on patches consisting of only one single element. Since the finite element solution is smooth on the interior of each element, the following equation is satisfied,

$$-\Delta e = -\Delta(u - u_h) = f + \Delta u_h \quad \text{on } T \, . \quad (5.11)$$

Obvious boundary conditions are

$$\begin{array}{rcll} e & = & 0 & \text{on } \partial T \cap \Gamma_D \\ \frac{\partial e}{\partial n_T} & = & g - \frac{\partial u_h}{\partial n_T} & \text{on } \partial T \cap \Gamma_N \\ \frac{\partial e}{\partial n_T} & = & \frac{\partial u}{\partial n_T} - \frac{\partial u_h}{\partial n_T}\Big|_T & \text{on } \partial T \setminus \partial\Omega \, . \end{array}$$

The first two boundary conditions are easily imposed, whereas the third one is more difficult since the normal derivative of the true solution is unknown. In the element residual method the approximation

$$\frac{\partial u}{\partial n_T} \approx \left\langle \frac{\partial u_h}{\partial n_T} \right\rangle$$

is chosen, with $\langle \cdot \rangle$ being the average between the two neighbouring elements,

$$\left\langle \frac{\partial u_h}{\partial n_T} \right\rangle = \begin{cases} \frac{1}{2}\mathbf{n}_T \cdot (\nabla u_h|_T + \nabla u_h|_K) & \text{on } \bar{T} \cap \bar{K} \\ g & \text{on } \bar{T} \cap \Gamma_N \\ \mathbf{n}_T \cdot \nabla u_h & \text{on } \bar{T} \cap \Gamma_D \,. \end{cases} \tag{5.12}$$

Multiplying (5.11) by a test function $v \in V(T)$, where we recall that

$$V(T) = \{v \in H^1(T) \;:\; v = 0 \;\text{ on } \partial T \cap \Gamma_D\} \,,$$

integrating and then using Green's formula on each element, yields

$$\begin{aligned} -\int_T \Delta e \, v \, dx &= \int_T f v \, dx + \int_T \Delta u_h \, v \, dx \\ \int_T \nabla e \cdot \nabla v \, dx - \int_{\partial T} \frac{\partial e}{\partial n_T} v \, ds &= F_T(v) - \int_T \nabla u_h \cdot \nabla v \, dx + \int_{\partial T} \frac{\partial u_h}{\partial n_T} v \, ds \\ a_T(e, v) &= F_T(v) - a_T(u_h, v) + \int_{\partial T} \frac{\partial u}{\partial n_T} v \, ds \,. \end{aligned} \tag{5.13}$$

Again the subscripts on $a(\cdot,\cdot)$ and $F(\cdot)$ indicate the integration domain, and by using the approximation described above, the problem to be solved is: Find $\phi_T \in V(T)$ such that

$$a_T(\phi_T, v) = F_T(v) - a_T(u_h, v) + \int_{\partial T} \left\langle \frac{\partial u_h}{\partial n_T} \right\rangle v \, ds \quad \forall v \in V(T) \,. \tag{5.14}$$

The element residual method is obtained from (5.14) by choosing a suitable finite dimensional subspace $\tilde{V}_h(T) \subset V(T)$ as the solution space. One difficulty arises due to the fact that the bilinear form may vanish for non-trivial test functions, so to make sure the finite dimensional counterpart of (5.14) is well posed, $\tilde{V}_h(T)$ needs to be chosen with care. One possible choice is to pick only functions that make the bilinear form coercive. What will be considered here is a space V_h that consists of piecewise (bi)linear functions and a space $\tilde{V}_h(T)$ that contains the interior- and edge bubble functions defined in Section 4.1.1. Note that since these functions vanish at the nodes, the local Neumann problem, which is usually only decided up to a constant, has a unique solution within this space. The element residual method is now: Find $\phi_T \in \tilde{V}_h(T)$ such that

$$a_T(\phi_T, v_h) = F_T(v_h) - a_T(u_h, v_h) + \int_{\partial T} \left\langle \frac{\partial u_h}{\partial n_T} \right\rangle v_h \, ds \quad \forall v_h \in \tilde{V}_h(T) \,. \tag{5.15}$$

The local error estimator is taken to be

$$\eta_T = |||\phi_T|||_T \ ,$$

and globally

$$\eta = \left\{ \sum_{T \in \mathcal{T}} \eta_T^2 \right\}^{1/2} .$$

This estimator is often referred to as the Bank-Weiser estimator, due to their work [9].

5.2.1 Equivalence of estimator

As was done for the previous estimators, equivalence for this implicit estimator will now be shown. The first step towards the proof is to relate the implicit estimator to the explicit energy norm estimator of Section 4.1. For simplicity, define

$$\tilde{R}_\gamma = \begin{cases} \frac{1}{2} R_\gamma & \gamma \not\subset \Gamma_N \\ R_\gamma & \gamma \subset \Gamma_N , \end{cases} \tag{5.16}$$

where R is the boundary residual known from Section 4.1. Applying Green's formula to the second term on the right hand side of (5.15) and taking advantage of the identity

$$\left.\frac{\partial u_h}{\partial n_T}\right|_\gamma = \left\langle \frac{\partial u_h}{\partial n_T} \right\rangle_\gamma - \tilde{R}_\gamma \, , \tag{5.17}$$

where $< \cdot >$ was defined in (5.12), gives

$$\begin{aligned} a_T(\phi_T, v_h) &= F_T(v_h) + \int_T \Delta u_h \, v_h \, dx - \int_{\partial T} \left.\frac{\partial u_h}{\partial n_T}\right|_T v_h \, ds + \int_{\partial T} \left\langle \frac{\partial u_h}{\partial n_T} \right\rangle v_h \, ds \\ &= \int_T r v_h \, dx + \int_{\partial T} \tilde{R} v_h \, ds \, . \end{aligned}$$

Here r is the interior residual defined in (4.3). Now, select $v_h = \phi_T$ to obtain

$$a_T(\phi_T, \phi_T) \equiv |||\phi_T|||^2 \equiv \eta_T^2 = \int_T r \phi_T \, dx + \int_{\partial T} \tilde{R} \phi_T \, ds \, .$$

Since ϕ_T belongs to a finite dimensional space and since it also vanishes at the nodes of the element, it can only be zero if its gradient vanishes on T. Intuitively the function ϕ_T can therefore be bounded by its gradient. The precise result is obtained from Poincaré's inequality, see e.g. [18],

$$||\phi_T||_{L^2(T)} \leq C h_T ||\nabla \phi_T||_{L^2(T)} . \tag{5.18}$$

Applying the following trace inequality, see e.g. [?, Lemma 3.1],

$$||\phi_T||_{L^2(\gamma)} \le C\Big\{h_T^{-1/2}||\phi_T||_{L^2(T)} + ||\phi_T||^{1/2}_{L^2(T)}||\nabla\phi_T||^{1/2}_{L^2(T)}\Big\} \tag{5.19}$$

together with (5.18), one obtains

$$||\phi_T||_{L^2(\gamma)} \quad\le\quad Ch_T^{1/2}||\nabla\phi_T||_{L^2(T)} \qquad \gamma\subset\partial T\ .$$

First using the Cauchy-Schwarz inequality, and thereafter inserting the above results, yields

$$\begin{aligned}\eta_T^2 &\le C\Big\{||r||_{L^2(T)}||\phi_T||_{L^2(T)} + \frac{1}{2}\sum_{\gamma\in\partial T}||R||_{L^2(\gamma)}||\phi_T||_{L^2(\gamma)}\Big\}\\ &\le C||\nabla\phi_T||_{L^2(T)}\left\{h_T||r||_{L^2(T)} + \frac{1}{2}h_T^{1/2}\sum_{\gamma\in\partial T}||R||_{L^2(\gamma)}\right\}.\end{aligned}$$

Recalling that $||\nabla\phi_T||_{L^2(T)} \equiv \eta_T$ and that $\sum\alpha_i \le C(\sum\alpha_i^2)^{1/2}$ where C only depends on the number of terms in the sum, the following result is obtained,

$$\eta_T^2 \le C\left\{h_T^2||r||^2_{L^2(T)} + \frac{1}{2}h_T||R||^2_{L^2(\partial T)}\right\}.$$

This shows that the estimator computed by the element residual method is bounded from above by a constant times the explicit estimator of Section 4.1. The explicit estimator was shown to be bounded by the energy norm of the error and terms arising from the approximation of the data. To obtain lower bounds on the element residual estimator it is once again enough to show that it is bounded by the explicit energy norm estimator. Let b_T and b_γ be bubble functions as in Section 4.1.1. If one proceeds exactly as in (4.14)–(4.17) and (4.18)–(4.25), starting with

$$a_T(\phi_T, \bar r_T b_T) = \int_T b_T r \bar r_T\, dx$$

and

$$a_T(\phi_T, \bar R_\gamma b_\gamma) = \int_T r\bar R_\gamma b_\gamma\, dx + \int_\gamma b_\gamma R\bar R_\gamma\, ds\ ,$$

respectively, one obtains

$$\begin{aligned}||r||_{L^2(T)} &\le C\Big\{h_T^{-1}\eta_T + ||\bar r_T - r||_{L^2(T)}\Big\}\\ ||R||_{L^2(\gamma)} &\le C\left\{h_T^{-1/2}\eta_T + h_T^{1/2}\sum_{T_*\in\omega_\gamma}||\bar r_T - r||_{L^2(T_*)} + ||\bar R_\gamma - R||_{L^2(\gamma)}\right\},\end{aligned}$$

with η_T being the implicit estimator of this section. Collecting the results exactly as previously, we have proved the following theorem.

Theorem 5.3 *Let η_T denote the local error estimator obtained using the element residual method. Then there exists a constant C, depending only on the shape regularity of the elements, such that*

$$|||e|||^2 \le C \sum_{T \in \mathcal{T}} \left\{ \eta_T^2 + h_T^2 ||f - \Pi_p f||^2_{L^2(T)} + h_T \sum_{\gamma \subset \partial T \cap \Gamma_N} ||g - \Pi_p g||^2_{L^2(\gamma)} \right\},$$

and

$$\eta_T^2 \le C \left\{ |||e|||^2_{\omega_T} + h_T^2 \sum_{T_* \in \omega_T} ||f - \Pi_p f||^2_{L^2(T_*)} + h_T \sum_{\gamma \in \partial T \cap \Gamma_N} ||g - \Pi_p g||^2_{L^2(\gamma)} \right\}.$$

□

Just as for the explicit estimator, the same equivalence is valid also for the numerical approximation to η_T.

5.2.2 Performance and asymptotic exactness

Some rather surprising results can be shown for the estimator described in this section. In previous sections the term *asymptotic exactness* was mentioned as a property for certain estimators, which means that the effectivity index tends to unity as the grid is refined. Assume that the local problems are solved "exactly" in the sense that $\tilde{V}_h(T) = V(T) \setminus \mathbb{R}$. With this choice of local solution space the bilinear form is coercive and the problem is therefore well posed. Then the element residual estimator is asymptotically exact for odd degree elements on especially regular grids, provided the true solution is smooth enough. However, there are examples showing that if any of these restrictions are violated the estimator is no longer asymptotically exact.

One might expect that solving the local problems using only a finite dimensional subspace of $V(T)$ would give even less satisfying results. This is, however, not necessarily the case. For the model problem it can in fact be proved [4] that using the Lobatto basis for the subspace $\tilde{V}_h(T)$ yields an asymptotic exact error estimator for degree $p > 1$ finite element approximations if the mesh is parallel and the true solution is smooth. The explanation for this rather surprising result is actually very simple. What makes the exact solution inconsistent, is the choice of boundary data. The true error does not exactly satisfy the boundary data that were chosen for each patch. This gives rise to some spurious modes in the exact solution to the element residual problem, which will not die out even if the grid is refined. The reason why it is still possible to make consistent estimators is that when choosing with care, it is possible to select a subspace that does not contain these spurious modes. With this approach such modes will never be present in the local error estimates, and the estimator will be consistent. A different and better choice for the boundary data will be considered in the next section.

5.3 The equilibrated residual method

The equilibrated residual method is very much related to the element residual method, and the only difference lies in the choice of boundary data for the local problems. The name of this section's method reflects its way of balancing, or *equilibrating*, the fluxes used as boundary data. What this means in practice will become clear during the presentation of the method.

The global problem describing the error is known from Section 1,

$$a(e,v) = L(v) - a(u_h,v) = F(v) - a(u_h,v) + \int_{\Gamma_N} gv\,ds, \tag{5.20}$$

and according to (5.13), the error locally satisfies

$$a_T(e,v) = F_T(v) - a_T(u_h,v) + \int_{\partial T} \frac{\partial u}{\partial n_T} v\;ds.$$

The boundary flux on ∂T is in general unknown, so the above problem cannot be solved without modifications. We will therefore develop a method which determines approximate fluxes that are in some sense good approximations to the true flux. Let g_T be some approximation to the boundary flux on element T, i.e.

$$g_T \approx \frac{\partial u}{\partial n_T} \quad \text{ on } \partial T.$$

Since the true flux is continuous across interelement edges, g_T will be required to satisfy

$$g_T + g_{T'} = 0 \quad \text{ on } \partial T \cap \partial T', \tag{5.21}$$

which makes sure no flux is generated on the interface. On the global Neumann boundary there is no need for approximations since the true flux is known, i.e.

$$g_T = g \quad \text{ on } \partial T \cap \Gamma_N. \tag{5.22}$$

Assume that the local boundary value problem

$$a_T(\phi_T,v) = F_T(v) - a_T(u_h,v) + \int_{\partial T} g_T v\;ds \qquad \forall v \in V(T) \tag{5.23}$$

has a solution $\phi_T \in V(T)$. Then, by the aid of (5.20),

$$\begin{aligned} a(e,v) &= \sum_{T\in\mathcal{T}} \Big\{F_T(v) - a_T(u_h,v)\Big\} + \int_{\Gamma_N} gv\,ds \\ &= \sum_{T\in\mathcal{T}} \Big\{F_T(v) - a_T(u_h,v) + \int_{\partial T} g_T v\,ds\Big\} \\ &= \sum_{T\in\mathcal{T}} a_T(\phi_T,v). \end{aligned} \tag{5.24}$$

Choosing $v = e$ and applying the Cauchy-Schwarz inequality, gives

$$\begin{aligned} |||e|||^2 &\leq \sum_{T\in\mathcal{T}} |||\phi_T|||_T \, |||e|||_T \\ &\leq \Big\{ \sum_{T\in\mathcal{T}} |||\phi_T|||_T^2 \Big\}^{1/2} |||e|||. \end{aligned}$$

Now, taking the local error estimator to be

$$\eta_T = |||\phi_T|||_T \, ,$$

and defining

$$\eta^2 = \sum_{T\in\mathcal{T}} \eta_T^2 \, ,$$

gives an estimator satisfying

$$|||e||| \leq \eta. \tag{5.25}$$

So far we have assumed that (5.23) has a solution, but this is not generally true. The problem arises from the nontrivial kernel of the bilinear form. Any constant v would make the bilinear form vanish, while this is not necessarily the case for the right hand side. However, imposing the compatibility condition

$$0 = F_T(1) - a_T(u_h, 1) + \int_{\partial T} g_T \, ds \tag{5.26}$$

guarantees that (5.23) is well posed. This is known as the (zeroth order) *equilibration condition*, and it gives conditions on g_T which makes sure the local problem has a solution, but as will be seen later, this does not determine g_T uniquely. The equilibration condition can also be viewed as a way of balancing the boundary flux and the interior load, which is the interior residual. Fluxes satisfying (5.26) will be referred to as equilibrated fluxes. Note that the equilibration condition is not necessary on an element T that has $\partial T \cap \Gamma_D \neq \emptyset$ because then the function $v|_T = 1$ is not in the test space $V(T)$.

The upper bound (5.25) is valid given that g_T satisfies (5.21), (5.22), and (5.26), but the quality of this bound depends on how the fluxes are chosen. For other problems the equilibration condition may not be needed for the local problems to be well posed, however, for some of those problems it can be shown that it is still needed to obtain a tight upper bound.

5.3.1 Equilibrated fluxes

In this section it will be assumed that the finite element space V_h is constructed using first order elements, and based on this assumption we will develop a way of determining the equilibrated fluxes g_T. The fluxes will be required to satisfy the so-called first order equilibration conditions,

$$\left.\begin{array}{rcll} F_T(\theta_n) - a_T(u_h, \theta_n) + \int_{\partial T} g_T \theta_n \, ds & = & 0 & \forall n \in \mathcal{N}(T) \\ g_T + g_{T'} & = & 0 & \text{on } \partial T \cap \partial T' \\ g_T & = & g & \text{on } \partial T \cap \Gamma_N \ , \end{array}\right\} \tag{5.27}$$

where θ_n is the Lagrangian basis function, defined in (5.5), associated with node n, and $\mathcal{N}(T)$ denotes the nodes of element T. The above conditions are referred to as first order conditions because the equilibration condition is satisfied for linear functions, θ_n, which of course incorporates also the zeroth order conditions (5.26).

The choice of degrees of freedom for the fluxes is crucial since we are seeking fluxes, satisfying (5.27), that can be determined without having to solve a global problem. We will follow the strategy employed by Ainsworth and Oden [5], which is closely related to the method suggested by Ladeveze and Leguillon [17]. The unknown fluxes are expressed as moments,

$$\mu_{T,n}^{\gamma} = \int_{\gamma} g_T \theta_n \, ds, \tag{5.28}$$

and the fluxes are taken to be linear functions on interelement edges, i.e.

$$g_T\Big|_{\gamma} = \left\{ \begin{array}{ll} g & \text{if } \gamma \subset \Gamma_N \\ \alpha_l \theta_l + \alpha_r \theta_r & \text{else,} \end{array} \right. \tag{5.29}$$

where the subscripts l and r indicate the two end nodes of γ, and where α_l and α_r are coefficients to be determined. Inserting the expression for g_T into (5.28) gives a 2×2 linear system relating $\mu_{T,l}^{\gamma}$ and $\mu_{T,r}^{\gamma}$ to α_l and α_r. Solving this system in terms of the α's and substituting it into (5.29), gives

$$g_T\Big|_{\gamma} = \frac{2}{h_{\gamma}} \Big\{ (2\mu_{T,l}^{\gamma} - \mu_{T,r}^{\gamma})\theta_l + (-\mu_{T,l}^{\gamma} + 2\mu_{T,r}^{\gamma})\theta_r \Big\},$$

where h_{γ} is the length of γ. Therefore, once the moments are in hand the fluxes are determined. Introduction of the moments in the equilibration condition (5.27) makes it possible to rewrite the conditions as

$$\left.\begin{array}{rcll} \sum_{\gamma \subset \partial T} \mu_{T,n}^{\gamma} & = & \Delta_T(\theta_n) & \forall n \in \mathcal{N}(T) \\ \mu_{T,n}^{\gamma} + \mu_{T',n}^{\gamma} & = & 0 & \forall n \in \mathcal{N}(\gamma), \quad \gamma = \partial T \cap \partial T' \\ \mu_{T,n}^{\gamma} & = & \int_{\gamma} g \theta_n \, ds & \forall n \in \mathcal{N}(\gamma), \quad \gamma = \partial T \cap \Gamma_N \ , \end{array}\right\} \tag{5.30}$$

in which

$$\Delta_T(\theta_n) = a_T(u_h, \theta_n) - F_T(\theta_n).$$

For each node the above set of conditions comprises a small independent system of linear equations where the unknowns are the moments associated with the

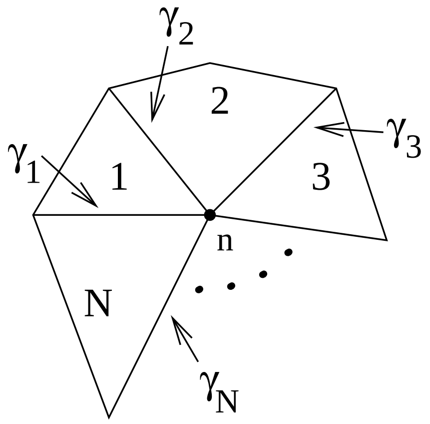

Figure 5.1: Local numbering of elements and edges in a patch associated with node n.

node. The apparently global problem of finding equilibrated fluxes has been broken up into small local problems which can be solved at low cost. This breaking up of the problem is mainly due to the choice of degrees of freedom for the fluxes, and not having to solve a global problem makes the equilibrated residual method about as cost effective as the other implicit methods.

Let n be an interior node connected to N edges, $\gamma_1, \ldots, \gamma_N$, and let these edges and the elements in the patch ω_n have a sequential local numbering, see Figure 5.1. Then the local linear system arising from the first equation in (5.30) takes the form

$$\begin{array}{rcl} \mu_{1,n}^{\gamma_1} + \mu_{1,n}^{\gamma_2} & = & \Delta_1(\theta_n) \\ \mu_{2,n}^{\gamma_2} + \mu_{2,n}^{\gamma_3} & = & \Delta_2(\theta_n) \\ & \vdots & \\ \mu_{N,n}^{\gamma_N} + \mu_{N,n}^{\gamma_1} & = & \Delta_N(\theta_n). \end{array} \tag{5.31}$$

Half of the unknowns can be eliminated from the constraints in the second equation of (5.30). However, the system does not have a unique solution, which can easily be seen as follows: The sum of all the left hand sides in (5.31) vanishes due to the second equation in (5.30). For the right hand sides we obtain

$$\sum_{T \in \omega_n} \Delta_T(\theta_n) = a_{\omega_n}(u_h, \theta_n) - F_{\omega_n}(\theta_n) = 0,$$

which is seen directly from the finite element discretisation (2.3) because n is an interior node, so $\theta_n \in V_h$. Since there is no unique solution, the equations are usually solved in a least squares sense. Ideally

$$g_T \approx \frac{\partial u}{\partial n_T} \quad \text{on } \partial T,$$

and if this holds as an equality it is easily seen that the estimator will be exact. Therefore, a natural choice for the quantity to be minimised is

$$\frac{1}{2}\sum_{T\in\omega_n}\sum_{\gamma\subset\partial T}\left(\mu_{T,n}^{\gamma}-\tilde{\mu}_{T,n}^{\gamma}\right)^2,$$

where

$$\tilde{\mu}_{T,n}^{\gamma}=\int_{\gamma}\theta_n\frac{\partial u_h}{\partial n_T}\,ds.$$

An equally good choice would be to use $<\partial u_h/\partial n_T>$ instead of the one sided flux in the expression for $\tilde{\mu}_{T,n}^{\gamma}$. The final results will be identical. The easiest way to solve this minimisation problem is probably by introducing Lagrangian multipliers, and a detailed description can be found in [5], which also describes the treatment of boundary nodes.

5.3.2 Equivalence of estimator

Earlier in this chapter it was shown that the equilibrated residual method provides an upper bound for the error. What remains is to show that it also provides a lower bound, which will be shown through an analogy with the explicit energy estimator of Section 4.1. We start by quoting a stability result similar to the one first obtained by Ainsworth and Babuška [2].

Theorem 5.4 *Let the finite element subspace be constructed using first order elements on a regular partitioning of the domain into triangular or quadrilateral elements. Let $\{g_K\}$ be the set of equilibrated fluxes satisfying (5.27), produced by the method described in Section 5.3.1. Let $\Pi_1^{\gamma}: L^2(\gamma)\to\mathbb{P}_1(\gamma)$ denote the L^2-projection on γ. Then, for each element T,*

$$\begin{aligned}\sum_{\gamma\subset\partial T\setminus\Gamma_N} & h_T\left|\left|g_T-\Pi_1^{\gamma}\left\langle\frac{\partial u_h}{\partial n_T}\right\rangle\right|\right|_{L^2(\gamma)}^2\\ &\le\quad C\Big\{|||e|||_{\tilde{\omega}_T}+h_T^2||r-\bar{r}||_{L^2(\tilde{\omega}_T)}^2+\ h_T\sum_{\gamma'\subset int(\tilde{\omega}_T)}||R-\bar{R}||_{L^2(\gamma')}^2\Big\},\end{aligned}$$

where $int(\tilde{\omega}_T)$ denotes the interior of the patch $\tilde{\omega}_T$ and the constant C is independent of the mesh size.

PROOF. See [5, Theorem 6.2]. □

With the aid of the above stability result, the following theorem can be proved.

Theorem 5.5 *Suppose that the partition $\mathcal{T}$ is regular and consists of affine elements. Let $\{g_T\}$ be the set of equilibrated fluxes satisfying (5.27), produced by the method described in Section 5.3.1. Let $\phi_T\in V(T)$ denote the solution of the local residual problem (5.23). Then*

$$|||e|||^2\le\sum_{T\in\mathcal{T}}|||\phi_T|||_T^2$$

and

$$|||\phi_T|||_T^2 \le C\Bigg\{ |||e|||^2_{\tilde{\omega}_T} + h_T^2||r-\bar{r}||^2_{L^2(\tilde{\omega}_T)} + h_T||R-\bar{R}||^2_{L^2(\tilde{\omega}_T)} + h_T\left|\left|\left\langle\frac{\partial u_h}{\partial n_T}\right\rangle - \Pi_1^\gamma\left\langle\frac{\partial u_h}{\partial n_T}\right\rangle\right|\right|^2_{L^2(\partial T\setminus\Gamma_N)}\Bigg\},$$

where Π_1^γ *is defined in Theorem 5.4,* $\bar{r}$ *and* $\bar{R}$ *are average values of* r *and* R *respectively, and the constant* C *is independent of the mesh size.*

PROOF. The first part of the theorem is identical to (5.25), which was proven under weaker assumptions. Turning now to the second part of the theorem, consider equation (5.23), and apply Green's theorem to the right hand side of the equation to obtain

$$a_T(\phi_T, v) = (r, v)_T + \int_{\partial T}\left(g_T - \frac{\partial u_h}{\partial n_T}\right) v\, ds.$$

Let $\tilde{R}$ *be defined as in (5.16). The identity (5.17) now implies*

$$a_T(\phi_T, v) = (r, v)_T + \int_{\partial T}\tilde{R}v\, ds + \int_{\partial T}\left(g_T - \left\langle\frac{\partial u_h}{\partial n_T}\right\rangle\right) v\, ds.$$

Then, by the Cauchy-Schwarz inequality,

$$|a_T(\phi_T,v)| \le ||r||_{L^2(T)}||v||_{L^2(T)} + ||\tilde{R}||_{L^2(\partial T)}||v||_{L^2(\partial T)} + \left|\left|g_T - \left\langle\frac{\partial u_h}{\partial n_T}\right\rangle\right|\right|_{L^2(\partial T)}||v||_{L^2(\partial T)}.$$

Let $\bar{v}$ *be the average of* v *over element* T. *Then Poincaré's inequality proves that*

$$||v-\bar{v}||_{L^2(T)} \le Ch_T|v|_{H^1(T)}, \tag{5.32}$$

and furthermore, by (5.19) and again using (5.32),

$$||v-\bar{v}||_{L^2(\partial T)} \le Ch_T^{1/2}|v|_{H^1(T)}.$$

Note that subtracting the average is necessary for the Poincaré inequality, whereas the trace inequality (5.19) holds for ϕ_T *replaced by* $v \in V(T)$. *Because the bilinear form only contains gradients, or more generally, from the zeroth order equilibration principle (5.26),*

$$a_T(\phi_T, v) = a_T(\phi_T, v-\bar{v}),$$

and hence

$$|a_T(\phi_T,v)| \le C\Bigg\{h_T||r||_{L^2(T)} + h_T^{1/2}||R||_{L^2(\partial T)} + h_T^{1/2}\left|\left|g_T - \left\langle\frac{\partial u_h}{\partial n_T}\right\rangle\right|\right|_{L^2(\partial T)}\Bigg\}|||v|||_T. \tag{5.33}$$

From the definition of g_T (5.29) and from the triangle inequality,

$$\begin{aligned}\Big|\Big|g_T-\Big\langle\frac{\partial u_h}{\partial n_T}\Big\rangle\Big|\Big|_{L^2(\partial T)} &= \Big|\Big|g_T-\Big\langle\frac{\partial u_h}{\partial n_T}\Big\rangle\Big|\Big|_{L^2(\Gamma_N)}\\ &\leq \Big|\Big|g_T-\Pi_1^\gamma\Big\langle\frac{\partial u_h}{\partial n_T}\Big\rangle\Big|\Big|_{L^2(\Gamma_N)}+\Big|\Big|\Pi_1^\gamma\Big\langle\frac{\partial u_h}{\partial n_T}\Big\rangle-\Big\langle\frac{\partial u_h}{\partial n_T}\Big\rangle\Big|\Big|_{L^2(\Gamma_N)}.\end{aligned}$$

After inserting this estimate into (5.33), the final result is obtained thanks to Lemma 4.3 and Theorem 5.4, by selecting $v = \phi_T$. $\square$

5.3.3 An a posteriori estimator

An algorithm for finding the equilibrated interelement fluxes has already been described, and once they are in hand the local infinite dimensional problem to be solved is: Find $\phi_T \in V(T)$ such that

$$a_T(\phi_T, v) = F_T(v) - a_T(u_h, v) + \int_{\partial T} g_T v\, ds \quad \forall v \in V(T)\ , \tag{5.34}$$

where g_T generally satisfies the equilibrium condition

$$0 = F_T(v_h) - a_T(u_h, v_h) + \int_{\partial T} g_T v_h\, ds \quad \forall v_h \in V_h(T). \tag{5.35}$$

Assume that the finite element approximation is of order p and let $V_h^{(q)}(T)$ be the family of finite dimensional subspaces obtained from mapping $\mathbb{P}_q$ (or $\mathbb{Q}_q$) from the reference element onto T. For $q = p, p+1, \dots$ a sequence of error estimators can now be defined: Find $\phi_T^{(q)} \in V_h^{(q)}(T)$ such that

$$a_T(\phi_T^{(q)}, v_h) = F_T(v_h) - a_T(u_h, v_h) + \int_{\partial T} g_T v_h\, ds \quad \forall v_h \in V_h^{(q)}(T)\ , \tag{5.36}$$

which leads to the estimator

$$\eta_T^2 = a_T(\phi_T^{(q)}, \phi_T^{(q)})\ . \tag{5.37}$$

Increasing the polynomial degree q will usually improve the approximate solution to the infinite dimensional problem (5.34), and high order approximations can effectively be obtained by constructing subspaces $\mathcal{B}^{(q)}(T) \subset V_h^{(q)}(T)$ that are orthogonal to $V_h^{(q-1)}(T)$ for increasing values of q. For a detailed description, see [4]. Note that the solution ϕ_T must somehow be fixed at least in one point since we only have Neumann boundary conditions. An alternative choice of subspace is the space of bubble functions which was suggested for the element residual method.

6 Goal oriented error estimators

So far we have only considered error estimates for global quantities such as the error measured in energy norm or L^2-norm. However, the goal of many finite element computations is to predict the value of certain local quantities such as fluxes, stresses, or point values, and meshes generated for controlling the error in global norms may not be appropriate for this. We will here describe techniques for computing estimators for the error in such local quantities. Many of the ideas date back to [?, ?, ?], and in recent years there has been a strong interest in this field [?, ?, ?, 19]. The estimators will be very similar to those developed in previous sections, and the main difference will be that the estimators of this section will contain information about how important the local error on each element is for the total error in the requested quantity.

It will be assumed that the quantity of interest can be expressed as a linear functional Q acting on the true solution, for example

$$Q(u) \quad = \quad \frac{1}{|\omega_\epsilon(x_0)|} \int_{\omega_\epsilon(x_0)} u \, dx$$

or

$$Q(u) \quad = \quad \int_\Gamma \mathbf{n} \cdot \nabla u \, ds,$$

where the first example gives the average value of u inside a ball centred at x_0 with radius ϵ, and the second example gives the flux across the surface Γ. It will also be assumed that

$$Q \in V' \, ,$$

such that Q can be evaluated for any $v \in V$. The error in the quantity of interest is then defined as

$$Q(e) = Q(u) - Q(u_h). \tag{6.1}$$

In order to obtain bounds for this quantity we will use the Aubin-Nitsche trick, which employs an argument involving a *dual* problem. If the original problem reads: Find $u \in V$ such that

$$b(u, v) = L(v) \qquad \forall v \in V \tag{6.2}$$

for some (possibly nonsymmetric) bilinear form $b(\cdot, \cdot)$ and linear functional $L(\cdot)$, the corresponding dual problem reads: Find $z \in V$ such that

$$b(v, z) = Q(v) \qquad \forall v \in V. \tag{6.3}$$

This formulation plays a key role in the following theorem.

Theorem 6.1 *Let $u_h \in V_h$ be the finite element approximation of (6.2), where $b : V \times V \to \mathbb{R}$ is a continuous, coercive bilinear form, and where $L : V \to \mathbb{R}$ is a bounded linear functional. Also, let $Q : V \to \mathbb{R}$ be a bounded linear functional, let z be the solution of the dual problem (6.3), and let $z_h \in V_h$ be the corresponding finite element solution. Then the error $Q(e)$ defined in (6.1) satisfies*

$$Q(e) = b(e, e_z),$$

where

$$\begin{aligned} e &= u - u_h \\ e_z &= z - z_h. \end{aligned}$$

PROOF. Selecting $v = e$ in (6.3) and thereafter taking advantage of the Galerkin orthogonality, gives the desired result,

$$Q(e) = b(e, z) = b(e, z - z_h).$$

□

Suppose that $b(\cdot, \cdot)$ defines a norm

$$|||v|||^2 = b(v, v),$$

and suppose that η and η_z are error estimators satisfying

$$\begin{aligned} |||e||| &\leq \eta \\ |||e_z||| &\leq \eta_z. \end{aligned}$$

Then, from Theorem 6.1 and the Cauchy-Schwarz inequality, one immediately obtains

$$|Q(e)| \leq \eta \eta_z,$$

which gives the following bounds for the quantity of interest

$$Q(u_h) - \eta\eta_z \leq Q(u) \leq Q(u_h) + \eta\eta_z.$$

However, this error estimate may be very conservative and we will therefore look at two other approaches that may lead to better estimates.

6.1 The parallelogram approach

We will here present a strategy pursued by several authors, see e.g. [19], and the approach is tailored towards problems where the bilinear form $b(\cdot, \cdot)$ is symmetric, such that the following parallelogram identity holds,

$$b(e, e_z) = \frac{1}{4}|||\alpha e + \alpha^{-1} e_z|||^2 - \frac{1}{4}|||\alpha e - \alpha^{-1} e_z|||^2, \qquad \alpha \neq 0.$$

This identity follows directly from writing the norms in terms of the bilinear form, and it gives rise to the following result.

Lemma 6.2 *Let α be a fixed, nonzero, real constant, and suppose that $\underline{\eta}_+, \overline{\eta}_+, \underline{\eta}_-$, and $\overline{\eta}_-$ satisfy*

$$\underline{\eta}_+ \leq |||\alpha e + \alpha^{-1} e_z||| \leq \overline{\eta}_+ \tag{6.4}$$

$$\underline{\eta}_- \leq |||\alpha e - \alpha^{-1} e_z||| \leq \overline{\eta}_-. \tag{6.5}$$

Then

$$\underline{\eta}_+^2 - \overline{\eta}_-^2 \leq 4Q(e) \leq \overline{\eta}_+^2 - \underline{\eta}_-^2.$$

PROOF. The parallelogram identity and Theorem 6.1 show that

$$Q(e) = \frac{1}{4}|||\alpha e + \alpha^{-1} e_z|||^2 - \frac{1}{4}|||\alpha e - \alpha^{-1} e_z|||^2,$$

and the result now follows directly from (6.4) and (6.5). □

The remaining task is to determine appropriate estimators $\underline{\eta}_+, \overline{\eta}_+, \underline{\eta}_-$, and $\overline{\eta}_-$. Possible estimators $\overline{\eta}_-$ and $\overline{\eta}_+$ can be found by applying the equilibrated residual method. Also other methods can be used, but they will generally not give an upper bound with the constant equal to unity. We will now go back to the test problem (2.1) and assume that after the finite element problem is solved the equilibrated residual method is applied, such that ϕ_T solves the local problem (5.23), and ϕ_T^z solves the corresponding local problem for the dual, i.e.

$$a_T(\phi_T^z, v) = Q_T(v) - a_T(z_h, v) + \int_{\partial T} g_T^z v \, ds \qquad \forall v \in V(T),$$

where g_T^z are the equilibrated fluxes for the dual problem. Then, similar to (5.24),

$$a(\alpha e \pm \alpha^{-1} e_z, v) = \sum_{T \in \mathcal{T}} a_T(\alpha \phi_T \pm \alpha^{-1} \phi_T^z, v) \qquad \forall v \in V. \tag{6.6}$$

Applying the Cauchy-Schwarz inequality twice, first the continuous form and then the discrete form, and thereafter expressing the energy norm as in (2.9), gives

$$|||\alpha e \pm \alpha^{-1} e_z|||^2 \leq \sum_{T \in \mathcal{T}} |||\alpha \phi_T \pm \alpha^{-1} \phi_T^z|||_T^2.$$

The upper bounds are taken to be

$$\overline{\eta}_-^2 = \sum_{T \in \mathcal{T}} |||\alpha \phi_T - \alpha^{-1} \phi_T^z|||_T^2$$

$$\overline{\eta}_+^2 = \sum_{T \in \mathcal{T}} |||\alpha \phi_T + \alpha^{-1} \phi_T^z|||_T^2.$$

Due to (2.9) and (6.6), a lower bound is easily found from the relation

$$|||\alpha e \pm \alpha^{-1} e_z||| \geq \frac{|\sum_{T \in \mathcal{T}} a_T(\alpha \phi_T \pm \alpha^{-1} \phi_T^z, v)|}{|||v|||} \qquad \forall v \in V. \tag{6.7}$$

However, to obtain a sharp and easily computable bound it is important to pick the test function $v = v^*$ with some care. Note that v^* cannot be chosen as

$$\sum_{T\in\mathcal{T}} (\alpha\phi_T \pm \alpha^{-1}\phi_T^z),$$

since this function is generally *not* in the space V. Therefore, define $V_0(T) \subset V(T)$ to be the subspace of functions that vanish on ∂T. It is now convenient to choose

$$v^* = \sum_{T\in\mathcal{T}} \alpha\xi_T + \alpha^{-1}\xi_T^z,$$

where ξ_T, $\xi_T^z \in V_0(T)$ satisfy

$$\begin{aligned} a_T(\xi_T, v) &= L(v) - a_T(u_h, v) \qquad && \forall v \in V_0(T) \\ a_T(\xi_T^z, v) &= Q(v) - a_T(z_h, v) \qquad && \forall v \in V_0(T). \end{aligned}$$

These problems are equivalent to the local problems solved in the equilibrated residual method (with a very restricted solution space), and they are also very similar to the local problems considered in the subdomain residual method. Note that the space of interior bubble functions is a possible choice of discrete subspace. Because of the boundary conditions of the subspace, the above problems can be solved independently for each element, and finally the lower bounds $\underline{\eta}_-$ and $\underline{\eta}_+$ are computed by selecting $v = v^*$ in (6.7).

The coefficient α may be chosen in different ways, and for instance $\alpha = 1$ is a perfectly valid choice. However, a more reasonable choice is to pick α such that the upper bounds $\overline{\eta}_-$ and $\overline{\eta}_+$ are minimised. Therefore, define

$$\begin{aligned} \psi(\alpha) &= \sum_{T\in\mathcal{T}} |||\alpha\phi_T \pm \alpha^{-1}\phi_T^z|||_T^2 \\ &= \sum_{T\in\mathcal{T}} \alpha^2 a_T(\phi_T, \phi_T) \pm a_T(\phi_T, \phi_T^z) + \alpha^{-2} a_T(\phi_T^z, \phi_T^z). \end{aligned}$$

The derivative of ψ with respect to α vanishes for

$$\alpha^4 = \frac{\sum_{T\in\mathcal{T}} |||\phi_T^z|||_T^2}{\sum_{T\in\mathcal{T}} |||\phi_T|||_T^2}.$$

The second derivative of ψ with respect to α is always positive for $\alpha \neq 0$, so the above value of α minimises $\overline{\eta}_-$ and $\overline{\eta}_+$.

6.2 The residual approach

The strategy presented in this section has a more explicit form than the parallelogram approach, and it has been pursued e.g. in [12]. The method applies to symmetrical as well as nonsymmetrical problems, and using this concept one may derive goal oriented error estimators also for nonlinear problems.

Following the proof of Theorem 6.1,

$$Q(e) = a(e, z - v_h^*),$$

where z satisfies the dual problem and $v_h^* \in V_h$ is a function to be chosen. Writing the above expression as a sum over all elements and thereafter applying Green's theorem, results in

$$Q(e) = \sum_{T \in \mathcal{T}} \left\{ \int_T r(z - v_h^*)\, dx + \int_{\partial T} \tilde{R}(z - v_h^*)\, ds \right\},$$

in which r is defined in (4.3) and $\tilde{R}$ is defined in (5.16). Application of the Cauchy-Schwarz inequality gives

$$|Q(e)| \leq \sum_{T \in \mathcal{T}} \left\{ ||r||_{L^2(T)} ||z - v_h^*||_{L^2(T)} + ||\tilde{R}||_{L^2(\partial T)} ||z - v_h^*||_{L^2(\partial T)} \right\},$$

which suggests the estimator

$$\begin{aligned} \eta_T &= ||r||_{L^2(T)} ||z - v_h^*||_{L^2(T)} + ||\tilde{R}||_{L^2(\partial T)} ||z - v_h^*||_{L^2(\partial T)} \\ \eta &= \sum_{T \in \mathcal{T}} \eta_T. \end{aligned}$$

Note that here the local contributions are *not* squared to obtain the global estimator. However, it is not clear how the dual weights should be evaluated since the solution z is generally unknown, and $v_h^* \in V_h$ may be chosen in different ways. Approximating z on a finer mesh is not an option since it is too expensive. Several approaches have been suggested, and the following two strategies are typical examples.

1. Assuming the dual solution is continuous and that V_h is comprised of first order finite elements, take v_h^* to be the nodal interpolant of the dual solution z and use the local interpolation properties of the finite elements to obtain

$$||z - I_{V_h} z||_T \leq C h_T^2 |z|_{H^2(T)}, \tag{6.8}$$

see [20, Theorem 3.4.1]. The interpolation constant C typically takes values from 0.1 to 1, see [21], and the seminorm of z may be approximated by second order finite differences taken at the centre point of the element, i.e. the weights are approximated by

$$||z - v_h^*||_T = ||z - I_{V_h} z||_T \lesssim h_T^2 |\nabla_h^2 z_h|_T.$$

2. Let $v_h^* = z_h$ be the finite element solution of the dual problem and for each element, T, approximate the true dual solution, z, by a higher order polynomial interpolation of z_h over a patch containing T. Denote the higher order approximation of order q by $I_h^{(q)} z_h$. Then the weights are approximated by

$$||z - v_h^*||_T = ||z - z_h||_T \approx ||I_h^{(q)} z_h - z_h||_T.$$

The above strategies may also be applied to the boundary weights, in which case the estimate (6.8) is replaced by

$$||z - I_{V_h} z||_\gamma \leq C h_T^{3/2} ||z||_{H^2(\tilde{\omega}_\gamma)}.$$

Note that option 2 does not involve interpolation constants. The presence of such a constant in option 1 makes the quality of this approximation less clear. Numerical tests in [12] suggest that option 2 may be more accurate than option 1.

7 Applications

So far the techniques described in this report have all been applied only to Poisson's equation. This is of course a convenient model problem, but it also turns out to be an important problem that needs to be solved in many applications. For instance when using certain splitting techniques for fluid flow problems, solving a Poisson equation is the crucial step. However, the methods can be extended to a number of other problems, such as elasticity, flow, and heat equations. In this chapter only one estimator for the stationary Stokes equations and one estimator for the stationary Navier-Stokes equations will be presented, but approaches leading to estimators for more general classes of problems can be found in e.g. [24] and [5].

7.1 Stokes equations

Very slow viscous fluid flow can be described by the Stokes equations, and in this section an explicit residual based error estimator for those equations will be derived. The stationary Stokes equations on differential form read

$$\begin{aligned} -\nu\Delta\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where $\mathbf{u}$ is the velocity, p is the pressure, $\nu > 0$ is a constant viscosity parameter, and $\mathbf{f} \in L^2(\Omega) \times L^2(\Omega)$ is a body force. It will be assumed that $\Omega \subset \mathbb{R}^2$ has a Lipschitz boundary.

We define the functional spaces $\mathbf{V}$ and Q by

$$\begin{aligned} \mathbf{V} &= H_0^1(\Omega) \times H_0^1(\Omega) \\ Q &= \{q \in L^2(\Omega) \, : \, \int_\Omega q \, dx = 0\}. \end{aligned}$$

Let the bilinear forms $a : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$ and $b : Q \times \mathbf{V} \to \mathbb{R}$ be defined by

$$\begin{aligned} a(\mathbf{v}, \mathbf{w}) &= \int_\Omega \nu \nabla\mathbf{v} \cdot \nabla\mathbf{w} \, dx \\ b(q, \mathbf{v}) &= -\int_\Omega q \, (\nabla \cdot \mathbf{v}) \, dx. \end{aligned}$$

Note that the inner product between gradients of vectors is a simplified notation which should be interpreted in the following way,

$$\nabla \mathbf{v} \cdot \nabla \mathbf{w} = \sum_{i=1}^{d} \nabla v_i \cdot \nabla w_i,$$

where d is the number of components in $\mathbf{v}$ and $\mathbf{w}$. Further, the L^2-inner product will be denoted by

$$(\mathbf{v}, \mathbf{w}) = \int_\Omega \mathbf{v} \cdot \mathbf{w}\, dx.$$

With the above definitions the weak form of the Stokes equations is: Find $\mathbf{u} \in \mathbf{V}$ and $p \in Q$ such that

$$\begin{array}{rcll} a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) & = & F(\mathbf{v}) & \forall \mathbf{v} \in \mathbf{V} \\ b(q, \mathbf{u}) & = & 0 & \forall q \in Q, \end{array} \tag{7.1}$$

where

$$F(\mathbf{v}) = (\mathbf{f}, \mathbf{v}).$$

For simplicity, define the space $\mathbf{X} \equiv \mathbf{V} \times Q$ and let $\tilde{a} : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$ be defined as

$$\tilde{a}\Big(\{\mathbf{u}, p\}, \{\mathbf{v}, q\}\Big) = a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) + b(q, \mathbf{u}),$$

such that (7.1) may be written as

$$\tilde{a}\Big(\{\mathbf{u}, p\}, \{\mathbf{v}, q\}\Big) = F(\mathbf{v}) \qquad \forall \{\mathbf{v}, q\} \in \mathbf{X}. \tag{7.2}$$

Let $\mathbf{V}_h \subset \mathbf{V}$ and $Q_h \subset Q$ be finite element spaces, and let $\{\mathbf{u}_h, p_h\} \in \mathbf{V}_h \times Q_h \equiv \mathbf{X}_h \subset \mathbf{X}$ be the finite element approximation to $\{\mathbf{u}, p\}$, i.e.

$$\tilde{a}\Big(\{\mathbf{u}_h, p_h\}, \{\mathbf{v}_h, q_h\}\Big) = F(\mathbf{v}_h) \qquad \forall \{\mathbf{v}_h, q_h\} \in \mathbf{X}_h, \tag{7.3}$$

such that the following Galerkin orthogonality holds,

$$\tilde{a}\Big(\{\mathbf{e}_u, e_p\}, \{\mathbf{v}_h, q_h\}\Big) = 0 \qquad \forall \{\mathbf{v}_h, q_h\} \in X_h,$$

where

$$\begin{array}{rcl} \mathbf{e}_u & = & \mathbf{u} - \mathbf{u}_h \\ e_p & = & p - p_h. \end{array}$$

Note that either one needs to apply a mixed finite element formulation, or one has to resort to stabilisation to obtain a stable finite element solution. For the Stokes equations, define the interior residual $\mathbf{r}$ and the boundary residual $\mathbf{R}$ by

$$\mathbf{r} = \mathbf{f} + \nu \Delta \mathbf{u}_h - \nabla p_h \tag{7.4}$$

$$\mathbf{R}_\gamma = \begin{cases} [p_h \mathbf{n}_\gamma - \nu \mathbf{n}_\gamma \cdot \nabla \mathbf{u}_h] & \gamma \not\subset \partial\Omega \\ \mathbf{0} & \gamma \subset \partial\Omega, \end{cases} \tag{7.5}$$

where the jump $[\cdot]$ was described in Section 4.1. The proof of the following theorem is partly based on [11], but we remark that the first results of this type were proved in [**?**].

Theorem 7.1 *Let $\{\mathbf{u}, p\} \in \mathbf{X}$ satisfy (7.2), and let $\{\mathbf{u}_h, p_h\} \in \mathbf{X}_h$ be the corresponding finite element approximation satisfying (7.3). Then*

$$\Big\{ ||\mathbf{u} - \mathbf{u}_h||^2_{H^1(\Omega)} + ||p - p_h||^2_{L^2(\Omega)} \Big\}^{1/2} \le C\eta,$$

where

$$\begin{aligned} \eta^2 &= \sum_{T \in \mathcal{T}} \eta_T^2 \\ \eta_T^2 &= h_T^2 ||\mathbf{r}||^2_{L^2(T)} + \frac{1}{2} h_T \sum_{\gamma \subset \partial T} ||\mathbf{R}_\gamma||^2_{L^2(\gamma)} + ||\nabla \cdot \mathbf{u}_h||^2_{L^2(T)}. \end{aligned}$$

PROOF. Let $\{\mathbf{z}_u, z_p\} \in \mathbf{X}$ be the solution of the following dual problem

$$\tilde{a}\Big(\{\mathbf{v}, q\}, \{\mathbf{z}_u, z_p\}\Big) = (\nabla \mathbf{e}_u, \nabla \mathbf{v}) + (e_p, q) \qquad \forall \{\mathbf{v}, q\} \in \mathbf{X}. \tag{7.6}$$

Selecting $\mathbf{v} = \mathbf{e}_u$ and $q = e_p$ in (7.6), and exploiting the Galerkin orthogonality gives

$$\begin{aligned} |\mathbf{e}_u|^2_{H^1(\Omega)} + ||e_p||^2_{L^2(\Omega)} &= \tilde{a}\Big(\{\mathbf{e}_u, e_p\}, \{\mathbf{z}_u, z_p\}\Big) \\ &= \tilde{a}\Big(\{\mathbf{e}_u, e_p\}, \{\mathbf{z}_u - I_{V_h}\mathbf{z}_u, z_p - I_{Q_h} z_p\}\Big), \end{aligned}$$

where I_{V_h} and I_{Q_h} are interpolation operators from the respective finite element spaces. By choosing interpolation operators with standard approximation properties, we obtain by the aid of Green's theorem,

$$\begin{aligned} &|\mathbf{e}_u|^2_{H^1(\Omega)} + ||e_p||^2_{L^2(\Omega)} \\ &\quad = \sum_{T \in \mathcal{T}} a(\mathbf{u} - \mathbf{u}_h, \mathbf{z}_u - I_{V_h}\mathbf{z}_u) + b(p - p_h, \mathbf{z}_u - I_{V_h}\mathbf{z}_u) + b(z_p - I_{Q_h} z_p, \mathbf{u}_h) \\ &\quad = \sum_{T \in \mathcal{T}} \int_T \mathbf{r} \cdot (\mathbf{z}_u - I_{V_h}\mathbf{z}_u)\, dx + \frac{1}{2}\int_{\partial T} \mathbf{R} \cdot (\mathbf{z}_u - I_{V_h}\mathbf{z}_u)\, ds + \int_T (z_p - I_{Q_h} z_p) \nabla \cdot \mathbf{u}_h \, dx \\ &\quad \le \sum_{T \in \mathcal{T}} h_T^2 ||\mathbf{r}||_{L^2(T)} ||\mathbf{z}_u||_{H^1(\tilde{\omega}_T)} + \frac{1}{2} h_T \sum_{\gamma \subset \partial T} ||\mathbf{R}_\gamma||_{L^2(\gamma)} ||\mathbf{z}_u||_{H^1(\tilde{\omega}_T)} \\ &\qquad\qquad + ||\nabla \cdot \mathbf{u}_h||_{L^2(T)} ||z_p||_{L^2(\tilde{\omega}_T)}. \end{aligned}$$

Application of the Cauchy-Schwarz inequality shows that

$$\begin{aligned} &|\mathbf{e}_u|^2_{H^1(\Omega)} \;+\; ||e_p||^2_{L^2(\Omega)} \le \\ &\quad C\Big\{ \sum_{T \in \mathcal{T}} h_T^2 ||\mathbf{r}||^2_{L^2(T)} + \frac{1}{2} h_T \sum_{\gamma \subset \partial T} ||\mathbf{R}_\gamma||^2_{L^2(\gamma)} + ||\nabla \cdot \mathbf{u}_h||^2_{L^2(T)} \Big\}^{1/2} \\ &\qquad \cdot \Big\{ ||\mathbf{z}_u||^2_{H^1(\Omega)} + ||z_p||^2_{L^2(\Omega)} \Big\}^{1/2}. \end{aligned}$$

According to [22, Prop.2.2], the following stability result holds for the dual problem,

$$||\mathbf{z}_u||_{H^1(\Omega)} + ||z_p||_{L^2(\Omega)} \leq C\Big(||\mathbf{e}_u||_{H^1(\Omega)} + ||e_p||_{L^2(\Omega)}\Big),$$

and thus

$$|\mathbf{e}_u|^2_{H^1(\Omega)} + ||e_p||^2_{L^2(\Omega)} \leq C\Big\{||\mathbf{e}_u||^2_{H^1(\Omega)} + ||e_p||^2_{L^2(\Omega)}\Big\}^{1/2}\eta. \tag{7.7}$$

Thanks to the coercivity of the bilinear form $a(\cdot,\cdot)$ on the space $\mathbf{V}\times\mathbf{V}$, there exists a constant α such that

$$\nu|\mathbf{v}|^2_{H^1(\Omega)} \equiv a(\mathbf{v},\mathbf{v}) \geq \alpha||\mathbf{v}||_{L^2(\Omega)} \qquad \forall \mathbf{v}\in\mathbf{V},$$

which shows that the H^1-norm and the H^1-seminorm are equivalent. Therefore, squaring both sides of (7.7) and simplifying gives the desired result. □

It can also be shown that the estimator η provides a lower bound for the error, see e.g. [24], but the proof involves a large machinery of estimates and notation, and the result is therefore not stated here. In [24] it is also shown how to treat the full Navier-Stokes equations with stabilisation terms. A similar estimator can also be shown to be an upper bound for the L^2-norm of the velocities, see e.g. [11].

7.2 The stationary incompressible Navier-Stokes equations

In this section an equilibrated residual estimator for the stationary and incompressible Navier-Stokes equations will be presented. The same kind of estimator can be derived for the Stokes equations, but for the Navier-Stokes equations some extra care is needed to treat the nonlinear term.

The following differential form of the Navier-Stokes equations will be considered,

$$\begin{aligned} -\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega \\ \nabla\cdot\mathbf{u} &= 0 && \text{in } \Omega \\ \mathbf{u} &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where the different variables are as described in Section 7.1. Also, let the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$, the linear functional $F(\cdot)$, and the functional spaces $\mathbf{V}$, $\mathbf{V}_h$, Q, Q_h, $\mathbf{X}$, and $\mathbf{X}_h$ be defined as in Section 7.1. Let the trilinear form $c\,:\,\mathbf{V}\times\mathbf{V}\times\mathbf{V}\to\mathbb{R}$ be defined by

$$c(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\,dx.$$

The weak form of the Navier-Stokes problem is now: Find $\{\mathbf{u},p\}\in\mathbf{X}$ such that

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + c(\mathbf{u},\mathbf{u},\mathbf{v}) + b(\mathbf{u},q) = F(\mathbf{v}) \qquad \forall\{\mathbf{v},q\}\in\mathbf{X}\;, \tag{7.8}$$

and the finite element approximation to (7.8) reads: Find $\{\mathbf{u}_h, p_h\} \in \mathbf{X}_h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{u}_h, q_h) = F(\mathbf{v}_h) \quad \forall \{\mathbf{v}_h, q_h\} \in \mathbf{X}_h \ . \tag{7.9}$$

It will be assumed that $\mathbf{u}_h$ converges to $\mathbf{u}$ as the grid is refined. We now define the residual $\rho : \mathbf{X} \to \mathbb{R}$ as

$$\begin{aligned} \rho(\{\mathbf{v}, q\}) &= F(\mathbf{v}) - \Big\{ a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) + c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) + b(\mathbf{u}_h, q) \Big\} \\ &= a(\mathbf{e}_u, \mathbf{v}) + b(\mathbf{v}, e_p) + b(\mathbf{e}_u, q) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) - c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}). \end{aligned}$$

The residual is a continuous linear functional, so we may define the pair $\{\boldsymbol{\phi}, \psi\} \in \mathbf{X}$ to be the Ritz projection of the residual, i.e.

$$a(\boldsymbol{\phi}, \mathbf{v}) + (\psi, q) = \rho(\{\mathbf{v}, q\}) \qquad \forall \{\mathbf{v}, q\} \in \mathbf{X},$$

where we recall that $(\cdot, \cdot)$ is the L^2 inner product. We now define the norm

$$\begin{aligned} |||\{\mathbf{e}_u, e_p\}||| &\equiv \Big\{ a(\boldsymbol{\phi}, \boldsymbol{\phi}) + (\psi, \psi) \Big\}^{1/2} \\ &\equiv \Big\{ \nu |\boldsymbol{\phi}|^2_{H^1(\Omega)} + ||\psi||^2_{L^2(\Omega)} \Big\}^{1/2}. \end{aligned} \tag{7.10}$$

Under the assumption that $\mathbf{u}_h \to \mathbf{u}$ as $h \to 0$ this norm can be shown to be equivalent to the usual norm on the space $\mathbf{X}$, which means there exist constants $C_1, C_2 > 0$ such that

$$C_1 |||\{\mathbf{e}_u, e_p\}|||^2 \le \nu |\mathbf{e}_u|^2_{H^1(\Omega)} + ||e_p||^2_{L^2(\Omega)} \le C_2 |||\{\mathbf{e}_u, e_p\}|||^2.$$

For a proof, see [4]. Note that the above equivalence of norms holds only for the pair $\{\mathbf{e}_u, e_p\}$. Similar to (5.21), let $\mathbf{g}_T \in L^2(\partial T) \times L^2(\partial T)$ satisfy

$$\mathbf{g}_T + \mathbf{g}_{T'} = 0 \qquad \text{on } \partial T \cap \partial T'.$$

Then the following result [4, Theorem 6.4] shows that the error measured in the new norm (7.10) can be bounded by the sum of certain local quadratic functionals.

Theorem 7.2 *Let $J_T : \mathbf{V}(T) \to \mathbb{R}$ be the quadratic functional*

$$\begin{aligned} J_T(\mathbf{v}_T) = &\ \frac{1}{2} a_T(\mathbf{v}_T, \mathbf{v}_T) - F(\mathbf{v}_T) + a_T(\mathbf{u}_h, \mathbf{v}_T) + b_T(\mathbf{v}_T, p_h) \\ &+ c_T(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_T) - \int_{\partial T} \mathbf{g}_T \cdot \mathbf{v}_T \, ds \ . \end{aligned}$$

Then,

$$|||\{\mathbf{e}_u, e_p\}|||^2 \le \sum_{T \in \mathcal{T}} \Big\{ -2 \inf_{\mathbf{v}_T \in \mathbf{V}(T)} J_T(\mathbf{v}_T) \ + \ ||\nabla \cdot \mathbf{u}_h||^2_{L^2(T)} \Big\} \ .$$

□

The local problem corresponding to minimising the functional $J_T(\cdot)$ in Theorem 7.2 is: Find $\boldsymbol{\phi}_T \in \mathbf{V}(T)$ such that

$$a_T(\boldsymbol{\phi}_T, \mathbf{v}_T) = F_T(\mathbf{v}_T) - a_T(\mathbf{u}_h, \mathbf{v}_T) - b_T(\mathbf{v}_T, p_h) \\ -c_T(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_T) + \int_{\partial T} \mathbf{g}_T \cdot \mathbf{v}_T \, ds \quad \forall \mathbf{v}_T \in \mathbf{V}(T). \quad (7.11)$$

This is easily seen, since for any constant ϵ and any $\mathbf{v}_T \in \mathbf{V}(T)$,

$$J_T(\boldsymbol{\phi}_T + \epsilon \mathbf{v}_T) = -\frac{1}{2} a_T(\boldsymbol{\phi}_T, \boldsymbol{\phi}_T) + \frac{\epsilon^2}{2} a_T(\mathbf{v}_T, \mathbf{v}_T) \geq J_T(\boldsymbol{\phi}_T). \quad (7.12)$$

For the local problem (7.11) to be well posed, it is necessary that the boundary fluxes satisfy the following zeroth order equilibration condition

$$0 = F_T(\boldsymbol{\theta}) - a_T(\mathbf{u}_h, \boldsymbol{\theta}) - b_T(\boldsymbol{\theta}, p_h) \\ - c_T(\mathbf{u}_h, \mathbf{u}_h, \boldsymbol{\theta}) + \int_{\partial T} \mathbf{g}_T \cdot \boldsymbol{\theta} \, ds \qquad \forall \boldsymbol{\theta} \in \mathrm{Ker}\,[a_T, \mathbf{V}(T)],$$

where $\mathrm{Ker}\,[a_T, \mathbf{V}(T)]$ is the kernel of $a_T(\cdot,\cdot)$, defined as

$$\mathrm{Ker}\,[a_T, \mathbf{V}(T)] = \left\{ \boldsymbol{\theta} \in \mathbf{V}(T) \,:\, a_T(\mathbf{v}, \boldsymbol{\theta}) = 0 \quad \forall \mathbf{v} \in \mathbf{V}(T) \right\}.$$

Since (7.12) shows that $\boldsymbol{\phi}_T$ is the minimiser of J_T and that

$$a_T(\boldsymbol{\phi}_T, \boldsymbol{\phi}_T) = -2 J_T(\boldsymbol{\phi}_T),$$

Theorem 7.2 suggests the local error estimator to be

$$\eta_T^2 = a_T(\boldsymbol{\phi}_T, \boldsymbol{\phi}_T) + ||\nabla \cdot \mathbf{u}_h||^2_{L^2(T)} \equiv \nu |\boldsymbol{\phi}_T|^2_{H^1(T)} + ||\nabla \cdot \mathbf{u}_h||^2_{L^2(T)}.$$

By defining the global error estimator as

$$\eta^2 = \sum_{T \in \mathcal{T}} \eta_T^2,$$

we conclude from Theorem 7.2 that

$$|||\{\mathbf{e}_u, e_p\}||| \leq \eta.$$

Note that this estimator is obtained without solving local Navier-Stokes problems, in fact the local problems are of Poisson type and they are very cheap to solve compared to the global Navier-Stokes problem. A more detailed description of this procedure and the corresponding procedure for the Stokes problem can be found in [4].

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