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A combined Filon/asymptotic quadrature method for highly oscillatory problems

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Abstract.

A cross between the asymptotic expansion of an oscillatory integral and the Filon-type methods is obtained by applying a Filon-type method on the error term in the asymptotic expansion, which is in itself an oscillatory integral. The efficiency of the approach is investigated through analysis and numerical experiments, and a potential for better methods than current ones is revealed. In particular can savings in the required number of potentially expensive moments be expected. The case of multivariate oscillatory integrals is discussed briefly.

AMS subject classification (2000): 65D30.

Key words: high oscillation, quadrature, asymptotic expansions, Filon integration.

1 Introduction

The quadrature of highly oscillatory integrals has been perceived as a hard problem. Traditionally one would have to resolve the oscillations by taking several sub-intervals for each period, resulting in a scheme whose complexity would grow linearly with the frequency of the oscillations. More careful analysis will however reveal that by exploiting the structure of certain classes of oscillatory integrals better discretisation schemes can be devised, schemes where the error actually decreases when the frequency of the oscillations increases. This is well known in asymptotic analysis with eg. saddle point methods and the method of stationary phase approximation[15, 13]. Recently attention has been directed at numerical methods with similar properties. Examples of such methods are Filon-type methods[7, 8] Levin-type methods[12, 14] and numerical steepest descent[6].

We are considering oscillatory integrals of the form

$$I[f] = \int_{-1}^{1} f(x)e^{i\omega g(x)}\mathrm{d}x, \tag{1.1}$$

where ω is a large parameter. It is well known that an ordinary Gaussian quadrature applied to this integral will have an error of $\mathcal{O}(1)$ as ω grows large. A much better approach to approximating $I[f]$ when ω is large is found through an asymptotic expansion: Assuming $g'(x) \neq 0,\ -1 \leq x \leq 1$, integration by parts yields

$$I[f] = \frac{1}{i\omega}\left[\frac{f(1)}{g'(1)}e^{i\omega g(1)} - \frac{f(-1)}{g'(-1)}e^{i\omega g(-1)}\right] - \frac{1}{i\omega}\int_{-1}^{1}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x)}{g'(x)}\right]e^{i\omega g(x)}\mathrm{d}x. \tag{1.2}$$

When ω becomes large the integral term in equation (1.2) vanishes faster than the boundary terms, by an extension of Riemann-Lebesgue's lemma, so the boundary terms can approximate the integral. Furthermore the process can be repeated on the integral remainder to obtain a full asymptotic expansion. This expansion will however not be perfect. As is often the case with asymptotic expansions the accuracy is limited due to the divergence of the series.

An even better approach is to choose a set of quadrature nodes $c_1, \ldots, c_\nu$, interpolate the function f by a polynomial $\tilde{f}$ at these points and let

$$Q_1^F[f] = \int_{-1}^{1}\tilde{f}(x)e^{i\omega g(x)}\mathrm{d}x = \sum_{j=1}^{\nu} b_j(\omega)f(c_j),$$

where $b_j(\omega) = \int_{-1}^{1} l_j(x)e^{i\omega g(x)}\mathrm{d}x$ for $l_j(x)$ the j-th Lagrange cardinal polynomial. A variant of this approach, then with piecewise quadratic interpolation in the Fourier-case when $g(x) = x$, dates back to L.N.G. Filon[4]. Schemes of this type are referred to as Filon-type methods. Constructing $b_j(\omega)$ requires the moments $\int_{-1}^{1} x^m e^{i\omega g(x)}\mathrm{d}x$. Moments are oscillatory integrals themselves that hopefully can be calculated by analytical means as in the Fourier case. If not, the numerical steepest descent method can be applied to compute moments for the Filon-type method, an approach which works well in practical applications[6, 2]. Iserles proved[7] that as long as the endpoints of the interval are included as quadrature nodes and $g'(x) \neq 0,\ -1 \leq x \leq 1$, this approach will carry an error

$$Q_1^F[f] - I[f] \sim \mathcal{O}(\omega^{-2}), \quad \omega \to \infty.$$

The superiority of this approach over the asymptotic expansion can be understood by realising that the method is exact for a class of problems, regardless of the size of ω. As for the behaviour for large ω it was proved by Iserles and Nørsett[10] that by applying Hermite interpolation to interpolate $f(x)$ with p derivatives at the endpoints, the asymptotic behaviour of the error can be expressed as

$$Q_p^F[f] - I[f] \sim \mathcal{O}(\omega^{-p-1}), \quad \omega \to \infty$$

The theory can be expanded to the cases where g has stationary points, that means points ξ with $g'(\xi) = 0$. What must be done to achieve good asymptotic properties is basically to include the stationary points among the quadrature nodes[8].

Considering the asymptotic expansion with the remainder term (1.2) one cannot fail to notice that the problem has really been transformed into boundary terms plus the remainder term, which is an integral of the same form as the original. A natural question to ask in light of this observation is whether treating the remainder term with a specialised technique, like the Filon-type quadrature, numerical steepest descent or a Levin-type method, could improve accuracy. In the following this question will be addressed, in particular for the choice of the Filon-type quadrature Q_p^F as quadrature method. In the above-mentioned case this would amount to a new method

$$Q^{FA}[f] = \frac{1}{i\omega}\left[\frac{f(1)}{g'(1)}e^{i\omega g(1)} - \frac{f(-1)}{g'(-1)}e^{i\omega g(-1)}\right] - \frac{1}{i\omega}Q_p^F\left[\frac{\mathrm{d}}{\mathrm{d}x}[f/g']\right].$$

We will refer to methods of this form as *combined Filon/asymptotic methods*. Observe that for $\omega \neq 0$ this method is consistent in the sense that accuracy can be improved by using a better quadrature method on the remainder term, a property which the asymptotic expansion does not have. Furthermore, because of the $1/\omega$-factor, the asymptotic error behaviour will be better than for the classical Filon-type method applied directly. This means that less work, in terms of moments, is needed to get high asymptotic order. The combined method is in this sense a true cross between the asymptotic method and the Filon-type method, combining good qualities of both methods. These observations will be elaborated on in the following with emphasis on the 1D case without stationary points, with stationary points and a brief look into the multivariate case.

2 The Asymptotic method and Filon-type methods

We begin the exposition by presenting an overview of the constituent parts of the combined method: The asymptotic expansion of the highly oscillatory integral and the Filon-type methods. In the following we will denote by $Q_p[f] \approx I[f]$ a highly oscillatory quadrature method of *asymptotic order* p, meaning that for smooth f

$$Q_p[f] - I[f] \sim \mathcal{O}(\omega^{-p-1}), \quad \omega \to \infty.$$

Note that in some parts of the literature this would be referred to as order $p+1$. This corresponds to absolute error decay, whereas ours is relative error decay in the case of no stationary points where $I[f] \sim 1/\omega$ [15]. In the presence of stationary points the picture is slightly different, and for simplicity we will then avoid the concept of asymptotic order.

2.1 The case of no stationary points

Assume for the time being that there are no stationary points in the interval of interest, that means $g'(x) \neq 0, \quad -1 \leq x \leq 1$. An asymptotic expansion of the highly oscillatory integral (1.1) is obtained by successively applying integration by parts. This approach gives us a full expansion through the following partial expansion

$$I[f] = -\sum_{m=1}^{s} \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right] + \frac{1}{(-i\omega)^s} \int_{-1}^{1} \sigma_s[f](x) e^{i\omega g(x)} \mathrm{d}x, \tag{2.1}$$

where

$$\sigma_0[f](x) = f(x)$$
$$\sigma_{m+1}[f](x) = \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sigma_m[f](x)}{g'(x)}, \quad k = 0, 1, \ldots. \tag{2.2}$$

The correctness of the above expansion can easily be checked through an induction argument. A full asymptotic expansion of the highly oscillatory integral (1.1) is then

$$I[f] \sim -\sum_{m=1}^{\infty} \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right]. \tag{2.3}$$

Truncating the series after s terms, yields the asymptotic method

$$Q_s^A[f] = -\sum_{m=1}^{s} \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right]. \tag{2.4}$$

The method has asymptotic order s. This can be seen by writing out the remainder term, which is an oscillatory integral $\mathcal{O}(\omega^{-1})$ multiplied by $(-i\omega)^{-s}$. Note that the concept of asymptotic order is rather useless for not-so-large ω. In fact the asymptotic expansion is divergent in the general case, and this divergence is more severe for smaller ω. Thus the asymptotic method is rather useless for small ω. Furthermore, divergence implies that only a fixed accuracy can be attained - adding terms will not always increase accuracy. This is problematic for practical applications where usually a given accuracy is sought.

The Filon-type methods will be accurate also for smaller ω and have controllable error, but that is at the cost of moments. We define the moments

$$\mu_k(\omega) = \int_{-1}^{1} x^k e^{i\omega g(x)} \mathrm{d}x,$$

and assume these can be computed, possibly at a significant cost. Then the Filon-type method is obtained by choosing a set of nodes $-1 = c_1 < c_2 < \cdots < c_\nu = 1$ and integer multiplicities $m_1, \ldots, m_\nu \geq 1$ associated with each node. Let $n = \sum_{j=1}^{\nu} m_j - 1$ and $\tilde{f}$ be the unique Hermite interpolation polynomial of degree n obtained by interpolating f at the points $\{c_j\}_{j=1}^{\nu}$ with the corresponding multiplicities,

$$\tilde{f}(x) = \sum_{l=1}^{\nu} \sum_{j=0}^{m_l - 1} \alpha_{l,j}(x) f^{(j)}(c_l).$$

The Filon-type method is defined as

$$Q_s^F[f] = \int_{-1}^{1} \tilde{f}(x) e^{i\omega g(x)} \mathrm{d}x = \sum_{l=1}^{\nu} \sum_{j=0}^{m_l - 1} \beta_{l,j}(\omega) f^{(j)}(c_l), \tag{2.5}$$

where $\beta_{l,j}(\omega) = \int_{-1}^{1} \alpha_{l,j}(x) e^{i\omega g(x)} \mathrm{d}x$ is obtained from linear combinations of moments. As for s, the asymptotic order of this method, we we state a theorem due to Iserles and Nørsett[10]:

Theorem 2.1. *Suppose $m_1, m_\nu \geq s$, then for every smooth f and smooth g with $g'(x) \neq 0,\ -1 \leq x \leq 1$*

$$Q_s^F[f] - I[f] \sim (\omega^{-s-1}), \quad \omega \to \infty.$$

The proof is obtained by expanding $f - \tilde{f}$ as in equation (2.3) and observing that the first s terms will cancel due to the interpolation criteria. This theorem can be summarised by saying that only by adding derivative information at the endpoints of the interval can the asymptotic order of the method be improved. Information about derivatives can also be supplied indirectly by clustering interpolation nodes near the endpoints. If the nodes approach the endpoints as $1/\omega$ high asymptotic order can be attained[9]. Note that increasing the order of the interpolating polynomial $\tilde{f}$ will increase the accuracy of the method for some fixed ω, at least when the interpolation nodes are the Chebychev points. This is indeed confirmed by numerical experiments[9]. This means that for any ω a prescribed accuracy can be attained, a property which is crucial for practical applications.

2.2 Generalized Filon and asymptotic method in the presence of stationary points

When g has stationary points Theorem 2.1 is no longer valid, a fact which is suggested by the singularity introduced in the integral in remainder term of the asymptotic expansion (1.2). Assume in the following that $g(x)$ has only one stationary point $\xi \in (-1, 1)$, which amounts to saying $g'(\xi) = 0,\quad g'(x) \neq 0, x \in [-1, 1] \backslash \{\xi\}$. Furthermore assume that $g'(\xi) = \cdots = g^{(r)}(\xi) = 0$, and $g^{(r+1)}(\xi) \neq 0$, this means that ξ is a rth order stationary point. The method of stationary phase[3, 13] states that in this case the leading order behaviour of the highly oscillatory integral (1.1) is of the form

$$I[f] \sim C\omega^{-1/(r+1)}, \quad \omega \to \infty. \tag{2.6}$$

This means that the main contribution to the value of the integral comes from the stationary point, suggesting that the interpolation nodes for the Filon-type methods should include stationary points as well as the endpoints.

Assume for simplicity that ξ is a first order stationary point meaning $g'(\xi) = 0$ and $g''(\xi) \neq 0$. Writing

$$\begin{aligned} I[f] &= f(\xi)I[1] + I[f - f(\xi)] \\ &= f(\xi)I[1] + \frac{1}{i\omega} \int_{1}^{-1} \frac{f(x) - f(\xi)}{g'(x)} \frac{\mathrm{d}}{\mathrm{d}x} e^{i\omega g(x)} \mathrm{d}x, \end{aligned}$$

then integrating by parts gives the following expression:

$$\begin{aligned} I[f] = f(\xi)I[1] + \frac{1}{i\omega} &\left[\frac{f(1) - f(\xi)}{g'(1)} e^{i\omega g(1)} - \frac{f(-1) - f(\xi)}{g'(-1)} e^{i\omega g(-1)} \right] \\ &- \frac{1}{i\omega} \int_{-1}^{1} \frac{\mathrm{d}}{\mathrm{d}x} \frac{f(x) - f(\xi)}{g'(x)} e^{i\omega g(x)} \mathrm{d}x. \end{aligned} \tag{2.7}$$

Now, since $g''(\xi) \neq 0$, the singularity is removable. The expansion can be continued giving a full expansion reminiscent of the expansion (2.3). More generally, for a rth order stationary point we introduce the generalized moments

$$\mu_k(\omega; \xi) = I[(\cdot - \xi)^k] = \int_{-1}^{1} (x - \xi)^k e^{i\omega g(x)} \mathrm{d}x, \quad k \geq 0.$$

Note that these can be written in terms of ordinary moments. Now write

$$I[f] = \sum_{j=0}^{r-1} \frac{1}{j!} f^{(j)}(\xi) \mu_j(\omega; \xi) + I\left[f(x) - \sum_{j=0}^{r-1} \frac{1}{j!} f^{(j)}(\xi)(x - \xi)^j \right]. \tag{2.8}$$

Again the singularity is removable, and the expansion can be formed. We will later need the expansion with the remainder term, so this will be formulated as a lemma[1]:

LEMMA 2.2. *Suppose ξ is a stationary point of order r, and that ξ is the only stationary point inside the interval $[-1,1]$. Then for every smooth f*

$$\begin{aligned}
I[f] =& \sum_{j=0}^{r-1} \frac{1}{j!}\mu_j(\omega;\xi) \sum_{m=1}^{s} \frac{1}{(-i\omega)^{m-1}} \rho_{m-1}[f]^{(j)}(\xi) \\
& - \sum_{m=1}^{s} \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \left(\rho_{m-1}[f](1) - \sum_{j=0}^{r-1} \frac{1}{j!} \rho_{m-1}[f]^{(j)}(\xi)(1-\xi)^j \right) \right. \\
& \qquad \left. - \frac{e^{i\omega g(-1)}}{g'(-1)} \left(\rho_{m-1}[f](-1) - \sum_{j=0}^{r-1} \frac{1}{j!} \rho_{m-1}[f]^{(j)}(\xi)(-1-\xi)^j \right) \right] \\
& + \frac{1}{(-i\omega)^s} I\left[\rho_s[f]\right],
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
\rho_0[f](x) &= f(x) \\
\rho_{m+1}[f](x) &= \frac{\mathrm{d}}{\mathrm{d}x} \frac{\rho_m[f](x) - \sum_{j=0}^{r-1} \frac{1}{j!} \rho_m[f]^{(j)}(\xi)(x-\xi)^j}{g'(x)}, \quad k = 0, 1, \ldots
\end{aligned} \tag{2.10}$$

PROOF. This is proved by induction. The Lemma is certainly true for $s = 0$. Now

$$\begin{aligned}
I\left[\rho_s[f]\right] =& \sum_{j=0}^{r-1} \frac{1}{j!} \rho_s[f]^{(j)}(\xi)\mu_j(\omega;\xi) \\
& + \frac{1}{i\omega} \int_{-1}^{1} \frac{\rho_s[f](x) - \sum_{j=0}^{r-1} \frac{1}{j!} \rho_s[f]^{(j)}(\xi)(x-\xi)^j}{g'(x)} \frac{\mathrm{d}}{\mathrm{d}x} e^{i\omega g(x)} \mathrm{d}x.
\end{aligned}$$

Integration by parts gives

$$\begin{aligned}
I\left[\rho_s[f]\right] =& \sum_{j=0}^{r-1} \frac{1}{j!} \rho_s[f]^{(j)}(\xi)\mu_j(\omega;\xi) \\
& - \frac{1}{(-i\omega)} \left[\frac{e^{i\omega g(1)}}{g'(1)} \left(\rho_s[f](1) - \sum_{j=0}^{r-1} \frac{1}{j!} \rho_s[f]^{(j)}(\xi)(1-\xi)^j \right) \right. \\
& \left. - \frac{e^{i\omega g(-1)}}{g'(-1)} \left(\rho_s[f](-1) - \sum_{j=0}^{r-1} \frac{1}{j!} \rho_s[f]^{(j)}(\xi)(-1-\xi)^j \right) \right] \\
& + \frac{1}{(-i\omega)} I\left[\rho_{s+1}[f]\right].
\end{aligned}$$

Inserting into equation (2.9) proves the Lemma. □

As before, truncating the expansion (2.9), that is the two m-summations after s terms, yields the asymptotic method. The asymptotic behaviour of the error in this method is found by the method of stationary phase applied to the remainder. Thus we get for the asymptotic method,

$$Q^A[f] - I[f] \sim \mathcal{O}(\omega^{-s-1/(r+1)}), \quad \omega \to \infty.$$

[1]Note that the conclusion in this lemma is different from that of Iserles & Nørsett in [10], Theorem 3.2, which we suggest is flawed.

For an even more general case, in the presence of more than one stationary point, the interval can be partitioned such that each sub interval contains only one stationary point, and then an expansion can be made for each sub interval. As before, truncating the expansion after s terms yields the asymptotic method.

Now to the Filon-type method: Let ξ be a unique stationary point of order r: $g'(\xi) = 0$ and $g'(x) \neq 0$ for $x \in [-1,1]\backslash\{\xi\}$, $g'(\xi) = \cdots = g^{(r)}(\xi) = 0$, and $g^{(r+1)}(\xi) \neq 0$. The *generalized Filon method*[10] is constructed by choosing nodes $-1 = c_1 < c_2 < \cdots < c_\nu = 1$ such that the stationary point is among the nodes, that is $c_q = \xi$ for some $q \in \{1, 2, \ldots, \nu\}$. Given multiplicities $m_1, m_2, \ldots, m_\nu \geq 1$ corresponding to each node, we let $\tilde{f}$ be the unique Hermite interpolation polynomial of degree $n = \sum_{j=1}^{\nu} m_j - 1$ obtained by interpolating f at the points $\{c_j\}_{j=1}^{\nu}$ with the corresponding multiplicities. The method is now simply

$$Q^F[f] = \int_{-1}^{1} \tilde{f}(x) e^{i\omega g(x)} \mathrm{d}x.$$

The above integral is computed from linear combinations of moments.

We present another theorem by Iserles and Nørsett[10] regarding the asymptotic error behaviour of the generalized Filon method.

THEOREM 2.3. *Let $m_1, m_\nu \geq s$ and $m_q \geq s(r+1) - 1$. Then*

$$Q^F[f] - I[f] \sim \mathcal{O}(\omega^{-s-1/(r+1)}), \quad \omega \to \infty.$$

This theorem is, like Theorem 2.1 proved by expanding $f - \tilde{f}$ and showing that terms up to order s cancel. The method is trivially expanded to cater for several stationary points, possibly of different order.

3 The combined Filon/asymptotic method

Let us for the moment assume that there are no stationary points of g in $[-1, 1]$. This assumption will be relaxed later on. A combined Filon/asymptotic method is constructed from the asymptotic expansion with the remainder term (2.1) by applying a Filon-type method on the remainder term, which is in itself an oscillatory integral. Denoting by $Q^{FA}_{p,s}$ a method which is obtained by applying a p-th order Filon-type method on the remainder of an s-term expansion we get

$$\begin{aligned} Q^{FA}_{p,s}[f] = &- \sum_{m=1}^{s} \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right] \\ &+ \frac{1}{(-i\omega)^s} Q^F_p\left[\sigma_s[f]\right], \end{aligned} \tag{3.1}$$

where the $\sigma_m[f]$ are defined as in equation (2.2). Note that this formula is consistent for $\omega \neq 0$ in the sense that if we resolve the remainder term exactly, then the formula is exact as well. Furthermore, note that the idea is not restricted to Filon-type methods. Any quadrature method Q_p can be applied:

THEOREM 3.1. *Let g be such that $g'(x) \neq 0$, $-1 \leq x \leq 1$. Applying a highly oscillatory quadrature method Q_p of asymptotic order p on the remainder in the s-term asymptotic expansion* (2.1) *yields a method $Q_{p,s}$. Applied to any smooth f this method is of order $p+s$, that is*

$$Q_{p,s}[f] - I[f] \sim \mathcal{O}(\omega^{-p-s-1}), \quad \omega \to \infty.$$

PROOF. Writing out the asymptotic expansion of $Q_{p,s}[f] - I[f]$ gives

$$\begin{aligned}
Q_{p,s}[f] - I[f] &\sim \frac{1}{(-i\omega)^s} Q_p[\sigma_s[f](x)] \\
&\quad + \sum_{m=s+1}^{\infty} \frac{1}{(i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right] \\
&= \frac{1}{(-i\omega)^s} \Bigg(Q_p[\sigma_s[f](x)] - \sum_{j=1}^{\infty} \frac{1}{(i\omega)^j} \Big[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{j-1}[\sigma_s[f]](1) \\
&\qquad - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{j-1}[\sigma_s[f]](-1) \Big] \Bigg) \\
&\sim \frac{1}{(-i\omega)^s} \mathcal{O}(\omega^{-p-1}) = \mathcal{O}(\omega^{-p-s-1}),
\end{aligned}$$

where the last line appears by using the asymptotic error property of the method Q_p. □

We will here limit our attention to the case where Q_p is a Filon-type method, and we call the combined method $Q_{p,s}^{FA}$ a Filon/asymptotic method.

EXAMPLE 3.1. *For the simplest case set $s = 1$ and get*

$$Q_{p,1}^{FA}[f] = \frac{1}{i\omega} \left[\frac{e^{i\omega g(1)}}{g'(1)} f(1) - \frac{e^{i\omega g(-1)}}{g'(-1)} f(-1) \right] - \frac{1}{i\omega} Q_p^F \left[\frac{\mathrm{d}}{\mathrm{d}x} \frac{f}{g'} \right], \tag{3.2}$$

which is a method of asymptotic order $p + 1$.

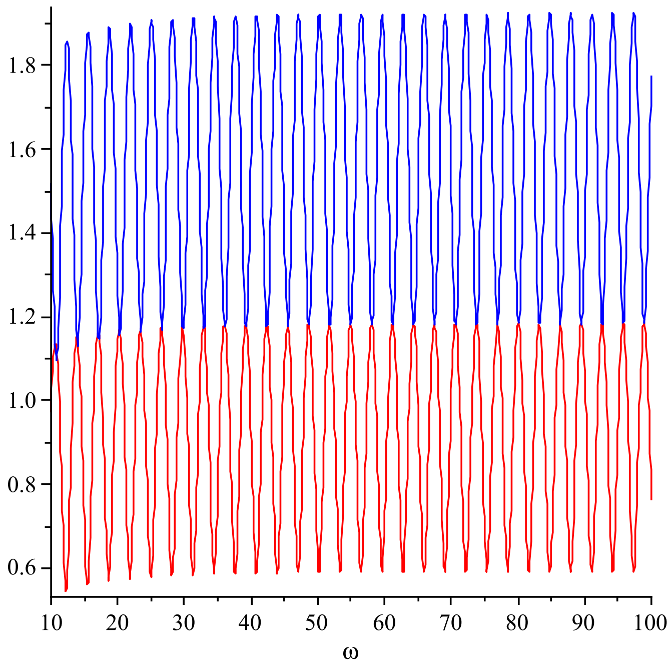


Figure 3.1: The absolute value of the error for the combined Filon/asymptotic method (top) and the classical Filon-type method (bottom) from example 3.2, all scaled by ω^3.

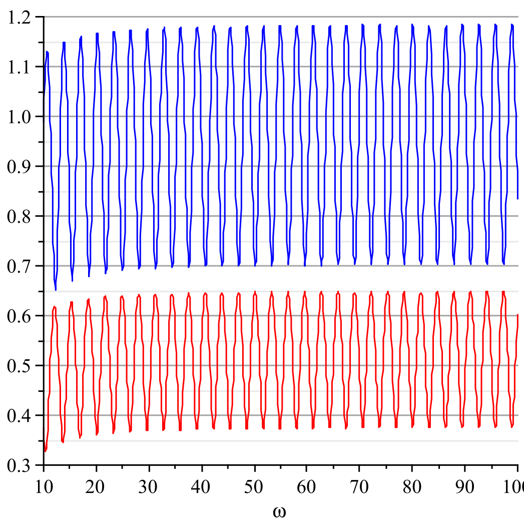
EXAMPLE 3.2. *We wish to compute*

$$\int_{-1}^{1} \frac{e^{i\omega x}}{2 + x} \mathrm{d}x.$$

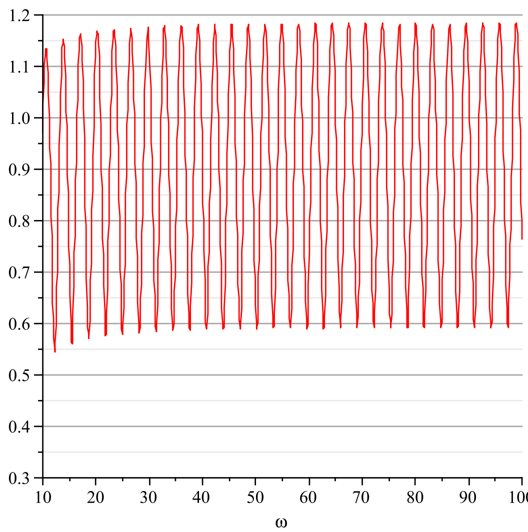
Interpolating $f(x) = 1/(2+x)$ and its first derivative at $x = -1$ and $x = 1$ will give us a Filon-type method of asymptotic order $s = 2$. This method requires four moments. Interpolating only the function value of $\sigma_1(x) = -1/(2+x)^2$ at the two endpoints gives the combined Filon/asymptotic scheme which is also of asymptotic order 2, but only needs two moments. We expect this to be at the cost of not that good approximation properties compared to the classical method, which is indeed confirmed by experiments, see figure 3.1. Note that the crests of the curve of one method seems to correspond with the troughs of the other, much like what was pointed out by Iserles & Nørsett in [10]. This behaviour will be discussed in section 5.

The key element in a discussion of the efficiency of this method is the need for moments. Recall that a classical asymptotic method needs no moments, but it breaks down for small ω and the error is not controllable. On the other hand a classical Filon-type method can be made precise also for moderately sized ω, but at the cost of moments. A Filon-type method needs a minimum of $2p$ moments to obtain asymptotic order p. The combined Filon/asymptotic method is situated between the Filon-type method and the asymptotic method, both in spirit and in terms of requirements. For example, this method can obtain any asymptotic order as well as accuracy for moderately sized ω with the use of only two moments. The asymptotic nature of the method is revealed by the $1/\omega^s$-factor which indicates that it will perform bad as $\omega \to 0$. For $\omega = 0$ the method does not work, as opposed to the classical Filon-type method which in this case reduces to a classical quadrature method. The combined method can, like the classical Filon-type method, be made precise to a prescribed tolerance by adding more moments. The usefulness is here dictated by the cost of computing moments, as well as the cost of computing $\sigma_m[f]$ and its derivatives. The following example, example 3.3, shows how a combined Filon/asymptotic method performs better than a classical Filon-type method with approximately the same input data. This observation will be elaborated on in section 5.1.

Example 3.3. *Once again we wish to compute with a combined Filon/asymptotic method the*



(a)



(b)

Figure 3.2: a) Error for the Filon/asymptotic method with interpolation nodes $[-1, 0, 1]$(top), and $[-1, -1/3, 1/3, 1]$(bottom), scaled by ω^3. b) Error for the classical Filon-type method scaled by ω^3 (same scale as (a).)

integral

$$\int_{-1}^{1} \frac{e^{i\omega x}}{2+x} \mathrm{d}x,$$

but this time we include internal nodes. Interpolating $\sigma_1(x) = -1/(2+x)^2$ at the nodes $[-1, 0, 1]$, and $[-1, -1/3, 1/3, 1]$ will result in combined schemes requiring three and four moments respectively. That means comparable to the classical Filon-type method from example 3.2, which is obtained by interpolating $f(x) = 1/(2 + x)$ with its first derivative at the endpoints requiring four moments. Both this classical method and the above described combined methods have asymptotic order 2. Comparing error plots for the methods(see figure 3.2) we see that the combined method with nodes $[-1, 0, 1]$ has almost exactly the asymptotic error constant as the classical method when ω increases, whereas the one with nodes $[-1, -1/3, 1/3, 1]$ has a significantly smaller error constant. In figure 3.3 we see how the different methods behaves for small ω. Note that including internal nodes reduces the severity of the singularity. Even for quite small ω the best Filon/asymptotic method is better than the classical method.

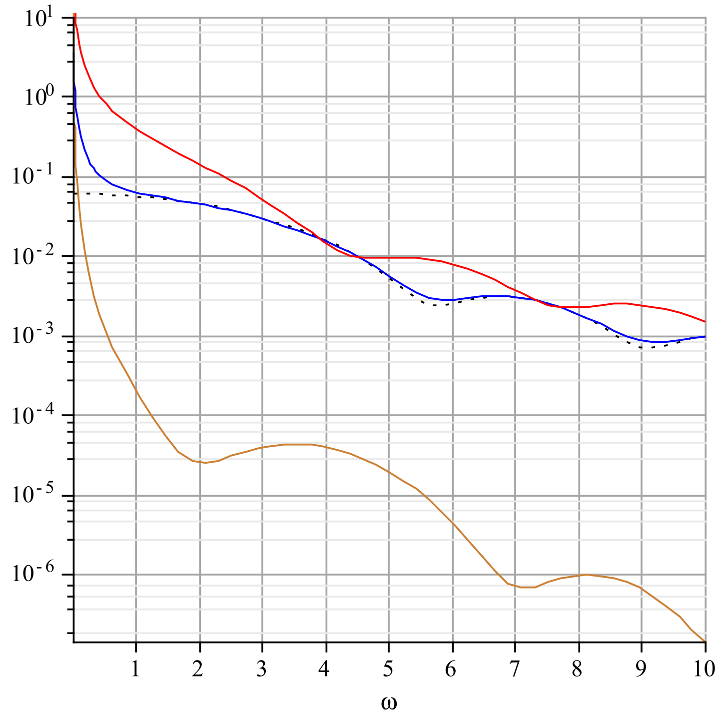


Figure 3.3: Log-plot of the error for the Filon/asymptotic method with interpolation nodes $[-1, 1]$(top), $[-1, 0, 1]$(middle) and $[-1, -1/3, 1/3, 1]$(bottom), not scaled. Error for the classical Filon-type method shown as a dotted line

3.1 The combined Filon/asymptotic method with stationary points

Extending the method to cater for stationary points is fairly straightforward given Lemma 2.2. Assume in the following that ξ is the only stationary point of order r in $[-1, 1]$. This requirement is not crucial, it will just simplify otherwise horrific expressions. In the following we will denote by Q_p a method tailored for this problem, like the generalized Filon-type quadrature, which for smooth f bears an error

$$Q_p[f] - I[f] \sim \mathcal{O}(\omega^{-p-1/(r+1)}), \quad \omega \to \infty$$

Applying the generalized Filon method Q_p^F on the expansion (2.9) yields the generalized combined Filon/asymptotic method

$$
\begin{aligned}
Q^{FA}_{p,s}[f] =& \sum_{j=0}^{r-1} \frac{1}{j!}\mu_j(\omega;\xi) \sum_{m=1}^{s} \frac{1}{(-i\omega)^{m-1}}\rho_{m-1}[f]^{(j)}(\xi) \\
& - \sum_{m=1}^{s} \frac{1}{(-i\omega)^m}\left[\frac{e^{i\omega g(1)}}{g'(1)}\Big(\rho_{m-1}[f](1) - \sum_{j=0}^{r-1}\frac{1}{j!}\rho_{m-1}[f]^{(j)}(\xi)(1-\xi)^j\Big)\right. \\
& \qquad \left. - \frac{e^{i\omega g(-1)}}{g'(-1)}\Big(\rho_{m-1}[f](-1) - \sum_{j=0}^{r-1}\frac{1}{j!}\rho_{m-1}[f]^{(j)}(\xi)(-1-\xi)^j\Big)\right] \\
& + \frac{1}{(-i\omega)^s} Q^F_p\left[\rho_s[f]\right]
\end{aligned}
\tag{3.3}
$$

$\rho_m[f]$ are defined as in equation (2.10). Recall that Q^F_p is constructed by interpolating f in the endpoints and ξ (c_1, c_ν and c_q) with multiplicities m_1, m_ν and m_q respectively. Using a generic method Q_p we have the following theorem:

Theorem 3.2. *Assume $g'(\xi) = \cdots = g^{(r)}(\xi) = 0$, $g^{(r+1)}(\xi) \neq 0$ and $g'(x) \neq 0$ for $x \in [-1,1]\backslash\{\xi\}$. Let Q_p be a method which for any smooth f has the asymptotic error*

$$Q_p[f] - I[f] \sim \mathcal{O}(\omega^{-p-1/(r+1)}), \quad \omega \to \infty.$$

For the combined method $Q_{p,s}$, constructed by applying Q_p on the remainder term in expansion (2.9)*, applied to any smooth f it is true that*

$$Q_{p,s}[f] - I[f] \sim \mathcal{O}(\omega^{-p-s-1/(r+1)}), \quad \omega \to \infty.$$

Proof. Completely analogous to the proof of Theorem 3.1 we get

$$
\begin{aligned}
Q_{p,s}[f] - I[f] &\sim \frac{1}{(-i\omega)^s}\Big(Q_p[\rho_s[f]] - \int_{-1}^{1}\rho_s[f](x)e^{i\omega g(x)}\mathrm{d}x\Big) \\
&\sim \frac{1}{(-i\omega)^s}\mathcal{O}(\omega^{-p-1/(r+1)}) = \mathcal{O}(\omega^{-p-s-1/(r+1)}).
\end{aligned}
$$

□

Again we restrict our treatment to the method $Q^{FA}_{p,s}$ constructed from a generalized Filon-type method.

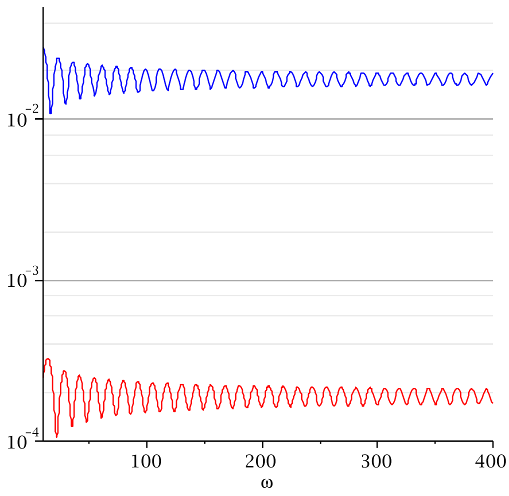
Example 3.4. *The simplest case is a problem with only one stationary point ξ of order one, expanded with one term(as in equation* (2.7)*). The combined Filon/asymptotic method* (3.3) *written out is then*

$$
\begin{aligned}
Q^{FA}_{p,1}[f] =& \mu_0(\omega)f(\xi) + \frac{1}{i\omega}\left(\frac{f(1)-f(\xi)}{g'(1)}e^{i\omega g(1)} - \frac{f(-1)-f(\xi)}{g'(-1)}e^{i\omega g(-1)}\right) \\
& - \frac{1}{i\omega}Q^F_p\left[\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)-f(\xi)}{g'(x)}\right]
\end{aligned}
\tag{3.4}
$$

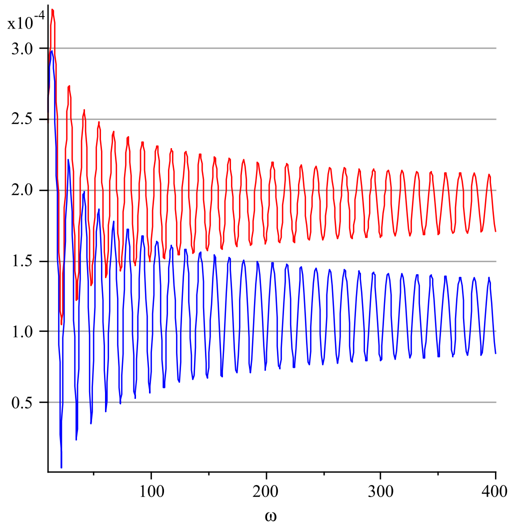
Example 3.5. *The oscillator of the integral*

$$\int_{-1}^{1} e^x e^{i\omega\frac{1}{2}x^2}\mathrm{d}x$$

has an order one stationary point at $x = 0$. Interpolating $\rho_1[f](x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)-f(\xi)}{g'(x)} = \frac{xe^x - e^x + 1}{x^2}$ at the nodes $[-1,0,1]$ (using l'Hospital's rule to obtain the value at the stationary point) gives a



(a)



(b)

Figure 3.4: a) The absolute value of the error for the combined Filon/asymptotic method with $\mathbf{c} = [-1, 0, 1]$ (top), together with classical Filon-type method (bottom) in logarithmic scale. b) Combined method with $\mathbf{c} = [-1, -1/2, 0, 1/2, 1]$ (bottom), and the classical Filon-type method(top). All curves are scaled by $\omega^{\frac{5}{2}}$. Logarithmic scale is used in (a) in order to properly represent both curves in the same plot.

combined Filon/asymptotic scheme on the form of (3.4). *The predicted error behaviour seems to be confirmed by experiments (see figure 3.4). The proposed scheme needs three moments plus the first generalized moment μ_0 which is constructed from these. A classical Filon-type method requires a total of seven to obtain the same asymptotic order. Figure 3.4 (a) shows that the proposed method has a much higher asymptotic error constant than the classical Filon-type method, however do we only need to add two interpolation nodes, that is two moments, to beat it. See figure 3.4 (b) for illustration.*

4 Extension to the multivariate case

For the model multivariate highly oscillatory integral we write

$$I[f, \Omega] = \int_\Omega f(\mathbf{x}) e^{i\omega g(\mathbf{x})} \mathrm{d}V,$$

where $\Omega \in \mathbb{R}^d$ and $f, g : \Omega \to \mathbb{R}$. Bringing the highly oscillatory quadrature methods into the multivariate setting presents us with a whole set of complications. For example we will have to take into account not only stationary points, $\mathbf{x}$ s.t $\nabla g(\mathbf{x}) = 0$, but also points of resonance, those are boundary points where ∇g is orthogonal to the boundary, ie. no oscillation along the boundary. For general smooth boundaries resonance will necessarily be a problem, in this case theory is not yet fully developed. Furthermore, computing moments will be even more expensive than in the univariate case. For oscillatory integrals on simplices and polygons we refer to [11] for a theoretical treatment.

In the following we assume that no stationary points or resonance points are present. Furthermore we restrict our treatment to the d-dimensional simplex[2]. The Filon-type method is in this case, like in the 1D case, constructed by interpolating in *critical* points, here being the vertices

[2]Note that polygons can be tiled by simplices, thus generalising the results for a simplex to the polygon case.

of the simplex. Increasing asymptotic order is done by increasing the number of interpolated derivatives at the vertices.

The point of departure for developing a combined formula will here be the Stokes-type formula for a simplex as presented in [11]:

$$
\begin{aligned}
I[f,\mathcal{S}_d] =& \frac{1}{i\omega}\int_{\partial\mathcal{S}_d} \mathbf{n}^T(\mathbf{x})\nabla g(\mathbf{x})\frac{f(\mathbf{x})}{||\nabla g(\mathbf{x})||^2}e^{i\omega g(\mathbf{x})}\mathrm{d}S \\
&- \frac{1}{i\omega}\int_{\mathcal{S}_d} \nabla^T\left[\frac{f(\mathbf{x})}{||\nabla g(\mathbf{x})||^2}\nabla g(\mathbf{x})\right]e^{i\omega g(\mathbf{x})}\mathrm{d}V.
\end{aligned}
\tag{4.1}
$$

Using the formula repeatedly on the remainder term yields an expansion with an integral remainder term. We here state this as a theorem:

THEOREM 4.1. *For any smooth f and smooth g without stationary points and subject to the non-resonance condition, it is true that*

$$
\begin{aligned}
I[f,\mathcal{S}_d] = &-\sum_{m=1}^{s}\frac{1}{(-i\omega)^m}\int_{\partial\mathcal{S}_d} \mathbf{n}^T(\mathbf{x})\nabla g(\mathbf{x})\frac{\sigma_{m-1}(\mathbf{x})}{||\nabla g(\mathbf{x})||^2}e^{i\omega g(\mathbf{x})}\mathrm{d}S \\
&+\frac{1}{(-i\omega)^s}\int_{\mathcal{S}_d}\sigma_s(\mathbf{x})e^{i\omega g(\mathbf{x})}\mathrm{d}V,
\end{aligned}
\tag{4.2}
$$

where

$$
\begin{aligned}
\sigma_0(\mathbf{x}) &= f(\mathbf{x}) \\
\sigma_{m+1}(\mathbf{x}) &= \nabla^T\left[\frac{\sigma_m(\mathbf{x})}{||\nabla g(\mathbf{x})||^2}\nabla g(\mathbf{x})\right].
\end{aligned}
$$

PROOF. The proof follows from an iterated use of formula (4.1). □

The expansion (4.2) can be carried on to obtain a full expansion for large ω, showing that the value of the integral is asymptotically determined by integrals over the faces of the simplex. Furthermore, by expanding the lower dimensional integrals one repeatedly "pushes" the integrals from faces to edges(lower dimensional faces), a process which terminates at the vertices, indicating that the value of the integral is asymptotically determined by data at the vertices of the simplex. The expansion can also be used to show that the value of the integral $I[f,\mathcal{S}_d]$ decays like $\mathcal{O}(\omega^{-d})$.

Now the combined method in all its glorious generality:

THEOREM 4.2. *Assume Q_p is a quadrature method with asymptotic order p, that is*

$$
I[f,\mathcal{S}_d] - Q_p[f,\mathcal{S}_d] \sim \mathcal{O}(\omega^{-d-p}), \quad \omega\to\infty.
$$

For any smooth f and g, without stationary points and subject to the non-resonance condition, the method

$$
\begin{aligned}
Q[f,\mathcal{S}_d] = &-\sum_{m=1}^{s}\frac{1}{(-i\omega)^m}\int_{\partial\mathcal{S}_d} \mathbf{n}^T(\mathbf{x})\nabla g(\mathbf{x})\frac{\sigma_{m-1}(\mathbf{x})}{||\nabla g(\mathbf{x})||^2}e^{i\omega g(\mathbf{x})}\mathrm{d}S \\
&+\frac{1}{(-i\omega)^s}Q_p[\sigma_s,\mathcal{S}_d]
\end{aligned}
\tag{4.3}
$$

is of asymptotic order $s+p$.

PROOF. As in proof of theorem 3.1, write out the expansion of the error and use the asymptotic error property of Q_p. □

This method is not really a quadrature rule per se, as we have not addressed the fact that also the boundary integrals have to be treated somehow. A lower dimensional, thus cheaper, quadrature method might be used. Using the Stokes-type formula to reduce the dimension of the boundary integrals until we are left with a formula incorporating data only at the vertices is a possibility, but then also treating the resulting remainder terms with a Filon-type method is preferable in order to retain control over the error.

4.1 Quadrature on the 2D simplex

To illustrate the combined Filon/asymptotic approach in the multivariate case we consider the case of the 2D simplex. Assume no stationary points or resonance points are present and write

$$I[f,\mathcal{S}_2] = \int_0^1 \int_0^{1-y} f(x,y)e^{i\omega g(x,y)}\mathrm{d}x\mathrm{d}y,$$

Applying the Stokes-type formula once yields:

$$\begin{aligned}
I[f,\mathcal{S}_2] = &\frac{1}{i\omega}\int_0^1 \mathbf{n_1}^T\nabla g(x,0)\frac{f(x,0)}{||\nabla g(x,0)||^2}e^{i\omega g(x,0)}\mathrm{d}x \\
&+\sqrt{2}\frac{1}{i\omega}\int_0^1 \mathbf{n_2}^T\nabla g(x,1-x)\frac{f(x,1-x)}{||\nabla g(x,1-x)||^2}e^{i\omega g(x,1-x)}\mathrm{d}x \\
&-\frac{1}{i\omega}\int_0^1 \mathbf{n_3}^T\nabla g(0,y)\frac{f(0,y)}{||\nabla g(0,y)||^2}e^{i\omega g(0,y)}\mathrm{d}y \\
&-\frac{1}{i\omega}\int_0^1\int_0^{1-y} \nabla^T\left[\frac{f(x,y)}{||\nabla g(x,y)||^2}\nabla g(x,y)\right]e^{i\omega g(x,y)}\mathrm{d}x\mathrm{d}y
\end{aligned} \tag{4.4}$$

with $\mathbf{n_1} = [0,-1], \mathbf{n_2} = [\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}]$ and $\mathbf{n_1} = [-1,0]$ being outer normals as illustrated in figure 4.1.

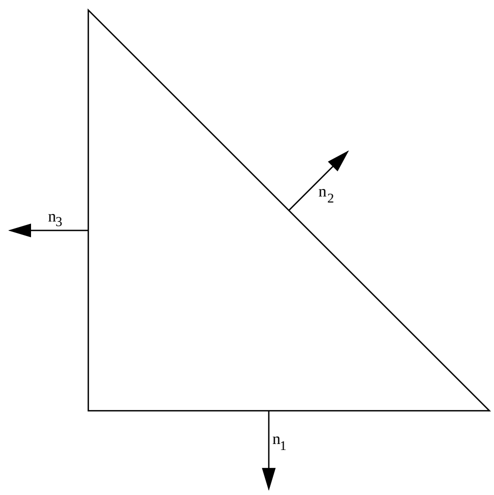
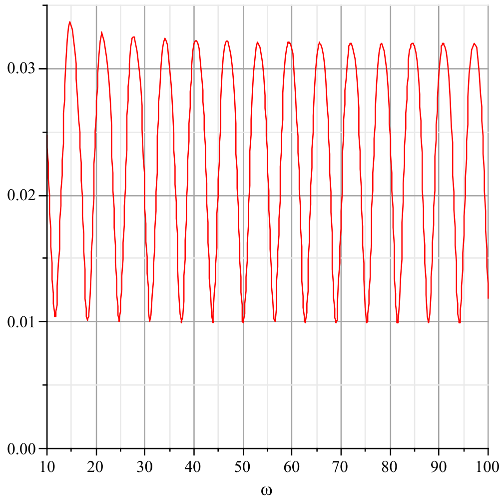


Figure 4.1

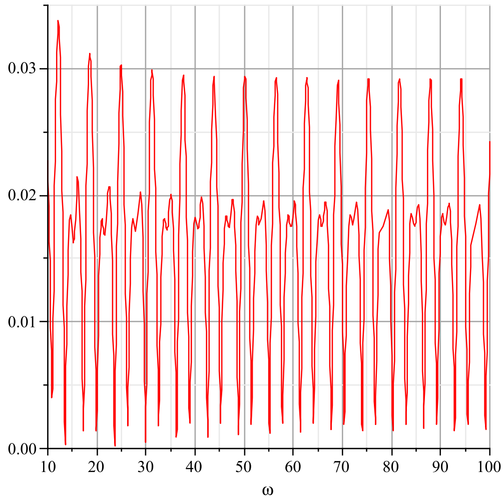
Example 4.1.
Considering the problem

$$I = \int_0^1 \int_0^{1-y} \sin(x+y)e^{i\omega(x-2y)}\mathrm{d}x\mathrm{d}y,$$

we construct a classical Filon-type method of order 2, meaning the error goes down like $\mathcal{O}(\omega^{-4})$*, by interpolating function values and derivatives at the vertices. An interpolation point at* $(1/4,1/4)$ *is included in order to fix the last parameter in a full third order interpolation polynomial. Thus 10 moments are required. Constructing a combined method with the same asymptotic order from the formula* (4.4)*, which consists of three univariate and one bivariate integrals, can be done with a first order multivariate method applied to the remainder term and a second order univariate method(error goes like* $\mathcal{O}(\omega^{-3})$*) on the boundary terms. In total we need four univariate moments per edge plus three bivariate moments for the remainder term. Adding an interpolation point in*



(a)



(b)

Figure 4.2: a) The error of the classical Filon-type method, scaled by ω^4. b) The combined method, also scaled by ω^4.

$(1/4, 1/4)$ *for the sake of comparison gives a method with similar accuracy as the classical method, see figure 4.2. We observe that in this case the combined method performs better than the classical method, for general problems the two methods will have comparable accuracy.*

Assuming bivariate moments are much harder to compute than univariate moments, the example shows a good improvement of efficiency. On the downside the combined method is harder to implement, and for error control, the error of four quadratures must be balanced, which can pose a problem.

The combined method can also be constructed in a more extreme way, sorting out all information at the vertices as simple terms, and all integrals as remainder terms. Carrying out the computations for the non-resonant 2D simplex problem without stationary points yields the following expression:

$$
\begin{aligned}
I = \frac{1}{(i\omega)^2}\Bigg[& \frac{e^{i\omega g(0,0)} f(0,0)}{||\nabla g(0,0)||^2}\left(\frac{g_y(0,0)}{g_x(0,0)} + \frac{g_x(0,0)}{g_y(0,0)}\right) \\
& - \frac{e^{i\omega g(1,0)} f(1,0)}{||\nabla g(1,0)||^2}\left(\frac{g_y(1,0)}{g_x(1,0)} - \frac{g_x(1,0)+g_y(1,0)}{g_x(1,0)-g_y(1,0)}\right) \\
& - \frac{e^{i\omega g(0,1)} f(0,1)}{||\nabla g(0,1)||^2}\left(\frac{g_x(0,1)}{g_y(0,1)} + \frac{g_x(0,1)+g_y(0,1)}{g_x(0,1)-g_y(0,1)}\right)\Bigg] \\
+ \frac{1}{(i\omega)^2}\Bigg[& \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x,0) g_y(x,0)}{||\nabla g(x,0)||^2 g_x(x,0)}\right] e^{i\omega g(x,0)} \mathrm{d}x \\
& - \int_0^1 \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{f(x,1-x)(g_x(x,1-x)+g_y(x,1-x))}{||\nabla g(x,1-x)||^2 (g_x(x,1-x)-g_y(x,1-x))}\right] e^{i\omega g(x,1-x)} \mathrm{d}x \\
& + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{f(0,y) g_x(0,y)}{||\nabla g(0,y)||^2 g_y(0,y)}\right] e^{i\omega g(0,y)} \mathrm{d}y\Bigg] \\
& - \frac{1}{i\omega}\int_0^1 \int_0^{1-y} \nabla^T \left[\frac{f(x,y)}{||\nabla g(x,y)||^2}\nabla g(x,y)\right] e^{i\omega g(x,y)} \mathrm{d}x\mathrm{d}y,
\end{aligned}
$$

Note however that this approach will potentially only reduce on the number of univariate moments needed, the bivariate remainder term is still at large. Therefore we will not pursue this approach further.

5 Error estimates

In Example 3.2 where the simple univariate case without stationary point was considered, we observed how the troughs in the error plot for a particular Filon/asymptotic method seem to correspond with the peaks of a classical Filon-type method. This is exactly the same observation Iserles and Nørsett made in [9], but then for two different Filon-type methods. The behaviour we have observed can be explained in a similar way. This investigation will also lead to a method for comparing classical Filon-type methods and Filon/asymptotic methods of the same asymptotic order.

Assume in the following that $g'(x) \neq 0, \quad -1 \leq x \leq 1$. From the discussion on the asymptotic order of a Filon-type method and equation (2.3) it is clear that

$$Q_p^F[f] - I[f] \sim \frac{e_p^F[f]}{\omega^{p+1}} + \mathcal{O}(\omega^{-p-2}), \quad \omega \to \infty.$$

$e_p^F[f]$ is basically the next term in the expansion of $f - \tilde{f}$, with $\tilde{f}$ being the interpolant of f:

$$e_p^F[f] = \frac{e^{i\omega g(1)}}{g'(1)}[\sigma_p[\tilde{f}](1) - \sigma_p[f](1)] - \frac{e^{i\omega g(-1)}}{g'(-1)}[\sigma_p[\tilde{f}](-1) - \sigma_p[f](-1)].$$

By arguing that $\sigma_p[f] = \frac{f^{(p)}}{(g')^p} +$ a linear combination of $f^{(k)}$ multiplied by a function involving derivatives of g, $k = 0, \ldots, p-1$, one states that for a Filon-type method the *asymptotic error constant* $|e_p^F|$ can be estimated by

$$\Lambda_-^F[f] \leq |e_p^F[f]| \leq \Lambda_+^F[f],$$

where

$$\Lambda_\pm^F[f] = \left| \frac{|\tilde{f}^{(p)}(1) - f^{(p)}(1)|}{|g'(1)|^{p+1}} \pm \frac{|\tilde{f}^{(p)}(-1) - f^{(p)}(-1)|}{|g'(-1)|^{p+1}} \right|.$$

The exact same reasoning can be used to estimate the asymptotic error constant for a combined Filon/asymptotic method $Q_{p,s}^{FA}$. Keeping in mind that the asymptotic order of this method is $p+s$ we can write

$$Q_{p,s}^{FA}[f] - I[f] \sim \frac{e_{p,s}^{FA}[f]}{\omega^{p+s+1}} + \mathcal{O}(\omega^{-p-s-2}), \quad \omega \to \infty.$$

Now the Filon-type method is applied to the remainder, so it should be clear that

$$e_{p,s}^{FA}[f] = \frac{e^{i\omega g(1)}}{g'(1)}[\tilde{\sigma}_s[f]^{(p)}(1) - \sigma_s[f]^{(p)}(1)] - \frac{e^{i\omega g(-1)}}{g'(-1)}[\tilde{\sigma}_s[f]^{(p)}(-1) - \sigma_s[f]^{(p)}(-1)].$$

Here $\tilde{\sigma}_s[f]$ denotes the interpolant of $\sigma_s[f]$, and $\tilde{\sigma}_s[f]^{(p)}(x)$ its p-th derivative evaluated in x. This gives

$$\Lambda_-^{FA}[f] \leq |e_{p,s}^{FA}[f]| \leq \Lambda_+^{FA}[f],$$

with

$$\Lambda_\pm^{FA}[f] = \left| \frac{|\tilde{\sigma}_s[f]^{(p)}(1) - \sigma_s[f]^{(p)}(1)|}{|g'(1)|^{p+1}} \pm \frac{|\tilde{\sigma}_s[f]^{(p)}(-1) - \sigma_s[f]^{(p)}(-1)|}{|g'(-1)|^{p+1}} \right|.$$

Example 5.1. *Example 3.2 concerns the problem* $\int_{-1}^{1} \frac{e^{i\omega x}}{2+x}\mathrm{d}x$, *whereby applying a Filon-type method we obtain*

$$\tilde{f}(x) = -\frac{1}{9}x^3 + \frac{2}{9}x^2 - \frac{2}{9}x + \frac{4}{9} \qquad \text{and} \qquad [\Lambda_-^F, \Lambda_+^F] = [0.5930, \ 1.1852].$$

The combined Filon/asymptotic method has

$$\tilde{\sigma}_1[f](x) = \frac{4}{9}x - \frac{5}{9} \qquad \text{and} \qquad [\Lambda_-^{FA}, \Lambda_+^{FA}] = [1.1852,\ 1.9259].$$

These estimates explain the most significant features of Figure 3.1. For the schemes in Example 3.3 we have:

$$\begin{array}{lll} \mathbf{c} = [-1, 0, 1]: & \tilde{\sigma}_1[f](x) = -\frac{11}{36}x^2 + \frac{4}{9}x - \frac{1}{4}, & [\Lambda_-^{FA}, \Lambda_+^{FA}] = [0.7037,\ 1.1852] \\ \mathbf{c} = [-1, -\frac{1}{3}, \frac{1}{3}, 1]: & \tilde{\sigma}_1[f](x) = \frac{248}{1225}x^3 - \frac{391}{1225}x^2 & \\ & \quad +\frac{2668}{11025}x - \frac{2606}{11025}, & [\Lambda_-^{FA}, \Lambda_+^{FA}] = [0.3754,\ 0.6492] \end{array}$$

These calculations fit well with what has been observed, note in particular how the method with $\mathbf{c} = [-1, 0, 1]$ *closely matches the classical Filon-type method.*

5.1 Comparing the classical Filon and Filon/asymptotic methods

Now it is time to address the important question: Will a combined Filon/ asymptotic method get better accuracy than the classical Filon-type method from the same information[3]? For simplicity, consider the Fourier case $g(x) = x$, and also assume derivatives of f are easily available. The maximum error for a Filon-type method and a combined Filon/asymptotic method, both of asymptotic order p, as ω becomes large are then

$$\begin{aligned} \Lambda_+^F[f] &= |\tilde{f}^{(p)}(1) - f^{(p)}(1)| + |\tilde{f}^{(p)}(-1) - f^{(p)}(-1)|, \\ \Lambda_+^{FA}[f] &= |\tilde{\sigma}_s[f]^{(p-s)}(1) - \sigma_s[f]^{(p-s)}(1)| + |\tilde{\sigma}_s[f]^{(p-s)}(-1) - \sigma_s[f]^{(p-s)}(-1)| \\ &= |\tilde{\sigma}_s[f]^{(p-s)}(1) - f^{(p)}(1)| + |\tilde{\sigma}_s[f]^{(p-s)}(-1) - f^{(p)}(-1)|. \end{aligned}$$

Now $g(x) = x$ implies that $\sigma_s[f] = f^{(s)}$, and $\tilde{\sigma}_s[f]$ is the interpolant of $f^{(s)}$. We see that both methods have an error which is determined by the interpolant's ability to approximate the pth derivative of f at the endpoints. The error constant in the Filon-type method comes from interpolating f and differentiating the interpolant, for the combined approach take s derivatives, interpolate, then differentiate. The possibility to more freely chose the placement of the interpolation nodes, not restricted to the endpoints, will also result in a better approximation of the pth derivative, explaining at least in part why the combined method performs better than the classical method with the same data. We wish to explore this a bit further.

In the following we will do a small computation to demonstrate what can be gained by using a combined method. Consider a method constructed from $2p$ nodes distributed equidistantly, including endpoints, to approximate the error in a $p-1$ term asymptotic expansion, that is a $Q_{p-1,1}^{FA}$-type method, compared to a Filon-type method of asymptotic order p of minimum complexity Q_p^F? By an order p method of minimum complexity we mean a method constructed by interpolating only p derivatives at the endpoints with no internal nodes, implying that we use the minimum number of moments to attain order p. Now bear in mind that equidistant points are by no means optimal, but are just used for the sake of demonstration. These two methods are both are of asymptotic order p and use $2p$ moments. Q_p^F requires p data at each endpoint to interpolate f, it is well known that the error of the Hermite interpolation is[5]

$$\tilde{f}(x) - f(x) = \frac{f^{(2p)}(c_1)}{(2p)!}(x+1)^p(x-1)^p,$$

where $c_1 \in [-1, 1]$. Then from Rodrigues' formula[1]

$$\tilde{f}^{(p)}(x) - f^{(p)}(x) = \frac{f^{(2p)}(c_1)}{(2p)!}P_p(x)2^p p!,$$

[3]Information here signifies moments.

with $P_p(x)$ being the pth Legendre polynomial. As $|P_n(\pm 1)| = 1$ we have

$$\Lambda_+^F[f] = 2^{p+1}p!\frac{|f^{(2p)}(c_1)|}{(2p)!} = |f^{(2p)}(c_1)|\frac{2^{1-p}\sqrt{\pi}}{\Gamma(p+\frac{1}{2})}. \tag{5.1}$$

For the $Q_{p-1,1}^{FA}$-type method, we consider the case with $n+1$ equidistant nodes, including endpoints. We interpolate $\sigma_{p-1}[f]$, and the interpolation error is now[5]:

$$\tilde{\sigma}_{p-1}[f](x) - f^{(p-1)}(x) = \frac{f^{(p-1+n+1)}(c_2)}{(n+1)!}\prod_{i=0}^{n}(x - 1 + i\frac{2}{n}),$$

for $c_2 \in [-1, 1]$. This simplifies to

$$\tilde{\sigma}_{p-1}[f](x) - f^{(p-1)}(x) = \frac{f^{(p+n)}(c_2)}{(n+1)!}\frac{2^{n+1}\Gamma(\frac{n}{2}(x+1))}{n^{n+1}\Gamma(\frac{n}{2}(x-1))}.$$

Differentiating gives

$$\tilde{\sigma}_{p-1}[f]'(x) - f^{(p)}(x) = \frac{f^{(p+n)}(c_2)}{(n+1)!}\frac{2^n}{n^n}\frac{(\Psi(\frac{n}{2}(x+1)+1) - \Psi(\frac{n}{2}(x-1))\Gamma(\frac{n}{2}(x+1)+1)}{\Gamma(\frac{n}{2}(x-1))},$$

with Ψ being the digamma function. The limit of the above expression as x tends to ± 1 can be found with a bit of effort:

$$\lim_{x\to\pm 1}[\tilde{\sigma}_{p-1}[f]'(x) - f^{(p)}(x)] = f^{(p+n)}(c_2)(\pm 1)^n\frac{2^n}{(n+1)n^n}.$$

Now

$$\Lambda_+^{FA}[f] = |f^{(p+n)}(c_2)|\frac{2^{n+1}}{(n+1)n^n}. \tag{5.2}$$

For the case where the two methods use the same moments $n = 2p - 1$, and then

$$\Lambda_+^{FA}[f] = |f^{(3p-1)}(c_2)|\frac{2^{2p}}{2p \cdot (2p-1)^{2p-1}}.$$

Now we investigate the relative sizes of the two asymptotic error constants.

$$\frac{\Lambda_+^{FA}[f]}{\Lambda_+^F[f]} = \frac{|f^{(3p-1)}(c_2)|\frac{2^{2p}}{(2p)(2p-1)^{2p-1}}}{|f^{(2p)}(c_1)|\frac{2^{1-p}\sqrt{\pi}}{\Gamma(p+\frac{1}{2})}} = \frac{|f^{(3p-1)}(c_2)|}{|f^{(2p)}(c_1)|}\frac{8^p}{4}\frac{\Gamma(p+1/2)}{\sqrt{\pi}p(2p-1)^{2p-1}}.$$

If we use no derivatives, that is $p = 1$, the ratio is one, and for increasing p the ration is decreasing. In general the derivatives can often be assumed to be of magnitude $|f^{(n)}| \sim L^n$, this will in the limit not alter the conclusion. The significance of the above calculations is most easily appreciated through a plot. Figure 5.1 shows that, assuming the derivatives of f are of the same order of magnitude, the combined Filon/asymptotic method will have a smaller error constant when using the same number of moments.

EXAMPLE 5.2. *As a final little calculation we once again investigate Example 3.3 and the close match between the* $\mathbf{c} = [-1, 0, 1]$ *combined Filon/asymptotic method and the classical Filon-type method, both of order* $p = 2$. *Equation* (5.1) *with* $p = 2$ *gives for the latter*

$$\Lambda_+^F[f] \sim \frac{\sqrt{\pi}}{2\frac{3}{4}\sqrt{\pi}} = \frac{2}{3}.$$

The $\mathbf{c} = [-1, 0, 1]$ *combined Filon/asymptotic method has three equidistant nodes, that is* $n = 2$. *Equation* (5.2) *gives,*

$$\Lambda_+^{FA}[f] \sim \frac{2^3}{3 \cdot 2^2} = \frac{2}{3}.$$

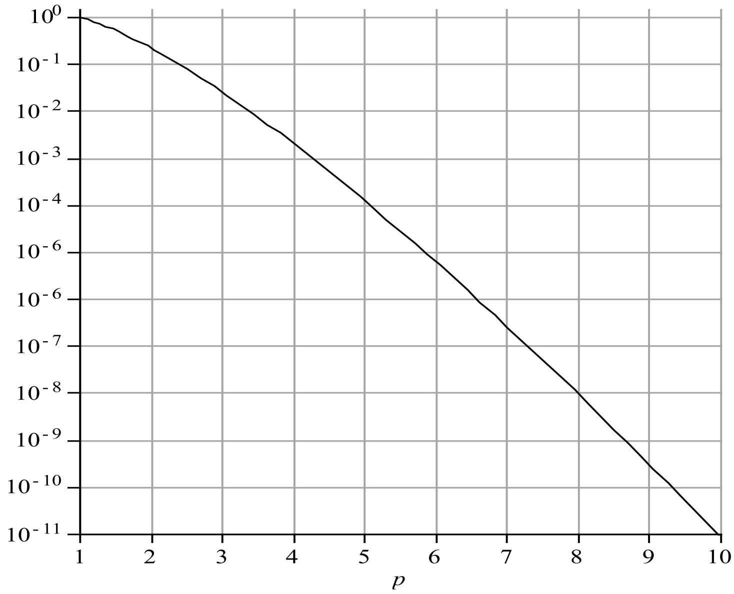


Figure 5.1: Log-plot of the ratio $\frac{8^p}{4}\frac{\Gamma(p+1/2)}{\sqrt{\pi}p(2p-1)^{2p-1}}$.

This fits well with the close match between the two methods that we observe in example 3.3. Provided that derivatives are of the same order, these methods will in general perform similarly.

We must remark that although the proposed method apparently performs better, it is by no means optimal. The freedom to choose interpolation nodes could be used to minimise the error, placing nodes closer to the boundary would generally be better as derivatives at the boundary would be better approximated, see [9], but this also depends on the size of ω. In the limit $\omega \to \infty$, placing all the nodes at the boundary, increasing the asymptotic order would be best. On the other hand, a more spread out distribution would probably be beneficial for smaller ω. All this seems to make the whole discussion about asymptotic error constants slightly artificial.

6 Conclusion

We have demonstrated the feasibility of combining the asymptotic expansion of highly oscillatory integrals and Filon-type methods. Experiments as well as theoretical calculations show that the combined method can achieve better precision than the classical Filon-type method with more or less the same information. The extra cost of the combined method lies mainly in more complicated expressions, especially for cases with several stationary points or in the multivariate case. In order to make a combined method for more general oscillatory integrals we must have an asymptotic expansion with an oscillatory integral remainder. However, such an expansion is not always available.

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