NORGES TEKNISK-NATURVITENSKAPELIGE UNIVERSITET

# Imposing free-surface boundary conditions using surface intrinsic coordinates

by

Tormod Bjøntegaard

PREPRINT
NUMERICS NO. 5/2007

NORWEGIAN UNIVERSITY OF
SCIENCE AND TECHNOLOGY
TRONDHEIM, NORWAY

This report has URL http://www.math.ntnu.no/preprint/numerics/2007/N5-2007.pdf
Address: Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway.

# Imposing free-surface boundary conditions using surface intrinsic coordinates

Tormod Bjøntegaard

September 6, 2007

We consider here the incompressible Navier-Stokes equations in three dimensions subject to free surface boundary conditions along part or all of the domain boundary. In [2] a surface integral for weak imposition of both normal and tangential surface tension boundary conditions was proposed. This integral was based on describing the free surface using surface-intrinsic coordinates. In this paper we derive the proposed surface integral using results from differential geometry. A key ingredient in this derivation is an expression for the curvature-normal product, and most of the paper will be devoted to deriving this expression.

## 1 Introduction

We consider here the incompressible Navier-Stokes equations in three dimensions subject to free surface boundary conditions along part or all of the domain boundary. One critical aspect of the numerical approximation of such problems is the incorporation of these boundary conditions. This is related to the fact that the shape of the free surface is generally unknown, and the normal and tangential stresses in the presence of surface tension depend on the local curvature and possibly the surface gradient of the surface tension. The free surface boundary conditions can be expressed as

$$F_n = n_i \sigma_{ij} n_j = \gamma \kappa,$$
$$F_t = t_i \sigma_{ij} n_j = t_k (\nabla_s \gamma)_k,$$

where $n_i, i = 1, 2, 3$, are the components of the outward unit normal vector ($\mathbf{n}$), $t_i, i = 1, 2, 3$, are the components of a tangent vector ($\mathbf{t}$), $F_n$ is the normal component of the stress force, $F_t$ is a tangential component of the stress force in the direction of $\mathbf{t}$, $\sigma_{ij}$ is the stress tensor, $\gamma$ is the surface tension and $\kappa$ is twice the mean curvature. Here, $\nabla_s \gamma$ represents the surface gradient of the surface tension and $t_k(\nabla_s \gamma)_k$ represents the component of the surface gradient in the direction of $\mathbf{t}$. Summation over repeated indices is assumed.

In order to naturally incorporate the free surface boundary conditions, we need to use the stress formulation of the Navier-Stokes equations. For the numerical treatment we will use the corresponding weak formulation of the free surface problem. This involves multiplying the governing equations with suitable test functions and integrating over our computational domain, $\Omega$. Integration by parts on the viscous term yields the surface integral

$$\int_{\Gamma_\gamma} v_i \sigma_{ij} n_j \, \mathrm{d}S, \tag{1.1}$$

where $\Gamma_\gamma = \partial\Omega_\gamma$ is the free surface, $v_i$ is a test function, and $\sigma_{ij}n_j$ are the total stress forces in the $i$'th direction; it is through this integral the imposition of the free-surface boundary conditions will be done. In [2] Ho and Patera proposed an alternative surface integral for weak imposition of both normal and tangential surface tension boundary conditions by the use of surface intrinsic coordinates. This alternative form reads,

$$-\int_{\Gamma_\gamma} \gamma \frac{\partial v_i}{\partial r^\alpha} g_i^\alpha \, \mathrm{d}S, \tag{1.2}$$

where $r^1$ and $r^2$ are surface parameters, and $\mathbf{g}^1$ and $\mathbf{g}^2$ are two vectors spanning the tangent plane which will be introduced later. To our knowledge a complete derivation of this surface integral has not been done, and the aim of this paper is to derive in detail all the necessary steps in order to go from (1.1) to (1.2). We remark that the total stress forces

$$\mathbf{F} = \mathbf{F}_n + \mathbf{F}_t = \gamma \mathbf{n} \kappa + \nabla_s \gamma,$$

where $\mathbf{F}_n$ is the normal force and $\mathbf{F}_t$ represents the tangential force. Thus, we see that in order to derive (1.2) we need to find an expression for the curvature normal product and the surface gradient in the context of surface intrinsic coordinates.

In Section 2 we will go through some basic concepts in differential geometry. This is by no means an exhaustive introduction to differential geometry, but more of a preliminary covering of the necessary tools needed in the later part of this paper. In Section 3 we cover some necessary quantities that we will need, like surface divergence, surface gradient and the mean curvature for a general surface, and in Section 4 we start from (1.1) and derive (1.2).

# 2 Differential geometry: preliminaries

The introduction to differential geometry in this section is mainly influenced by [1], both in contents and notation.

## 2.1 Cartesian coordinate system

A vector in the cartesian coordinate system, $x^{i'}$, can be written on the form,

$$\mathbf{V} = V_1 \mathbf{i}_{x^{1'}} + V_2 \mathbf{i}_{x^{2'}} + V_3 \mathbf{i}_{x^{3'}}.$$

Here, $\mathbf{i}^{i'}, i = 1, 2, 3$, are an orthonormal basis for $\mathbb{R}^d$ (Figure 2.1) such that $\mathbf{i}_{x^{m'}} \cdot \mathbf{i}_{x^{n'}} = \delta_{m'n'}$. As an example, the inner-product between two vectors written in this notation can be expressed as,

$$\mathbf{V} \cdot \mathbf{u} = V_1 u_1 + V_2 u_2 + V_3 u_3,$$

since the orthogonality makes the cross-terms zero.

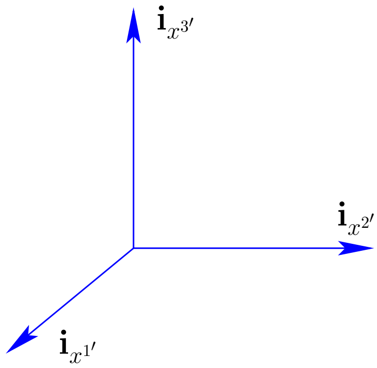



**Figure 2.1:** Cartesian coordinate system with orthonormal base vectors.

In the following, we will use $x^{i'}$ for the cartesian coordinate system and $x^i$ or $x^{i^*}$ for a general coordinate system.

## 2.2 General coordinate system

If we instead consider another basis for $\mathbb{R}^d$,

$$\mathbb{R}^d = \text{span}\{\mathbf{g}_i\}_{i=1}^d,$$

a vector in $\mathbb{R}^3$ can be written

$$\mathbf{V} = V^1 \mathbf{g}_1 + V^2 \mathbf{g}_2 + V^3 \mathbf{g}_3.$$

For this basis, we have no restriction on orthogonality between the base vectors nor that they have unit length. The only requirement is that they are linearly independent. The inner-product between two vectors in $\mathbb{R}^3$, $\mathbf{V}$ and $\mathbf{u}$, expressed in this basis reads

$$\begin{aligned}\mathbf{V}\cdot\mathbf{u} = &V^1u^1\mathbf{g}_1\cdot\mathbf{g}_1 + V^1u^2\mathbf{g}_1\cdot\mathbf{g}_2 + V^1u^3\mathbf{g}_1\cdot\mathbf{g}_3\\ &+ V^2u^1\mathbf{g}_2\cdot\mathbf{g}_1 + V^2u^2\mathbf{g}_2\cdot\mathbf{g}_2 + V^2u^3\mathbf{g}_2\cdot\mathbf{g}_3\\ &+ V^3u^1\mathbf{g}_3\cdot\mathbf{g}_1 + V^3u^2\mathbf{g}_3\cdot\mathbf{g}_2 + V^3u^3\mathbf{g}_3\cdot\mathbf{g}_3.\end{aligned}$$

We see that all nine entries are included, since the basis vectors in general are not orthogonal.

## 2.3 The base-vectors

We will consider a point, $P$, with an associated position vector, $\mathbf{p}$. A small increment, $\mathrm{d}\mathbf{p}$, may be expressed as

$$\mathrm{d}\mathbf{p} = \frac{\partial \mathbf{p}}{\partial x^i}\mathrm{d}x^i, \tag{2.1}$$

where $x^i$ $(i = 1, \ldots, d)$ denotes the reference frame, see Figure 2.2. However, if we assume

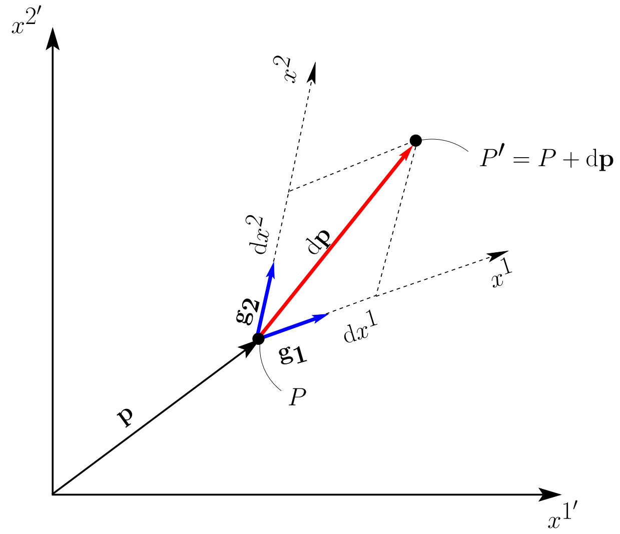



**Figure 2.2:** A differential expressed in a general coordinate system in two dimensions.

that each base-vector, $\mathbf{g}_i$, points in the same directions as $x^i$, we may choose $\{\mathbf{g}_i\}$ such that

$$\mathrm{d}\mathbf{p} = \mathbf{g}_i\mathrm{d}x^i, \tag{2.2}$$

and thus, an expression for the $i$'th base-vector is

$$\mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial x^i}. \tag{2.3}$$

We now introduce another basis for $\mathbb{R}^d$,

$$\mathbb{R}^d = \mathrm{span}\{\mathbf{g}^i\}_{i=1}^d,$$

with the property that $\mathbf{g}^m \cdot \mathbf{g}_n = \delta_{mn} = \delta^m_n$, the Kronecker delta. This is displayed in two dimensions in Figure 2.3.

A vector in $\mathbb{R}^3$ may now be expressed as

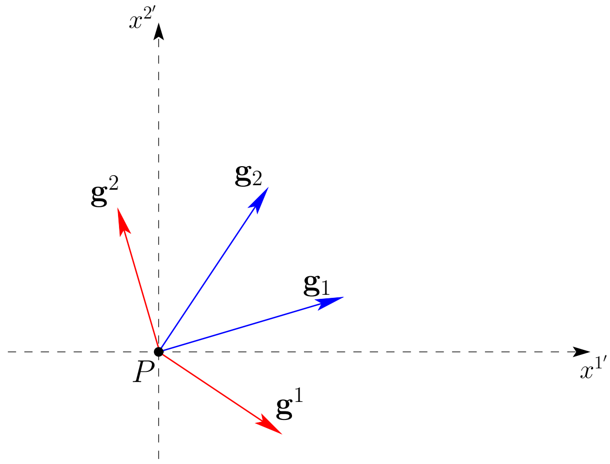



**Figure 2.3:** Covariant and contravariant basis vectors associated with the point $P$ in two dimensions.

$$\mathbf{V} = V_1\mathbf{g}^1 + V_2\mathbf{g}^2 + V_3\mathbf{g}^3.$$

If we return to the example of the inner-product of two vectors, $\mathbf{u}$ and $\mathbf{v}$, where $\mathbf{u}$ is expressed in the basis $\{\mathbf{g}_i\}$ and $\mathbf{v}$ is expressed in $\{\mathbf{g}^j\}$, we get the simplified formula,

$$\mathbf{u} \cdot \mathbf{v} = u^1 v_1 + u^2 v_2 + u^3 v_3 = u^i v_i.$$

In literature, $\mathbf{g}_i$ is often denoted as the $i$'th *covariant* base-vector, and $\mathbf{g}^j$ as the $j$'th *contravariant* base-vector.

### 2.3.1 The metric tensor

Since both the covariant and contravariant base-vectors represent a basis for $\mathbb{R}^d$, the covariant vector, $\mathbf{g}_i$, can be expressed in terms of contravariant base-vectors,

$$\mathbf{g}_i = g_{ij}\mathbf{g}^j.$$

Here, $g_{ij}$ are the components of the *covariant metric tensor*. Similarly, we have

$$\mathbf{g}^i = g^{ij}\mathbf{g}_j.$$

By the use of the orthogonality relation, we get

$$\begin{aligned} \mathbf{g}_i \cdot \mathbf{g}_j &= g_{ik}\mathbf{g}^k \cdot \mathbf{g}_j = g_{ij} = g_{ji}, \\ \mathbf{g}^i \cdot \mathbf{g}^j &= g^{ik}\mathbf{g}_k \cdot \mathbf{g}^j = g^{ij} = g^{ji}, \end{aligned}$$

and

$$\delta_i^j = \mathbf{g}_i \cdot \mathbf{g}^j = g_{ik}\mathbf{g}^k \cdot g^{jl}\mathbf{g}_l = g_{ik}g^{jl}\mathbf{g}^k \cdot \mathbf{g}_l = g_{ik}g^{jk}.$$

Thus, if the covariant components of the metric tensor is known, it is straightforward to find the contravariant components.

## 2.4 Transformation between coordinate systems

We now assume that we have two reference frames, $\{x_i\}_{i=1}^d$ and $\{x_i^*\}_{i=1}^d$, with two corresponding sets of covariant base-vectors, $\mathbf{g}_i$ and $\mathbf{g}_{i^*}$ (see Figure 2.4). We also have contravariant vectors which satisfy the relations

$$\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j,$$
$$\mathbf{g}_{i^*} \cdot \mathbf{g}^{j^*} = \delta_{i^*}^{j^*}.$$

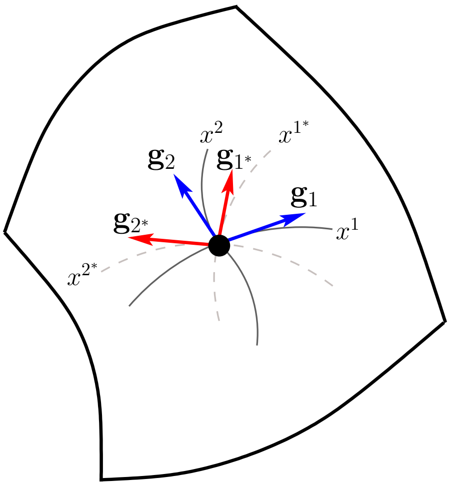



**Figure 2.4:** Two sets of base-vectors in two dimensions.

These four sets of base-vectors span the same space, and can therefore be expressed in terms of each other. Thus, we can write,

$$\mathbf{g}_{i^*} = \beta_{i^*}^j \mathbf{g}_j,$$
$$\mathbf{g}^{i^*} = \beta_j^{i^*} \mathbf{g}^j.$$

Notice that the summation is over the superscript of $\beta_i^j$ for the first case, and the subscript for the second case. Further,

$$\delta_{i^*}^{k^*} = \mathbf{g}_{i^*} \cdot \mathbf{g}^{k^*} = \beta_{i^*}^j \mathbf{g}_j \cdot \beta_l^{k^*} \mathbf{g}^l = \beta_{i^*}^j \beta_l^{k^*} \mathbf{g}_j \cdot \mathbf{g}^l = \beta_{i^*}^j \beta_l^{k^*} \delta_j^l$$
$$\Downarrow$$
$$\beta_{i^*}^j \beta_j^{k^*} = \delta_{i^*}^{k^*}, \tag{2.4}$$

and in the same manner we have

$$\beta_i^{j^*} \beta_{j^*}^k = \delta_i^k. \tag{2.5}$$

### 2.4.1 The components, $\beta_i^{j^*}$

As before, we may write a small vector, $\mathrm{d}\mathbf{s}$, in terms of the covariant base-vectors,

$$\mathrm{d}\mathbf{s} = \mathbf{g}_i \,\mathrm{d}x^i = \beta_i^{k^*} \mathbf{g}_{k^*} \,\mathrm{d}x^i,$$

by the use of two reference frames, $\{x_i\}_{i=1}^d$ and $\{x_i^*\}_{i=1}^d$. The same vector may also be expressed as

$$\mathrm{d}\mathbf{s} = \mathbf{g}_{k^*} \,\mathrm{d}x^{k^*} = \mathbf{g}_{k^*} \frac{\partial x^{k^*}}{\partial x^i} \,\mathrm{d}x^i.$$

Thus,

$$\beta_i^{k^*} \mathbf{g}_{k^*} \,\mathrm{d}x^i = \mathbf{g}_{k^*} \frac{\partial x^{k^*}}{\partial x^i} \,\mathrm{d}x^i$$

should be valid for all $\mathrm{d}\mathbf{s}$. By systematically choosing,

$$\begin{aligned}
\mathrm{d}x^1 &= 1, \quad \mathrm{d}x^2 = 0, \quad \mathrm{d}x^3 = 0, \\
\mathrm{d}x^1 &= 0, \quad \mathrm{d}x^2 = 1, \quad \mathrm{d}x^3 = 0, \\
\mathrm{d}x^1 &= 0, \quad \mathrm{d}x^2 = 0, \quad \mathrm{d}x^3 = 1,
\end{aligned}$$

we find that

$$\beta_i^{k^*} \mathbf{g}_{k^*} = \mathbf{g}_{k^*} \frac{\partial x^{k^*}}{\partial x^i}.$$

Taking the inner-product of both sides with $\mathbf{g}^{j^*}$, we get

$$\beta_i^{k^*} \mathbf{g}_{k^*} \cdot \mathbf{g}^{j^*} = \mathbf{g}_{k^*} \cdot \mathbf{g}^{j^*} \frac{\partial x^{k^*}}{\partial x^i},$$

$$\Downarrow$$

$$\beta_i^{j^*} = \frac{\partial x^{j^*}}{\partial x^i}.$$

## 2.5 The permutation tensor

We wish to be able to evaluate cross-products in general coordinates. This is closely related to a tensor often called the *permutation tensor*, and in order to derive this tensor we will first introduce the *permutation symbols*.

### 2.5.1 The permutation symbols

The permutation symbols, $e_{ijk} = e^{ijk}$, are defined by

$e_{ijk} = +1$, if $i, j, k$ is a cyclic sequence $(1, 2, 3;\ 2, 3, 1$ or $3, 1, 2$ for the $3 \times 3$ case.)

$e_{ijk} = -1$, if $i, j, k$ is a anticyclic sequence $(3, 2, 1;\ 2, 1, 3$ or $1, 3, 2$ for the $3 \times 3$ case.)

$e_{ijk} = 0$, if $i, j, k$ is acyclic (two or more subscripts are equal.)

### 2.5.2 The determinant

We define $a^2$ to be the determinant of a matrix $a_i^j$:

$$a^2 = |a_i^j| = \begin{vmatrix} a_1^1 & a_2^1 & a_3^1 \\ a_1^2 & a_2^2 & a_3^2 \\ a_1^3 & a_2^3 & a_3^3 \end{vmatrix}.$$

By the use of the permutation symbols, this can also be written as

$$a^2 = a_1^i a_2^j a_3^k e_{ijk},$$

or

$$a^2 = a_l^1 a_m^2 a_n^3 e^{lmn}.$$

### 2.5.3 The permutation tensor

We now define the permutation tensor connected to the cartesian coordinate system, $x^{i'}$, as

$$\varepsilon_{i'j'k'} = e_{i'j'k'}. \tag{2.6}$$

If we wish to transform this tensor to another coordinate system, $x^i$, we may do this in the same way as introduced earlier,

$$\varepsilon_{ijk} = \varepsilon_{i'j'k'} \beta_i^{i'} \beta_j^{j'} \beta_k^{k'} = e_{i'j'k'} \beta_i^{i'} \beta_j^{j'} \beta_k^{k'}.$$

However, we recognize this as $|\beta_i^{i'}|$ if $\{i, j, k\}$ is a cyclic sequence and $-|\beta_i^{i'}|$ if $\{i, j, k\}$ is an anticyclic sequence,

$$\begin{aligned} \varepsilon_{ijk} &= |\beta_i^{i'}|, \qquad \text{if } i, j, k \text{ is cyclic,} \\ \varepsilon_{ijk} &= -|\beta_i^{i'}|, \qquad \text{if } i, j, k \text{ is anticyclic.} \end{aligned} \tag{2.7}$$

From (2.4), we know that

$$\beta_{i'}^j \beta_j^{k'} = \delta_{i'}^{k'},$$

thus,

$$|\beta_{i'}^j \beta_j^{k'}| = 1,$$

$$\Downarrow$$

$$|\beta_{i'}^j| = \Delta, \qquad |\beta_j^{i'}| = \frac{1}{\Delta},$$

where $\Delta$ is currently unknown. We define $g^2$ to be the determinant of the metric tensor,

$$g^2 = |g_{ij}| = \begin{vmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{vmatrix}.$$

Since,

$$g_{ij}g^{jk} = \delta_i^k,$$

we have

$$|g_{ij}g^{jk}| = 1,$$
$$\Downarrow$$
$$|g_{ij}||g^{jk}| = 1,$$
$$\Downarrow$$
$$|g^{jk}| = \frac{1}{g^2}.$$

We transform the metric tensor to the cartesian reference frame,

$$g_{i'j'} = g_{ij}\beta^i_{i'}\beta^j_{j'},$$

which leads to

$$(g')^2 = |g_{i'j'}| = |g_{ij}\beta^i_{i'}||\beta^j_{j'}| = |g_{ij}||\beta^i_{i'}||\beta^j_{j'}| = g^2\Delta\Delta. \tag{2.8}$$

For the cartesian coordinate system, we know that

$$g' = 1,$$

since the metric tensor is simply the identity matrix, and this leads to

$$\frac{1}{\Delta} = g.$$

Thus, we find that the tensor $|\beta^{i'}_i|$, which is used for expressing a base vector in a general coordinate system in terms of base vectors in the cartesian coordinate system, has the determinant $g$. We arrive at,

$$\begin{aligned} \varepsilon_{ijk} &= |\beta^{i'}_i| = g, && \text{if } i,j,k \text{ is cyclic,} \\ \varepsilon_{ijk} &= -|\beta^{i'}_i| = -g, && \text{if } i,j,k \text{ is anticyclic,} \\ \varepsilon_{ijk} &= 0, && \text{if } i,j,k \text{ is acyclic.} \end{aligned} \tag{2.9}$$

## 2.6 Representation of a three-dimensional surface

We will use surface intrinsic coordinates for describing three-dimensional surfaces. Such surfaces will in general be described by two surface parameters and written on the form $x^{i'} = f_i(r^1, r^2)$, $i = 1, 2, 3$. We are free to choose the way we describe the surface in terms of $r^1$ and $r^2$, but it is natural to choose $r^1$ and $r^2$ as variables on a pre-determined reference domain, which is connected to the physical surface through a one-to-one mapping. The covariant base-vectors are then given by

$$\mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial r^i}.$$

Note that, independent of the choice of $r^1$ and $r^2$, the covariant base-vector, $\mathbf{g}_i$, will at each point on the surface point in the mapped direction of $r^i$ while $\mathbf{g}^i$ will generally have a component in both the $r^1$ and $r^2$ direction.

In the following we will use the second approach, where $r^1$ and $r^2$ are variables on a reference domain, $\hat{\Gamma}$ (see Figure 2.5.) Here, we see that a line on the reference domain in which $r^1$ varies and $r^2$ is constant corresponds to a curve on the physical surface, $\Gamma$, through the mapping $\mathcal{F}$. The vector $\mathbf{g}_1$ will be a tangent to this curve at all grid points, and similarly $\mathbf{g}_2$ will be the tangent vector along a curve on $\Gamma$ corresponding to constant $r^1$ and varying $r^2$ on the reference domain. Thus, $\{\mathbf{g}_1, \mathbf{g}_2\}$ and $\{\mathbf{g}^1, \mathbf{g}^2\}$ will both span the tangent plane at all points on the surface. For the third direction, we will in the following choose $\mathbf{g}_3 = \mathbf{g}^3 = \mathbf{n}$, such that

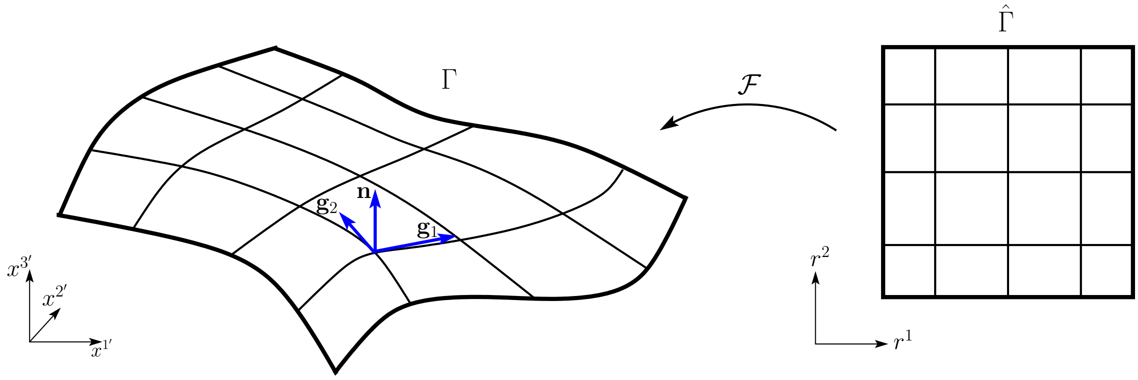



**Figure 2.5:** Mapping between reference domain and physical domain for a surface in three dimensions.

the third base vector is orthogonal to the tangent vectors as well as normalized,

$$\begin{aligned}
\mathbf{g}_3 \cdot \mathbf{g}_i &= 0, \qquad i = 1, 2, \\
\sqrt{\mathbf{g}_3 \cdot \mathbf{g}_3} &= 1, \\
&\Downarrow \\
\mathbf{g}_3 &= \mathbf{n}.
\end{aligned}$$

Note that for this choice of $\mathbf{g}_3$ we get,

$$g^2 = \begin{vmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix}.$$

## 2.7 Cross product

### 2.7.1 Cross product between base vectors

If we choose $\mathbf{g}_{i'}$ as the orthonormal basis in the cartesian coordinate system,

$$\mathbf{g}_{i'} = \mathbf{i}_{x^{i'}}, \qquad i = 1, 2, 3,$$

we know that for this case

$$\mathbf{g}_{i'} \times \mathbf{g}_{j'} = \varepsilon_{i'j'k'}\mathbf{g}^{k'} \tag{2.10}$$

is valid. Here $\varepsilon_{i'j'k'}$ is the permutation tensor associated with the cartesian coordinate system, defined in (2.6) and $\mathbf{g}^{3'} = \mathbf{g}_{3'} = \mathbf{i}_{x^{3'}}$. For general coordinate directions, we may now write

$$\begin{aligned}\mathbf{g}_i \times \mathbf{g}_j &= (\beta_i^{i'}\mathbf{g}_{i'}) \times (\beta_j^{j'}\mathbf{g}_{j'}) \\ &= \beta_i^{i'}\beta_j^{j'}(\mathbf{g}_{i'} \times \mathbf{g}_{j'}),\end{aligned}$$

which by the use of (2.10) can be written as

$$\begin{aligned}&= \beta_i^{i'}\beta_j^{j'}\varepsilon_{i'j'k'}\mathbf{g}^{k'} \\ &= \beta_i^{i'}\beta_j^{j'}\beta_k^{k'}\varepsilon_{i'j'k'}\mathbf{g}^{k} \\ &\Downarrow \\ \mathbf{g}_i \times \mathbf{g}_j &= \varepsilon_{ijk}\mathbf{g}^k.\end{aligned} \tag{2.11}$$

### 2.7.2 Area-element on the surface

We now denote $s^1$ and $s^2$ to be surface coordinates pointing in the same directions as $r^1$ and $r^2$, respectively, such that $s^1 = s^1(r^1)$ and $s^2 = s^2(r^2)$, see Figure 2.6. We are interested in expressing an area-element, $\mathrm{d}S$, on a general three-dimensional surface in terms of the reference variables, $r^1$ and $r^2$.

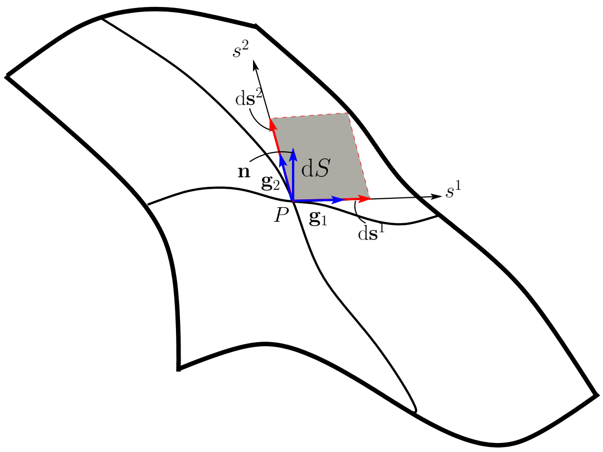



**Figure 2.6:** A surface element.

We assign $\mathrm{d}\mathbf{s}^1$ to be a vector originating from $P$ associated with a small increment $\mathrm{d}r^1$ on the reference domain, and similarily $\mathrm{d}\mathbf{s}^2$ is the vector associated with the increment $\mathrm{d}r^2$. Thus, $\mathrm{d}\mathbf{s}^1$ points in the direction of $\mathbf{g}_1$ and $\mathrm{d}\mathbf{s}^2$ points in the direction of $\mathbf{g}_2$ and we have the relation

$$\begin{aligned} \mathrm{d}\mathbf{s}^1 \times \mathrm{d}\mathbf{s}^2 &= \mathrm{d}S\mathbf{n}, \\ &\Downarrow \\ \mathrm{d}S &= |\,\mathrm{d}\mathbf{s}^1 \times \mathrm{d}\mathbf{s}^2|. \end{aligned} \tag{2.12}$$

We also have

$$\begin{aligned} \mathrm{d}\mathbf{s}^1 &= \frac{\partial \mathbf{s}^1}{\partial r^1}\,\mathrm{d}r^1 = \frac{\partial \mathbf{p}}{\partial r^1}\,\mathrm{d}r^1 = \mathbf{g}_1\,\mathrm{d}r^1, \\ \mathrm{d}\mathbf{s}^2 &= \frac{\partial \mathbf{s}^2}{\partial r^2}\,\mathrm{d}r^2 = \frac{\partial \mathbf{p}}{\partial r^2}\,\mathrm{d}r^2 = \mathbf{g}_2\,\mathrm{d}r^2. \end{aligned}$$

Inserted into (2.12), we get

$$\begin{aligned} \mathrm{d}S &= |(\mathbf{g}_1\,\mathrm{d}r^1 \times \mathbf{g}_2\,\mathrm{d}r^2)|, \\ &= |\mathbf{g}_1 \times \mathbf{g}_2|\,\mathrm{d}r^1\,\mathrm{d}r^2. \end{aligned}$$

From (2.11) and our choice of $\mathbf{g}^3 = \mathbf{n}$, we get

$$\begin{aligned} |\mathbf{g}_1 \times \mathbf{g}_2| &= |\varepsilon_{123}\mathbf{n}|, \\ &= |\varepsilon_{123}||\mathbf{n}|, \\ &= \varepsilon_{123}, \\ &\Downarrow \\ \mathrm{d}S &= g\,\mathrm{d}r^1\,\mathrm{d}r^2, \end{aligned} \tag{2.13}$$

by the use of (2.9) and the fact that $g$ is always positive.

### 2.7.3 Cross-product between two general vectors

The cross product between two general vectors expressed in a covariant basis

$$\begin{aligned} \mathbf{a} &= a^i\mathbf{g}_i, \\ \mathbf{b} &= b^j\mathbf{g}_j, \end{aligned}$$

becomes

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= a^i\mathbf{g}_i \times b^j\mathbf{g}_j, \\ &= a^ib^j\mathbf{g}_i \times \mathbf{g}_j, \\ &= a^ib^j\varepsilon_{ijk}\mathbf{g}^k. \end{aligned} \tag{2.14}$$

### 2.7.4 Normal vector of a line element

We consider a line element vector, $\mathrm{d}\mathbf{s}$, associated with a curve, $C$, on an arbitrary curved surface, $S$. In Figure 2.7 this curve is the boundary of the surface, but in general, $C$ could be anywhere on $S$. We will consider a point $P$ on $C$ where $\mathrm{d}\mathbf{s}$ is a tangent vector, $\mathbf{g}^3 = \mathbf{n}$ is the normal to $S$ at $P$ and $\mathrm{d}\mathbf{n}$ is an outer normal to $C$. We express $\mathrm{d}\mathbf{s}$ in terms of covariant base-vectors and $\mathrm{d}\mathbf{n}$ in terms of contravariant base-vectors,

$$\begin{aligned} \mathrm{d}\mathbf{s} &= \mathrm{d}r^\alpha \mathbf{g}_\alpha, \\ \mathrm{d}\mathbf{n} &= \mathrm{d}n_\beta \mathbf{g}^\beta. \end{aligned} \tag{2.15}$$

Then, the outer normal to $C$ at $P$ may be written

$$\begin{aligned} \mathrm{d}\mathbf{n} &= \mathrm{d}\mathbf{s} \times \mathbf{g}^3 \\ &= \mathrm{d}r^\alpha (\mathbf{g}_\alpha \times \mathbf{g}_3) \\ &= \mathrm{d}r^\alpha \varepsilon_{\alpha 3 \beta} \mathbf{g}^\beta. \end{aligned} \tag{2.16}$$

Comparing (2.15) and (2.16) leads to

$$\begin{aligned} \mathrm{d}n_\beta &= \mathrm{d}r^\alpha \varepsilon_{\alpha 3 \beta}, \\ \mathrm{d}n_\alpha &= \varepsilon_{\beta 3 \alpha}\, \mathrm{d}r^\beta, \\ &\Downarrow \\ &= \tilde{\varepsilon}_{\alpha\beta}\, \mathrm{d}r^\beta, \end{aligned} \tag{2.17}$$

where $\tilde{\varepsilon}_{12} = g$, $\tilde{\varepsilon}_{21} = -g$ and $\tilde{\varepsilon}_{11} = \tilde{\varepsilon}_{22} = 0$. Since $\mathbf{g}^3$ is of unit length, we observe that $|\,\mathrm{d}\mathbf{n}| = |\,\mathrm{d}\mathbf{s}|$.

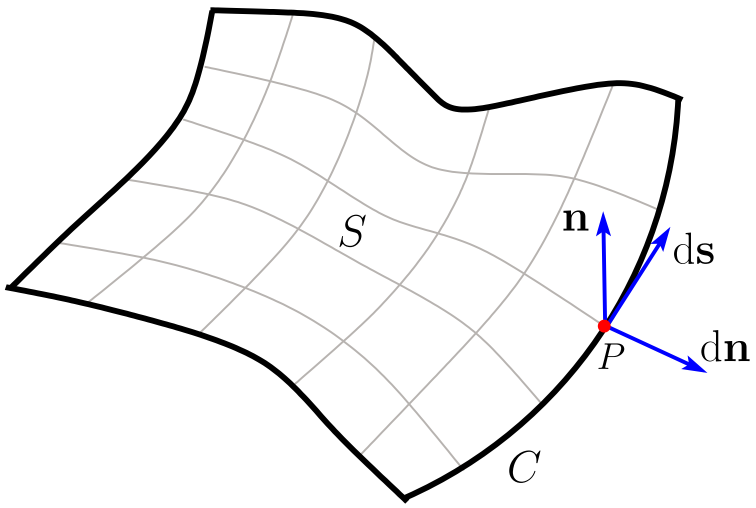



**Figure 2.7:** Outward normal to the point $P$ on the curve $C$.

## 2.8 Summary

In summary, we have some results and definitions which will be used in the following sections:

$$\mathbf{g}_\alpha = \frac{\partial \mathbf{p}}{\partial r^\alpha}, \tag{2.18}$$

$$\mathbf{g}_\alpha \cdot \mathbf{g}_\beta = g_{\alpha\beta}, \tag{2.19}$$

$$g_{\alpha\gamma} g^{\gamma\beta} = \delta_\alpha^\beta, \tag{2.20}$$

$$\mathbf{g}^\alpha = g^{\alpha\beta} \mathbf{g}_\beta, \tag{2.21}$$

$$g = \sqrt{[\det(g_{\alpha\beta})]}, \tag{2.22}$$

$$b_{\alpha\beta} = \mathbf{n} \cdot \frac{\partial \mathbf{g}_\alpha}{\partial r^\beta} = -\mathbf{g}_\alpha \cdot \frac{\partial \mathbf{n}}{\partial r^\beta} \tag{2.23}$$

The relation between the contravariant and covariant metric tensor is given by

$$\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} = \frac{1}{g^2} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{g^2} \begin{bmatrix} g_{22} & -g_{12} \\ -g_{21} & g_{11} \end{bmatrix}$$

$$\Downarrow$$

$$g^{11} = \frac{1}{g^2} g_{22} \tag{2.24}$$

$$g^{12} = -\frac{1}{g^2} g_{12} \tag{2.25}$$

$$g^{21} = g^{12} \tag{2.26}$$

$$g^{22} = \frac{1}{g^2} g_{11} \tag{2.27}$$

# 3 Mean curvature and related operators

The main objective in this section is to find an expression for the curvature-normal product. In order to do this we will derive expressions for some important quantities like gradient, divergence and principal directions related to the mean curvature in the context of a general coordinate system. The contents of this section is mainly influenced by [4] and [3], while the notation is mostly the same as used in [1] and [3].

## 3.1 Surface divergence of a vector in general coordinates

If we have a vector, $\mathbf{F} = \mathbf{F}(r^1, r^2)$, expressed in general coordinates, the divergence at a given point is given by [4]

$$\nabla_s \cdot \mathbf{F} = \frac{1}{g^2}\mathbf{g}_1 \cdot \left( g_{22}\frac{\partial \mathbf{F}}{\partial r^1} - g_{12}\frac{\partial \mathbf{F}}{\partial r^2} \right) + \frac{1}{g^2}\mathbf{g}_2 \cdot \left( g_{11}\frac{\partial \mathbf{F}}{\partial r^2} - g_{12}\frac{\partial \mathbf{F}}{\partial r^1} \right). \tag{3.1}$$

We now wish to find another expression for $\nabla_s \cdot \mathbf{F}$. Expressing $\mathbf{F}$ in covariant base-vectors, we may write $\mathbf{F} = a\mathbf{g}_1 + b\mathbf{g}_2 + c\mathbf{n}$. Thus, the divergence may be written as

$$\nabla_s \cdot \mathbf{F} = \nabla_s \cdot (a\mathbf{g}_1) + \nabla_s \cdot (b\mathbf{g}_2) + \nabla_s \cdot (c\mathbf{n}).$$

By the help of (3.1), we get

$$\begin{aligned}
\nabla_s \cdot (a\mathbf{g}_1) &= \frac{1}{g^2}\mathbf{g}_1 \cdot \left[ g_{22}\left( \frac{\partial a}{\partial r^1}\mathbf{g}_1 + a\frac{\partial \mathbf{g}_1}{\partial r^1} \right) - g_{12}\left( \frac{\partial a}{\partial r^2}\mathbf{g}_1 + a\frac{\partial \mathbf{g}_1}{\partial r^2} \right) \right] \\
&+ \frac{1}{g^2}\mathbf{g}_2 \cdot \left[ g_{11}\left( \frac{\partial a}{\partial r^2}\mathbf{g}_1 + a\frac{\partial \mathbf{g}_1}{\partial r^2} \right) - g_{12}\left( \frac{\partial a}{\partial r^1}\mathbf{g}_1 + a\frac{\partial \mathbf{g}_1}{\partial r^1} \right) \right] \\
&= \frac{1}{g^2}\left[ g_{22}\frac{\partial a}{\partial r^1}\left( \mathbf{g}_1 \cdot \mathbf{g}_1 \right) + g_{22}a\left( \mathbf{g}_1 \cdot \frac{\partial \mathbf{g}_1}{\partial r^1} \right) - g_{12}\frac{\partial a}{\partial r^2}\left( \mathbf{g}_1 \cdot \mathbf{g}_1 \right) - g_{12}a\left( \mathbf{g}_1 \cdot \frac{\partial \mathbf{g}_1}{\partial r^2} \right) \right] \\
&+ \frac{1}{g^2}\left[ g_{11}\frac{\partial a}{\partial r^2}\left( \mathbf{g}_2 \cdot \mathbf{g}_1 \right) + g_{11}a\left( \mathbf{g}_2 \cdot \frac{\partial \mathbf{g}_1}{\partial r^2} \right) - g_{12}\frac{\partial a}{\partial r^1}\left( \mathbf{g}_1 \cdot \mathbf{g}_2 \right) - g_{12}a\left( \mathbf{g}_2 \cdot \frac{\partial \mathbf{g}_1}{\partial r^1} \right) \right]
\end{aligned}$$

To proceed, we need the following quantities (note that $\frac{\partial \mathbf{g}_1}{\partial r^2} = \frac{\partial \mathbf{g}_2}{\partial r^1}$),

$$\begin{aligned}
\mathbf{g}_1 \cdot \frac{\partial \mathbf{g}_1}{\partial r^1} &= \frac{1}{2}\frac{\partial(\mathbf{g}_1 \cdot \mathbf{g}_1)}{\partial r^1} &&= \frac{1}{2}\frac{\partial g_{11}}{\partial r^1}, \\
\mathbf{g}_1 \cdot \frac{\partial \mathbf{g}_1}{\partial r^2} &= \frac{1}{2}\frac{\partial(\mathbf{g}_1 \cdot \mathbf{g}_1)}{\partial r^2} &&= \frac{1}{2}\frac{\partial g_{11}}{\partial r^2}, \\
\mathbf{g}_2 \cdot \frac{\partial \mathbf{g}_1}{\partial r^2} &= \mathbf{g}_2 \cdot \frac{\partial \mathbf{g}_2}{\partial r^1} = \frac{1}{2}\frac{\partial(\mathbf{g}_2 \cdot \mathbf{g}_2)}{\partial r^1} &&= \frac{1}{2}\frac{\partial g_{22}}{\partial r^1}, \\
\mathbf{g}_2 \cdot \frac{\partial \mathbf{g}_1}{\partial r^1} &= \frac{\partial(\mathbf{g}_1 \cdot \mathbf{g}_2)}{\partial r^1} - \mathbf{g}_1 \cdot \frac{\partial \mathbf{g}_2}{\partial r^1} &&= \frac{\partial g_{12}}{\partial r^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial r^2}.
\end{aligned}$$

Thus,

$$
\begin{aligned}
\nabla_s \cdot (a\mathbf{g}_1) &= \frac{1}{g^2}\left[g_{22}\frac{\partial a}{\partial r^1}g_{11} + g_{22}a\left(\frac{1}{2}\frac{\partial g_{11}}{\partial r^1}\right) - g_{12}\frac{\partial a}{\partial r^2}g_{11} - g_{12}a\left(\frac{1}{2}\frac{\partial g_{11}}{\partial r^2}\right)\right] \\
&+ \frac{1}{g^2}\left[g_{11}\frac{\partial a}{\partial r^2}g_{12} + g_{11}a\left(\frac{1}{2}\frac{\partial g_{22}}{\partial r^1}\right) - g_{12}\frac{\partial a}{\partial r^1}g_{12} - g_{12}a\left(\frac{\partial g_{12}}{\partial r^1} - \frac{1}{2}\frac{\partial g_{11}}{\partial r^2}\right)\right] \\
&= \frac{1}{g^2}\overbrace{(g_{11}g_{22} - g_{12}^2)}^{g^2}\frac{\partial a}{\partial r^1} + \frac{a}{2g^2}\left(g_{11}\frac{\partial g_{22}}{\partial r^1} + g_{22}\frac{\partial g_{11}}{\partial r^1} - 2g_{12}\frac{\partial g_{12}}{\partial r^1}\right), \\
&= \frac{\partial a}{\partial r^1} + \frac{a}{2g^2}\left(g_{11}\frac{\partial g_{22}}{\partial r^1} + g_{22}\frac{\partial g_{11}}{\partial r^1} - 2g_{12}\frac{\partial g_{12}}{\partial r^1}\right), \\
&= \frac{\partial a}{\partial r^1} + \frac{a}{2g^2}\frac{\partial}{\partial r^1}(g_{11}g_{22} - g_{12}^2), \\
&= \frac{1}{g}\frac{\partial (ga)}{\partial r^1},
\end{aligned}
$$

where $g = \sqrt{g_{11}g_{22} - g_{12}^2}$. Similarly, we have

$$
\nabla_s \cdot (b\mathbf{g}_2) = \frac{1}{g}\frac{\partial (gb)}{\partial r^2},
$$

and we will later show that

$$
\nabla_s \cdot (c\mathbf{n}) = -\kappa c,
$$

where $\kappa$ is twice the mean curvature. This gives us another formula for the divergence of a vector, $\mathbf{F} = a\mathbf{g}_1 + b\mathbf{g}_2 + c\mathbf{n}$,

$$
\nabla_s \cdot \mathbf{F} = \frac{1}{g}\left[\frac{\partial (ga)}{\partial r^1} + \frac{\partial (gb)}{\partial r^2}\right] - \kappa c. \tag{3.2}
$$

### 3.1.1 Example

As a simple example we will consider the surface of a sphere with radius $r = 1$.

We will choose basis vectors

$$
\begin{aligned}
\mathbf{g}_1 &= -\sin\theta \mathbf{i}_{x^{1'}} + \cos\theta \mathbf{i}_{x^{2'}} + 0\mathbf{i}_{x^{3'}}, \\
\mathbf{g}_2 &= \cos\phi\cos\theta \mathbf{i}_{x^{1'}} + \cos\phi\sin\theta \mathbf{i}_{x^{2'}} - \sin\phi \mathbf{i}_{x^{3'}},
\end{aligned}
$$

where $\{\mathbf{i}_{x^{i'}}\}_{i=1}^3$ as usual are the standard basis vectors in the cartesian coordinate system. We observe that

$$
\begin{aligned}
g_{12} &= \mathbf{g}_1 \cdot \mathbf{g}_2 = 0, \\
g_{11} &= \mathbf{g}_1 \cdot \mathbf{g}_1 = 1, \\
g_{22} &= \mathbf{g}_2 \cdot \mathbf{g}_2 = 1, \\
&\Downarrow \\
g &= 1.
\end{aligned} \tag{3.3}
$$

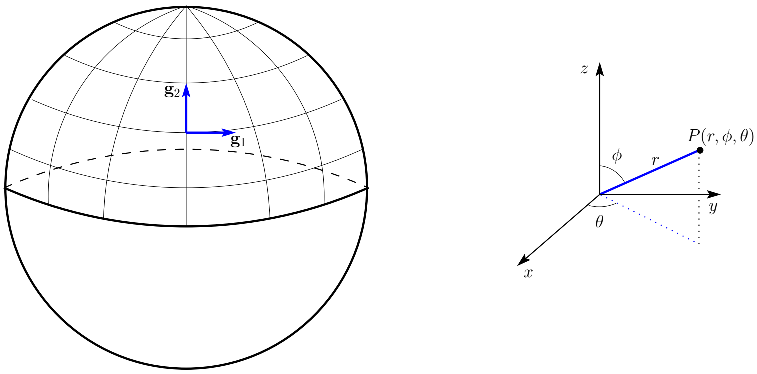


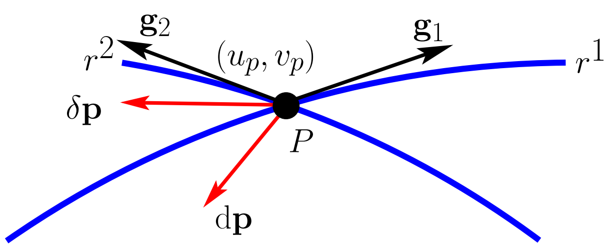

**Figure 3.1:** Sphere with orthonormal base-vectors.

Inserted into (3.2), we find that the surface divergence to a vector $\mathbf{F} = a\mathbf{g}_1 + b\mathbf{g}_2$ for this case is given by

$$\nabla_s \cdot \mathbf{F} = \frac{\partial a}{\partial r^1} + \frac{\partial b}{\partial r^2},$$

which is what we would expect.

## 3.2 Surface gradient of a scalar field




We now consider a scalar function, $\phi(r^1, r^2)$, on a surface, $S$, parameterized by the reference variables $r^1$ and $r^2$. We assume that $\phi(r^1, r^2) = C$ is a *level curve* on the surface and that $P$ is a point on this curve. If $(\delta r^1, \delta r^2)$ is a small displacement from $P$ such that

$$\delta\mathbf{p} = \mathbf{g}_1 \delta r^1 + \mathbf{g}_2 \delta r^2 \tag{3.4}$$

is a tangent to this curve, then

$$\phi_{,1} \delta r^1 + \phi_{,2} \delta r^2 = 0, \tag{3.5}$$

where $\phi_{,1} = \frac{\partial \phi}{\partial r^1}$ and $\phi_{,2} = \frac{\partial \phi}{\partial r^2}$.

We consider another displacement $(\mathrm{d}r^1, \mathrm{d}r^2)$, with associated displacement vector,

$$\mathrm{d}\mathbf{p} = \mathbf{g}_1\, \mathrm{d}r^1 + \mathbf{g}_2\, \mathrm{d}r^2, \tag{3.6}$$

and find that the inner-product between $\delta\mathbf{p}$ and $\mathrm{d}\mathbf{p}$ is given by

$$
\begin{aligned}
\mathrm{d}\mathbf{p}\cdot\delta\mathbf{p} &= \Big(\mathbf{g}_1\,\mathrm{d}r^1 + \mathbf{g}_2\,\mathrm{d}r^2\Big)\cdot\Big(\mathbf{g}_1\delta r^1 + \mathbf{g}_2\delta r^2\Big)\\
&= g_{11}\,\mathrm{d}r^1\delta r^1 + g_{12}(\,\mathrm{d}r^1\delta r^2 + \,\mathrm{d}r^2\delta r^1) + g_{22}\,\mathrm{d}r^2\delta r^2.
\end{aligned}
$$

We now assume that $\mathrm{d}\mathbf{p}$ and $\delta\mathbf{p}$ are perpendicular, which leads to the relation

$$
g_{11}\frac{\mathrm{d}r^1}{\mathrm{d}r^2}\frac{\delta r^1}{\delta r^2} + g_{12}\left(\frac{\mathrm{d}r^1}{\mathrm{d}r^2} + \frac{\delta r^1}{\delta r^2}\right) + g_{22} = 0. \tag{3.7}
$$

From (3.5) we see that

$$
\frac{\delta r^1}{\delta r^2} = -\frac{\phi_{,2}}{\phi_{,1}}.
$$

We insert this into (3.7) and find that

$$
\frac{\mathrm{d}r^1}{\mathrm{d}r^2} = \frac{g_{22}\phi_{,1} - g_{12}\phi_{,2}}{g_{11}\phi_{,2} - g_{12}\phi_{,1}}. \tag{3.8}
$$

The displacement vector, $\delta\mathbf{p}$, is a tangent to the curve $\phi(r^1, r^2) = C$, and we know that the surface gradient points in a normal direction to this curve along the surface. Thus, from (3.8) we see that the vector

$$
\mathbf{V} = k(g_{22}\phi_{,1} - g_{12}\phi_{,2})\mathbf{g}_1 + k(g_{11}\phi_{,2} - g_{12}\phi_{,1})\mathbf{g}_2
$$

is parallel to $\nabla_s\phi$. In order to determine $k$, we require that

$$
\mathbf{V}\cdot\frac{\mathbf{g}_1}{\sqrt{g_{11}}} = \frac{\partial\phi}{\partial r^1}\frac{\partial r^1}{\partial s^1} = \frac{1}{\sqrt{g_{11}}}\phi_{,1},
$$

where $s_1$ is an arc-length coordinate which runs in the same "direction" as the reference variable, $r^1$. We find that $k = \frac{1}{g^2}$, so the gradient of the scalar function $\phi(r^1, r^2)$ may be written

$$
\begin{aligned}
\nabla_s\phi &= \frac{(g_{22}\phi_{,1} - g_{12}\phi_{,2})}{g^2}\mathbf{g}_1 + \frac{(g_{11}\phi_{,2} - g_{12}\phi_{,1})}{g^2}\mathbf{g}_2,\\
&= (g^{11}\phi_{,1} + g^{12}\phi_{,2})\mathbf{g}_1 + (g^{12}\phi_{,1} + g^{22}\phi_{,2})\mathbf{g}_2, && (3.9)\\
&= \phi_{,\alpha}\mathbf{g}^\alpha, && (3.10)
\end{aligned}
$$

where we have used (2.21) in the last step, and $\alpha = 1, 2$.

### 3.2.1 Example

By using the same example as for the surface divergence, (see Figure 3.1), where the geometric factors are given by (3.3), we get the simplified formula

$$
\nabla_s\phi = \frac{\partial\phi}{\partial r^1}\mathbf{g}_1 + \frac{\partial\phi}{\partial r^2}\mathbf{g}_2,
$$

which again seems reasonable.

## 3.3 Curvature of a curve

We will consider a curve, $C$, spanning three dimensions given by

$$\mathbf{p}(s) = \begin{bmatrix} x^{1'}(s) \\ x^{2'}(s) \\ x^{3'}(s) \end{bmatrix},$$

where $x^{i'}$ is the $i$'th cartesian coordinate and $s$ is an arc-length variable along the curve. Such a curve is depicted in Figure 3.2. Here, we have used

- **Normal plane**, $N$

  The plane spanned by all vectors normal to the unit tangent vector $\mathbf{t}(s) = \dot{\mathbf{p}} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s}$ at the point $P$.

- **Osculating plane**, $O$

  The plane spanned by $\mathbf{t} = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}s}$ and $\dot{\mathbf{t}} = \frac{\mathrm{d}^2\mathbf{p}}{\mathrm{d}s^2}$.

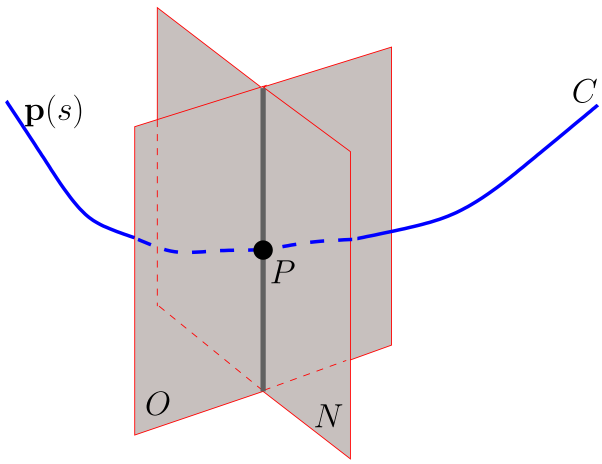



**Figure 3.2:** A curve in three dimensions. $N$ is the normal plane and $O$ is the osculating plane.

We observe that the vector

$$\mathbf{n}_c = \frac{\dot{\mathbf{t}}(s)}{|\dot{\mathbf{t}}(s)|}$$

is a unit normal to $C$. This vector lies in both the normal plane, $N$, and in the osculating plane, $O$, and therefore points in the direction of the line of intersection between $N$ and $O$. The curvature of the curve $C$ at the point $P(s)$ is given by

$$\kappa_c = |\dot{\mathbf{t}}(s)|,$$
$$\Downarrow$$
$$\kappa_c \mathbf{n}_c = \ddot{\mathbf{p}}(s). \tag{3.11}$$

Figure 3.3 shows a plot of the curve $C$ projected to the osculating plane at the point $P$. The point $M$ at a distance $\rho = \frac{1}{\kappa_c}$ from $P$ in the direction of $\mathbf{n}_c$ is called the *centre of curvature*.

The circle in the osculating plane with centre $M$ and radius $\rho$ is called the *circle of curvature* of $C$ at $P$.

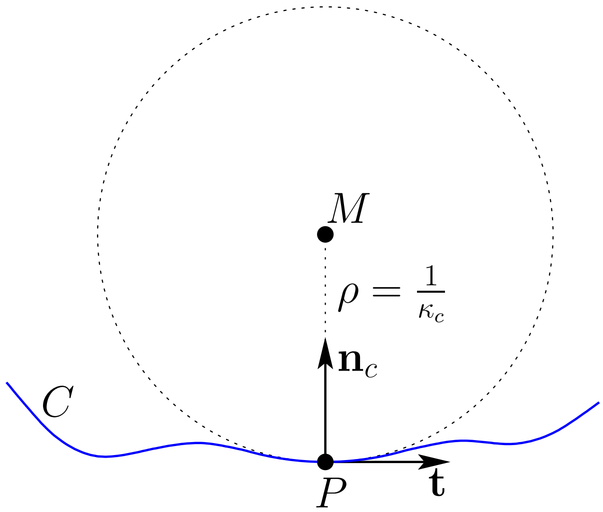



**Figure 3.3:** The curve, $C$, projected to the osculating plane at $P$.

We observe that the circle of curvature is only dependent on $\dot{\mathbf{t}}$ at the point $P$, such that any curve through $P$ with the same local behavior will have the same circle of curvature.

## 3.4 Orthogonal curves on a surface

In Section 3.2 we found that (3.7) must be satisfied for the two directions $\frac{\mathrm{d}r^1}{\mathrm{d}r^2}$ and $\frac{\delta r^1}{\delta r^2}$ to be orthogonal. We will later encounter equations on the form

$$a_1 \left(\frac{\mathrm{d}r^1}{\mathrm{d}r^2}\right)^2 + a_2 \left(\frac{\mathrm{d}r^1}{\mathrm{d}r^2}\right) + a_3 = 0, \tag{3.12}$$

where the solutions represent two directions on a surface associated with a point, $P$. If we assume that $\frac{\mathrm{d}r^1}{\mathrm{d}r^2}$ and $\frac{\delta r^1}{\delta r^2}$ are the two solutions of (3.12), we find that

$$\begin{aligned} \frac{\mathrm{d}r^1}{\mathrm{d}r^2} + \frac{\delta r^1}{\delta r^2} &= -\frac{a_2}{a_1}, \\ \frac{\mathrm{d}r^1}{\mathrm{d}r^2}\frac{\delta r^1}{\delta r^2} &= \frac{a_3}{a_1}. \end{aligned}$$

Combining this with (3.7), we get the required relation for orthogonality,

$$g_{11}a_3 - g_{12}a_2 + g_{22}a_1 = 0. \tag{3.13}$$

### 3.4.1 Example

If we again consider a situation with orthonormal base vectors, for instance the case in Figure 3.1, we get the required relation,

$$a_3 + a_1 = 0.$$

Choosing $a_1 = -a_3$ in (3.12), we get the two solutions,

$$\frac{\mathrm{d}r^1}{\mathrm{d}r^2} = -\frac{a_2}{2a_3} + \frac{\sqrt{a_2^2 + 4a_3^2}}{2a_3},$$
$$\frac{\delta r^1}{\delta r^2} = -\frac{a_2}{2a_3} - \frac{\sqrt{a_2^2 + 4a_3^2}}{2a_3}.$$

Defining the two vectors,

$$\mathbf{V}_1 = \left( -\frac{a_2}{2a_3} + \frac{\sqrt{a_2^2 + 4a_3^2}}{2a_3} \right) \mathbf{g}_1 + \mathbf{g}_2,$$
$$\mathbf{V}_2 = \left( -\frac{a_2}{2a_3} - \frac{\sqrt{a_2^2 + 4a_3^2}}{2a_3} \right) \mathbf{g}_1 + \mathbf{g}_2,$$

we find that

$$\mathbf{V}_1 \cdot \mathbf{V}_2 = 0,$$

which shows that the two directions are orthogonal.

## 3.5 Principal directions and mean curvature

We now wish to find an expression for the mean curvature at a point on a surface. In Section 3.3 we showed that the curvature at the point $P$ of a curve $C$ with the parametric representation $\mathbf{p}(s)$ is given by

$$\kappa_c \mathbf{n}_c = \ddot{\mathbf{p}}(s),$$

where $\mathbf{n}_c$ is the unit normal to $C$ which points in the direction of the intersection of the osculating plane and the normal plane, and $s$ is an arc-length variable. If we now set $\mathbf{n}_c = \mathbf{n}$, where $\mathbf{n}$ is the unit normal to a surface $S$ at $P$, we get the formula

$$\kappa_c \mathbf{n} = \ddot{\mathbf{p}}(s). \tag{3.14}$$

This formula will give us the curvature for all curves $\mathbf{p}(s)$ on $S$ for which the intersection between the osculating plane and the normal plane points in the same direction as $\mathbf{n}$. This leads us to a type of curves on $S$ called *normal sections*:

- **Normal section**

  A normal section is a *plane curve* associated with a general curve on $S$ which passes through $P$. A normal section is defined by the intersection of $S$ and a plane containing the normal $\mathbf{n}$ of $S$ at $P$ and a tangent vector, $\mathbf{t}$, to the curve. The normal section at $P$ will then automatically have $\mathbf{t}$ as a tangent vector and $\mathbf{n}$ as a principal normal. For such a curve (3.14) will be the formula for the curvature. We will denote the curvature of a normal section by $\kappa_n$.

An example of a normal section is displayed in Figure 3.4.

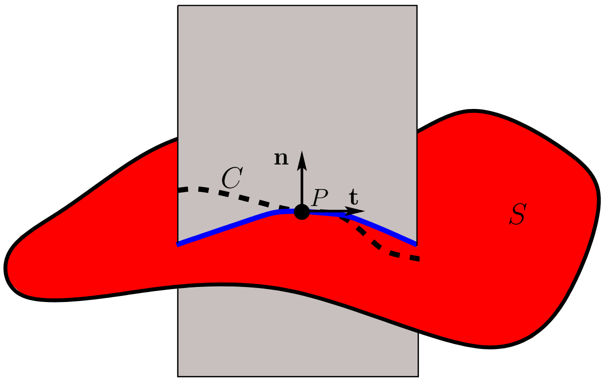



**Figure 3.4:** A normal section associated with a curve, $C$, passing through the point, $P$..

### 3.5.1 Curvature of a normal section

We now wish to derive another expression for the curvature of a normal section which involves tensors. From (3.14) we have

$$\begin{aligned}\kappa_n \mathbf{n} &= \ddot{\mathbf{p}}(s) = \frac{\mathrm{d}^2\mathbf{p}}{\mathrm{d}s^2} \\ &= \frac{\partial^2 \mathbf{p}}{\partial r^\alpha \partial r^\beta}\dot{r}^\alpha \dot{r}^\beta + \frac{\partial \mathbf{p}}{\partial r^\alpha}\ddot{r}^\alpha \\ &\Downarrow \\ \kappa_n &= \left(\frac{\partial^2 \mathbf{p}}{\partial r^\alpha \partial r^\beta} \cdot \mathbf{n}\right)\dot{r}^\alpha \dot{r}^\beta \\ &= \left(\frac{\partial \mathbf{g}_\alpha}{\partial r^\beta} \cdot \mathbf{n}\right)\dot{r}^\alpha \dot{r}^\beta. \end{aligned} \tag{3.15}$$

By the use of (2.23), we find that

$$\kappa_n = b_{\alpha\beta}\dot{r}^\alpha \dot{r}^\beta.$$

If we now assume that we parameterize the curve C by a parameter $t$ instead of the arc-length, $s$, we find,

$$\dot{r}^\alpha = \frac{\mathrm{d}r^\alpha}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{r^{\alpha\prime}}{s'}.$$

Thus, we may write

$$\kappa_n = \frac{b_{\alpha\beta} r^{\alpha\prime} r^{\beta\prime}}{(s')^2} \tag{3.16}$$

We know that

$$\mathrm{d}s^2 = \mathrm{d}\mathbf{p} \cdot \mathrm{d}\mathbf{p} = g_{\alpha\beta}\,\mathrm{d}r^\alpha\,\mathrm{d}r^\beta$$

and thus

$$(s')^2 = g_{\alpha\beta} r^{\alpha'} r^{\beta'}.$$

(3.16) now becomes

$$\begin{aligned}\kappa_n &= \frac{b_{\alpha\beta} r^{\alpha'} r^{\beta'}}{g_{\alpha\beta} r^{\alpha'} r^{\beta'}} \\ &= \frac{b_{\alpha\beta}\, \mathrm{d}r^\alpha\, \mathrm{d}r^\beta}{g_{\alpha\beta}\, \mathrm{d}r^\alpha\, \mathrm{d}r^\beta}.\end{aligned} \tag{3.17}$$

Thus, (3.17) gives us an expression for the curvature of a normal section whose tangent direction is given by $(\mathrm{d}r^1, \mathrm{d}r^2)$.

### 3.5.2 Principal directions

Twice the mean curvature is given by

$$\kappa = \kappa_{n_{\min}} + \kappa_{n_{\max}},$$

where $\kappa_{n_{\min}}$ and $\kappa_{n_{\max}}$ are the extremas for $\kappa_n$ when we consider all possible curves on $S$ passing through a point, $P$. We now wish to find which directions on $S$ for which $\kappa_n$ has its extremas. In (3.17) we have a formula to find the curvature at a point $P$ of the *normal section.* In Figure 3.5, we see that for a general vector, $\mathbf{v} = \mathrm{d}r^1\mathbf{g}_1 + \mathrm{d}r^2\mathbf{g}_2$, the ratio $\frac{\mathrm{d}r^1}{\mathrm{d}r^2}$ defines a direction on the surface. Thus, we wish to find the directions, $\frac{\mathrm{d}r^1}{\mathrm{d}r^2}$, such that $\frac{\partial \kappa_n}{\partial s} = 0$, where $s$ is a variable in the angular direction, see Figure 3.6. We see that a vector in the angular direction may be expressed

$$\delta\mathbf{s} = \delta r^1\mathbf{g}_1 + \delta r^2\mathbf{g}_2.$$

Thus, if we require that

$$\begin{aligned}\frac{\partial \kappa_n}{\partial l^1} &= 0, \\ \frac{\partial \kappa_n}{\partial l^2} &= 0,\end{aligned} \tag{3.18}$$

where $l^\alpha = \mathrm{d}r^\alpha$, we will also have $\frac{\partial \kappa_n}{\partial s} = 0$.

(3.17) may be written as

$$(b_{\alpha\beta} - \kappa_n g_{\alpha\beta}) l^\alpha l^\beta = 0. \tag{3.19}$$

If we set

$$a_{\alpha\beta} = b_{\alpha\beta} - \kappa_n g_{\alpha\beta},$$

differentiating (3.19) yields

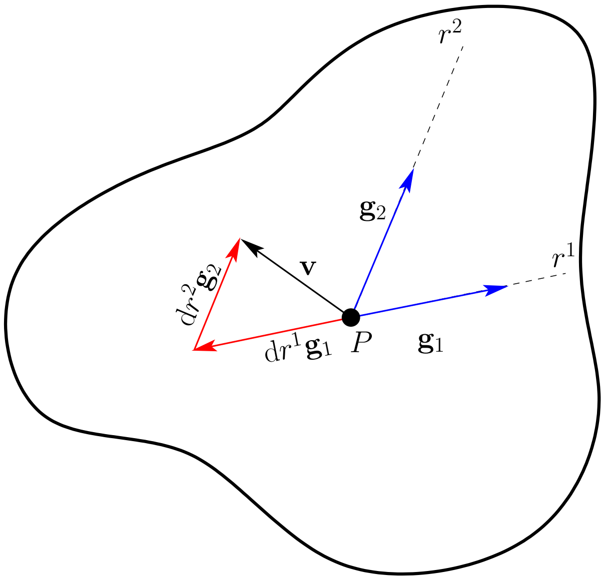



**Figure 3.5:** A vector, $\mathbf{v}$ , in a general direction on a surface.

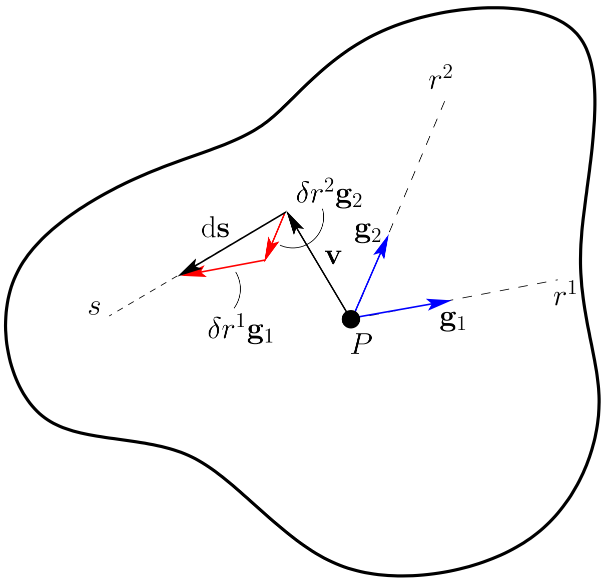



**Figure 3.6:** A vector, $\mathrm{d}\mathbf{s}$ , in the angular direction on a surface.

$$\frac{\partial}{\partial l^\gamma}(a_{\alpha\beta} l^\alpha l^\beta) = a_{\alpha\beta}\left(\frac{\partial l^\alpha}{\partial l^\gamma} l^\beta + l^\alpha \frac{\partial l^\beta}{\partial l^\gamma}\right)$$
$$= a_{\alpha\beta}(\delta^\alpha_\gamma l^\beta + l^\alpha \delta^\beta_\gamma) = a_{\gamma\beta} l^\beta + a_{\alpha\gamma} l^\alpha = (a_{\gamma\alpha} + a_{\alpha\gamma}) l^\alpha, \qquad \gamma = 1, 2.$$

(Note here that $\frac{\partial}{\partial l^\gamma}$ denotes differentiation with respect to the direction $\mathrm{d}r^\gamma$ on the surface. Thus, the point $P$ is constant, and therefore $\frac{\partial b_{\alpha\beta}}{\partial l^\gamma} = 0$ and $\frac{\partial g_{\alpha\beta}}{\partial l^\gamma} = 0$.) $a_{\alpha\beta}$ is symmetric, and we get the two equations

$$\begin{aligned} (b_{\alpha 1} - \kappa_n g_{\alpha 1})\, \mathrm{d}r^\alpha &= 0, \\ (b_{\alpha 2} - \kappa_n g_{\alpha 2})\, \mathrm{d}r^\alpha &= 0. \end{aligned} \tag{3.20}$$

If we eliminate $\kappa_n$ from (3.20), we end up with the second order equation

$$(g_{11}b_{12} - g_{12}b_{11})\left(\frac{\mathrm{d}r^1}{\mathrm{d}r^2}\right)^2 + (g_{11}b_{22} - g_{22}b_{11})\left(\frac{\mathrm{d}r^1}{\mathrm{d}r^2}\right) + (g_{12}b_{22} - g_{22}b_{12}) = 0. \tag{3.21}$$

We see that (3.21) satisfies (3.13) with

$$\begin{aligned} a_1 &= (g_{11}b_{12} - g_{12}b_{11}), \\ a_2 &= (g_{11}b_{22} - g_{22}b_{11}), \\ a_3 &= (g_{12}b_{22} - g_{22}b_{12}), \end{aligned}$$

and thus we have shown that the two principal curves are orthogonal.

### 3.5.3 Mean curvature

From (3.20) we find that

$$\begin{aligned} \frac{\mathrm{d}r^1}{\mathrm{d}r^2} &= \frac{(\kappa_n g_{21} - b_{21})}{(b_{11} - \kappa_n g_{11})}, \\ \frac{\mathrm{d}r^1}{\mathrm{d}r^2} &= \frac{(\kappa_n g_{22} - b_{22})}{(b_{12} - \kappa_n g_{12})}. \end{aligned}$$

Eliminating $\frac{\mathrm{d}r^1}{\mathrm{d}r^2}$ leads to the second order equation

$$(g_{11}g_{22} - g_{12}^2)\kappa_n^2 + (2g_{12}b_{12} - b_{11}g_{22} - g_{11}b_{22})\kappa_n + (b_{11}b_{22} - b_{12}^2) = 0, \tag{3.22}$$

from which we obtain $\kappa_{n_{max}}$ and $\kappa_{n_{min}}$. Twice the mean curvature is given by

$$\begin{aligned} \kappa &= \kappa_{n_{max}} + \kappa_{n_{min}} \\ &= \frac{2(g_{22}b_{11} - 2g_{12}b_{12} + g_{11}b_{22})}{2g^2} \\ &= b_{11}g^{11} + 2b_{12}g^{12} + b_{22}g^{22}. \end{aligned} \tag{3.23}$$

### 3.5.4 Surface divergence of the unit normal

From (3.1) and (2.23), we find that the divergence of the unit normal may be expressed as

$$
\begin{aligned}
\nabla_s \cdot \mathbf{n} &= \frac{1}{g^2}\mathbf{g}_1 \cdot \left( g_{22}\frac{\partial \mathbf{n}}{\partial r^1} - g_{12}\frac{\partial \mathbf{n}}{\partial r^2} \right) + \frac{1}{g^2}\mathbf{g}_2 \cdot \left( g_{11}\frac{\partial \mathbf{n}}{\partial r^2} - g_{12}\frac{\partial \mathbf{n}}{\partial r^1} \right) \\
&= \frac{1}{g^2}\left( g_{22}\mathbf{g}_1 \cdot \frac{\partial \mathbf{n}}{\partial r^1} - g_{12}\mathbf{g}_1 \cdot \frac{\partial \mathbf{n}}{\partial r^2} + g_{11}\mathbf{g}_2 \cdot \frac{\partial \mathbf{n}}{\partial r^2} - g_{12}\mathbf{g}_2 \cdot \frac{\partial \mathbf{n}}{\partial r^1} \right) \\
&= -\frac{(g_{22}b_{11} - g_{12}b_{12} + g_{11}b_{22} - g_{12}b_{21})}{g^2} \\
&= -\frac{(g_{11}b_{22} - 2g_{12}b_{12} + g_{22}b_{11})}{g^2} \\
&= -(b_{11}g^{11} + 2b_{12}g^{12} + b_{22}g^{22}) \\
&= -\kappa.
\end{aligned}
$$

## 3.6 The surface Laplacian of the position vector, $\nabla_s^2 \mathbf{p}$

If we use (3.9) with (3.2), we find

$$
\begin{aligned}
\nabla_s^2 p_i &= \nabla_s \cdot \nabla_s p_i \\
&= \frac{1}{g}\frac{\partial}{\partial r^1}\left( \frac{g_{22}\frac{\partial p_i}{\partial r^1} - g_{12}\frac{\partial p_i}{\partial r^2}}{g} \right) + \frac{1}{g}\frac{\partial}{\partial r^2}\left( \frac{g_{11}\frac{\partial p_i}{\partial r^2} - g_{12}\frac{\partial p_i}{\partial r^1}}{g} \right) \\
&= \frac{1}{g}\frac{\partial}{\partial r^1}\left( \frac{g_{22}g_{1_i} - g_{12}g_{2_i}}{g} \right) + \frac{1}{g}\frac{\partial}{\partial r^2}\left( \frac{g_{11}g_{2_i} - g_{12}g_{1_i}}{g} \right) \\
&= \frac{1}{g}\left( \frac{\partial}{\partial r^1}\left(\frac{g_{22}}{g}\right) g_{1_i} + \left(\frac{g_{22}}{g}\right)\frac{\partial g_{1_i}}{\partial r^1} - \frac{\partial}{\partial r^1}\left(\frac{g_{12}}{g}\right) g_{2_i} - \left(\frac{g_{12}}{g}\right)\frac{\partial g_{2_i}}{\partial r^1} \right) \\
&+ \frac{1}{g}\left( \frac{\partial}{\partial r^2}\left(\frac{g_{11}}{g}\right) g_{2_i} + \left(\frac{g_{11}}{g}\right)\frac{\partial g_{2_i}}{\partial r^2} - \frac{\partial}{\partial r^2}\left(\frac{g_{12}}{g}\right) g_{1_i} - \left(\frac{g_{12}}{g}\right)\frac{\partial g_{1_i}}{\partial r^2} \right).
\end{aligned} \tag{3.24}
$$

Thus, we need other expressions for $\frac{\partial \mathbf{g}_1}{\partial r^1}$, $\frac{\partial \mathbf{g}_2}{\partial r^1}$, $\frac{\partial \mathbf{g}_1}{\partial r^2}$ and $\frac{\partial \mathbf{g}_2}{\partial r^2}$. We may show that

$$
\begin{aligned}
\frac{\partial \mathbf{g}_1}{\partial r^1} &= a_1\mathbf{g}_1 + b_1\mathbf{g}_2 + c_1\mathbf{n} \\
\frac{\partial \mathbf{g}_2}{\partial r^1} = \frac{\partial \mathbf{g}_1}{\partial r^2} &= a_2\mathbf{g}_1 + b_2\mathbf{g}_2 + c_2\mathbf{n} \\
\frac{\partial \mathbf{g}_2}{\partial r^2} &= a_3\mathbf{g}_1 + b_3\mathbf{g}_2 + c_3\mathbf{n}
\end{aligned}
$$

Taking the inner-product of $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{n}$ with these three equations leads to

$$
\begin{aligned}
a_1 &= \frac{(g_{22}\frac{\partial g_{11}}{\partial r^1} - 2g_{12}\frac{\partial g_{12}}{\partial r^1} + g_{12}\frac{\partial g_{11}}{\partial r^2})}{2g^2} \\
b_1 &= \frac{(2g_{11}\frac{\partial g_{12}}{\partial r^1} - g_{11}\frac{\partial g_{11}}{\partial r^2} - g_{12}\frac{\partial g_{11}}{\partial r^1})}{2g^2} \\
c_1 &= b_{11} \\
a_2 &= \frac{(g_{22}\frac{\partial g_{11}}{\partial r^2} - g_{12}\frac{\partial g_{22}}{\partial r^1})}{2g^2} \\
b_2 &= \frac{(g_{11}\frac{\partial g_{22}}{\partial r^1} - g_{12}\frac{\partial g_{11}}{\partial r^2})}{2g^2} \\
c_2 &= b_{12} \\
a_3 &= \frac{(2g_{22}\frac{\partial g_{12}}{\partial r^2} - g_{22}\frac{\partial g_{22}}{\partial r^1} - g_{12}\frac{\partial g_{22}}{\partial r^2})}{2g^2} \\
b_3 &= \frac{(g_{11}\frac{\partial g_{22}}{\partial r^2} - 2g_{12}\frac{\partial g_{12}}{\partial r^2} + g_{12}\frac{\partial g_{22}}{\partial r^1})}{2g^2} \\
c_3 &= b_{22}
\end{aligned}
$$

Inserted into (3.24), we see that all the tangential components cancel, and we end up with

$$
\begin{aligned}
\nabla_s^2 p_i &= \frac{(g_{11}b_{22} - 2g_{12}b_{12} + g_{22}b_{11})}{g^2} n_i \\
&= \kappa n_i.
\end{aligned}
$$

### 3.7 Curvature-normal product

From (3.24) we have that

$$
\kappa n_i = \frac{1}{g}\frac{\partial}{\partial r^1}\left(\frac{g_{22}\frac{\partial p_i}{\partial r^1} - g_{12}\frac{\partial p_i}{\partial r^2}}{g}\right) + \frac{1}{g}\frac{\partial}{\partial r^2}\left(\frac{g_{11}\frac{\partial p_i}{\partial r^2} - g_{12}\frac{\partial p_i}{\partial r^1}}{g}\right).
$$

Finally, by the use of (2.24)-(2.27) and (2.21)

$$
\begin{aligned}
\kappa n_i &= \frac{1}{g}\frac{\partial}{\partial r^1}\left(g\left(\frac{g_{22}g_{1,i} - g_{12}g_{2,i}}{g^2}\right)\right) + \frac{1}{g}\frac{\partial}{\partial r^2}\left(g\left(\frac{-g_{12}g_{1,i} + g_{11}g_{2,i}}{g^2}\right)\right) \\
&= \frac{1}{g}\frac{\partial}{\partial r^1}(g(g^{11}g_{1,i} + g^{12}g_{2,i})) + \frac{1}{g}\frac{\partial}{\partial r^2}(g(g^{21}g_{1,i} + g^{22}g_{2,i})) \\
&= \frac{1}{g}(gg_i^{\alpha})_{,\alpha}. \qquad (3.25)
\end{aligned}
$$

## 4 Derivation of the surface integral

We have now obtained all the necessary expressions in order to find another expression for (1.1) by using surface intrinsic coordinates. By the use of (2.13), (3.25) and (3.10),

$$\begin{aligned}
\int_\Gamma v_i \sigma_{ij} n_j \, \mathrm{d}S &= \int_\Gamma v_i (\gamma n_i \kappa + (\nabla_s \gamma)_i) \, \mathrm{d}S \\
&= \int_{\hat{\Gamma}} v_i (\gamma g^{-1} (g g_i^\alpha)_{,\alpha} + \gamma_{,\alpha} g_i^\alpha) g \, \mathrm{d}r^1 \, \mathrm{d}r^2 \\
&= \int_{\hat{\Gamma}} v_i (\gamma g_{i,\alpha}^\alpha + \gamma g^{-1} g_{,\alpha} g_i^\alpha + \gamma_{,\alpha} g_i^\alpha) g \, \mathrm{d}r^1 \, \mathrm{d}r^2 \\
&= \int_{\hat{\Gamma}} v_i (\gamma g g_{i,\alpha}^\alpha + \gamma g_{,\alpha} g_i^\alpha + \gamma_{,\alpha} g g_i^\alpha) \, \mathrm{d}r^1 \, \mathrm{d}r^2 \\
&= \int_{\hat{\Gamma}} v_i (\gamma g g_i^\alpha)_{,\alpha} \, \mathrm{d}r^1 \, \mathrm{d}r^2 \\
&= \oint_{\partial\hat{\Gamma}} \gamma v_i g_i^\alpha \, \mathrm{d}n_\alpha - \int_{\hat{\Gamma}} v_{i,\alpha} \gamma g_i^\alpha g \, \mathrm{d}r^1 \, \mathrm{d}r^2 .
\end{aligned}$$

where $\mathrm{d}n_\alpha = \tilde{\varepsilon}_{\alpha\beta} \, \mathrm{d}r_\alpha$, and $\tilde{\varepsilon}_{11} = \tilde{\varepsilon}_{22} = 0, \tilde{\varepsilon}_{12} = -\tilde{\varepsilon}_{21} = g$.

For the cases we will consider, $\oint_{\partial\hat{\Gamma}} \gamma v_i g_i^\alpha \, \mathrm{d}n_\alpha = 0$, such that

$$\int_\Gamma v_i \sigma_{ij} n_j \, \mathrm{d}S = - \int_{\hat{\Gamma}} v_{i,\alpha} \gamma g_i^\alpha g \, \mathrm{d}r^1 \, \mathrm{d}r^2 .$$

This is the same integral as proposed in [2].

**Acknowledgement**

I would like to thank Professor Einar Rønquist for proposing to work on this topic. I would also like to thank Norwegian University of Science and Technology for the financial support of this project.

## References

[1] W. Flügge. *Tensor Analysis and Continuum Mechanics*. Springer, Berlin, 1972.

[2] L.W. Ho and A.T. Patera. Variational formulation of three-dimensional viscous free-surface flows: Natural imposition of surface tension boundary conditions. *International Journal for Numerical Methods in Fluids*, 13:691–698, 1991.

[3] E. Kreyszig. *Differential Geometry*. Dover Publications, Inc., 1991.

[4] C.E. Weatherburn. *Differential Geometry of Three Dimensions*. Cambridge University Press, 1927.