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A combined Filon/asymptotic quadrature method for highly oscillatory problems

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A cross between the asymptotic expansion of an oscillatory integral and the Filon-type methods is obtained by applying a Filon-type method on the error term in the asymptotic expansion, which is in itself an oscillatory integral. The efficiency of the method is investigated through analysis and numerical experiments. The case of multivariate oscillatory integrals is treated with an example.

1 Introduction

The quadrature of highly oscillatory integrals has been perceived as a hard problem. Traditionally one would have to resolve the oscillations by taking several sub intervals for each period, resulting in a scheme whose complexity would grow linearly with the frequency of the oscillations. More careful analysis will however reveal that by exploiting the structure of certain classes of oscillatory integrals better discretisation schemes can be devised, schemes where the error actually decreases when the period of the oscillations increase. Examples of such methods are Filon-type methods[8, 9] Levin-type methods[10, 11] and numerical steepest descent[4].

We are considering oscillatory integrals on the form

$$I_g[f] = \int_{-1}^{1} f(x) e^{i\omega g(x)} \mathrm{d}x, \tag{1}$$

where ω is a large parameter. It is well known that an ordinary Gauss quadrature applied to this integral will have an error of $\mathcal{O}(1)$ as ω grows large. A much better approach to approximating $I_g[f]$ when ω is large is found through an asymptotic expansion: Assume g is strictly monotone, then applying integration by parts on $I_g[f]$ yields

$$I_g[f] = \frac{1}{i\omega}\left[\frac{f(1)}{g'(1)}e^{i\omega g(1)} - \frac{f(-1)}{g'(-1)}e^{i\omega g(-1)}\right] - \frac{1}{i\omega}\int_{-1}^{1}\frac{f(x)}{g'(x)}e^{i\omega g(x)}\mathrm{d}x. \tag{2}$$

When ω becomes large the integral in equation (2) vanishes faster than the boundary terms (by an extension of Riemann-Lebesgue's lemma), so the boundary terms can approximate the integral. Furthermore the process can be repeated on the integral remainder to obtain a full asymptotic expansion. This expansion will however not be optimal; as is often the case with asymptotic expansions the method will break down for moderately sized ω.

Perhaps an even better approach, in its most basic form first proposed by Louis Napoleon Filon[3] is to choose a set of quadrature nodes $c_1, \ldots, c_\nu$, interpolate the function f by a polynomial $\tilde{f}$ at these points and let

$$Q_1^F[f] = \int_{-1}^{1} \tilde{f}(x) e^{i\omega g(x)} \mathrm{d}x = \sum_{j=1}^{\nu} b_j(\omega) f(c_j),$$

where $b_j(\omega) = \int_{-1}^{1} l_j(x) e^{i\omega g(x)} \mathrm{d}x$ for $l_j(x)$ the j-th Lagrange cardinal polynomial. Constructing $b_j(\omega)$ requires the moments $\int_{-1}^{1} x^m e^{i\omega g(x)} \mathrm{d}x$. Moments are oscillatory integrals themselves, hopefully these can be calculated by analytical means or approximated. Iserles proved[8] that as long as the endpoints of the interval are included as quadrature nodes and g is strictly monotone, this approach will carry an error

$$Q_1^F[f] - I_g[f] \sim \mathcal{O}(\omega^{-2}), \quad \omega \to \infty.$$

The superiority of this approach over the asymptotic expansion can be understood by realising that the method is exact for a class of problems, regardless of the size of ω. As for the behaviour for large ω it was proved by Iserles and Nørsett[6] that by interpolating $f(x)$ with a number of its derivatives at the endpoints the asymptotic behaviour of the error can be expressed as

$$Q_p^F[f] - I_g[f] \sim \mathcal{O}(\omega^{-s-1}), \quad \omega \to \infty,$$

for any s. The theory can be expanded to the cases where g has stationary points, that means points ξ with $g'(\xi) = 0$. What must be done to achieve good asymptotic properties is basically to include the stationary points among the quadrature nodes[9].

This report will suggest a variation on the Filon-type quadrature, or rather a mix between the asymptotic expansion and the Filon-type quadrature. The idea is based on the observation that the remainder term in the asymptotic expansion (2) is an oscillatory integral on the same form as the original oscillatory integral (1). Using a Filon-type quadrature Q_p^F on this remainder term yields a method

$$Q^{FA}[f] = \frac{1}{i\omega}\left[\frac{f(1)}{g'(1)} e^{i\omega g(1)} - \frac{f(-1)}{g'(-1)} e^{i\omega g(-1)}\right] - \frac{1}{i\omega} Q_p^F[f/g'],$$

which in the following will be called a *combined Filon/asymptotic method*. This method will require fewer moments than Filon-type methods to achieve high asymptotic order, while retaining accuracy for moderately sized ω, in this sense appearing as a true marriage between the asymptotic method and the Filon-type methods. The aim of this report is to explore the properties of this method and assess the efficiency of the method compared to that of the classical Filon-type method.

2 The Asymptotic method and Filon-type methods

Assume for the time being that there are no stationary points in the interval of interest, that means $g'(x) \neq 0, \quad -1 \leq x \leq 1$. An asymptotic expansion of the highly oscillatory integral (1) is obtained by successively applying integration by parts. This approach gives us a full expansion trough the following partial expansion

$$\begin{aligned} I_g[f] = & -\sum_{m=1}^{s} \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right] \\ & + \frac{1}{(-i\omega)^s} \int_{-1}^{1} \sigma_s[f](x) e^{i\omega g(x)} \mathrm{d}x, \end{aligned} \tag{3}$$

where

$$\begin{aligned} \sigma_0[f](x) &= f(x) \\ \sigma_{k+1}[f](x) &= \frac{\mathrm{d}}{\mathrm{d}x} \frac{\sigma_k[f](x)}{g'(x)}, \quad k = 0, 1, \ldots. \end{aligned} \tag{4}$$

The full asymptotic expansion of the highly oscillatory integral (1) is then

$$I_g[f] \sim -\sum_{m=1}^{\infty} \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right]. \tag{5}$$

Truncating the series after s terms, yields the asymptotic method

$$Q_s^A[f] = -\sum_{m=1}^{s} \frac{1}{(-i\omega)^m} \left[\frac{e^{i\omega g(1)}}{g'(1)} \sigma_{m-1}[f](1) - \frac{e^{i\omega g(-1)}}{g'(-1)} \sigma_{m-1}[f](-1) \right]. \tag{6}$$

This method has an *asymptotic order* of s, that means for large ω and smooth f the error goes like

$$Q_s^A[f] - I_g[f] \sim \mathcal{O}(\omega^{-s-1}).$$

This can be seen by writing out the remainder term, which is an oscillatory integral $\mathcal{O}(\omega^{-1})$ multiplied by $(-i\omega)^{-s}$. Note that the concept of asymptotic order is rather useless for not-so-large ω. In fact the asymptotic method will break down for smaller ω.

The Filon-type methods will be accurate also for smaller ω, but at the cost of moments. We define the moments

$$\mu_k(\omega) = \int_a^b x^k e^{i\omega g(x)} \mathrm{d}x,$$

and assume these can be computed, possibly at a significant cost, then the Filon-type method is obtained by choosing a set of nodes $-1 = c_1 < c_2 \leq \cdots < c_\nu = 1$ and integer multiplicities $m_1, \ldots, m_\nu \geq 1$ associated with each node. Let $n = \sum_{j=1}^{\nu} m_j - 1$ and $\tilde{f}$ be the unique Hermite interpolation polynomial of degree n obtained by interpolating f at the points $\{c_j\}_{j=1}^{\nu}$ with the corresponding multiplicities

$$\tilde{f}(x) = \sum_{l=1}^{\nu} \sum_{j=0}^{m_l} \alpha_{l,j}(x) f^{(j)}(c_l).$$

The Filon-type method is defined as

$$Q_s^F[f] = \int_{-1}^{1} \tilde{f}(x) e^{i\omega g(x)} \mathrm{d}x = \sum_{l=1}^{\nu} \sum_{j=0}^{m_l} \beta_{l,j}(\omega) f^{(j)}(c_l), \tag{7}$$

where $\beta_{l,j}(\omega) = \int_a^b \alpha_{l,j}(x) e^{i\omega g(x)} \mathrm{d}x$ is obtained from linear combinations of moments

Now we state a theorem due to Iserles and Nørsett[6] regarding the asymptotic order of this method:

Theorem 1. *Suppose $m_1, m_\nu \geq s$, then for every smooth f and smooth strictly monotonous g*

$$Q_s^F[f] - I_g[f] \sim (\omega^{-s-1})$$

The proof is obtained by expanding $f - \tilde{f}$ as in equation (5) and observing that the first s terms will cancel due to the interpolation criteria. This theorem can be summarised by saying that only by adding derivative information at the endpoints of the interval can the asymptotic order of the method be improved. Note that increasing the order of the interpolating polynomial $\tilde{f}$ should normally increase the accuracy of the method for fixed ω, at least when no Runge-phenomena are present. This is indeed confirmed by numerical experiments (see for example [5]).

2.1 Generalised Filon and asymptotic method in the presence of stationary points

When g has stationary points Theorem 1 is no longer valid, a fact which is suggested by the singularity introduced in the integral in remainder term of the asymptotic expansion (2). Assume in the following that $g(x)$ has only one stationary point ξ, which amounts to saying $g'(\xi) = 0, \;\; g'(x) \neq 0, x \in [-1,1] \backslash \{\xi\}$. Furthermore assume that $g'(\xi) = \cdots = g^{(r)}(\xi) = 0$, and $g^{(r+1)}(\xi) \neq 0$, this means that ξ is a rth order stationary point. The method of stationary phase(see for example [2]) states that in this case the leading order behaviour of the highly oscillatory integral (1) is on the form

$$I_g[f] \sim C\omega^{-1/(r+1)}, \quad \omega. \to \infty \tag{8}$$

This means that the main contribution to the value of the integral comes from the stationary point, suggesting that the interpolation nodes for the Filon-type methods should include stationary points as well as the endpoints.

Assume for simplicity that ξ is a first order stationary point, $g''(\xi) \neq 0$. Writing

$$I_g[f] = f(\xi) I_g[1] + I_g[f - f(\xi)],$$

then integrating by parts gives the following expression

$$I_g[f] = f(\xi) I_g[1] + \frac{1}{i\omega} \left[\frac{f(1) - f(\xi)}{g'(1)} e^{i\omega g(1)} - \frac{f(-1) - f(\xi)}{g'(-1)} e^{i\omega g(-1)} \right] - \frac{1}{i\omega} \int_{-1}^{1} \frac{f(x) - f(\xi)}{g'(x)} e^{i\omega g(x)} \mathrm{d}x. \tag{9}$$

Now, since $g''(\xi) \neq 0$, the singularity is removable. The expansion can be continued giving a full expansion reminiscent of the expansion (5). More generally, for a rth order stationary point we introduce the generalised moments

$$\mu_k(\omega;\xi) = I_g[(\cdot - x)^k] = \int_{-1}^{1} (x-\xi)^k e^{i\omega g(x)} \mathrm{d}x, \quad k \geq 0,$$

and write

$$I_g[f] = \sum_{j=0}^{r-1} \frac{1}{j!} f^{(j)}(\xi)\mu_j(\omega;\xi) + I_g\left[f(x) - \sum_{j=0}^{r-1} \frac{1}{j!} f^{(j)}(\xi)(x-\xi)^j \right]. \tag{10}$$

Again the singularity is removable, and the expansion can be formed. We will later need the expansion with the remainder term, so this will be formulated as a lemma[1]:

Lemma 1. *Suppose ξ is a stationary point of order r, and that ξ is the only stationary point in the interval $[-1,1]$. Then for every smooth f*

$$\begin{aligned} I_g[f] = & \sum_{j=0}^{r-1} \frac{1}{j!}\mu_j(\omega;\xi) \sum_{m=0}^{s-1} \frac{1}{(-i\omega)^m} \rho_m^{(j)}[f](\xi) \\ & - \sum_{m=0}^{s-1} \frac{1}{(-i\omega)^{m+1}} \left[\frac{e^{i\omega g(1)}}{g'(1)} \left(\rho_m[f](1) - \sum_{j=0}^{r-1} \frac{1}{j!}\rho_m[f]^{(j)}(\xi)(1-\xi)^j \right) \right. \\ & \qquad \left. - \frac{e^{i\omega g(-1)}}{g'(-1)} \left(\rho_m[f](-1) - \sum_{j=0}^{r-1} \frac{1}{j!}\rho_m[f]^{(j)}(\xi)(-1-\xi)^j \right) \right] \\ & + \frac{1}{(-i\omega)^s} I_g\Big[\rho_s[f]\Big], \end{aligned} \tag{11}$$

where

$$\begin{aligned} \rho_0[f](x) &= f(x) \\ \rho_{k+1}[f](x) &= \frac{\mathrm{d}}{\mathrm{d}x} \frac{\rho_k[f](x) - \sum_{j=0}^{r-1}\frac{1}{j!}\rho_k[f]^{(j)}(\xi)(x-\xi)^j}{g'(x)}, \quad k = 0,1,\ldots \end{aligned} \tag{12}$$

Proof. This is proved by induction. The Lemma is certainly true for $s = 0$. Now

$$\begin{aligned} I_g\Big[\rho_s[f]\Big] = & \sum_{j=0}^{r-1} \frac{1}{j!} \rho_s[f]^{(j)}(\xi)\mu_j(\omega;\xi) \\ & + \frac{1}{i\omega} \int_{-1}^{1} \frac{\rho_s[f](x) - \sum_{j=0}^{r-1}\frac{1}{j!}\rho_s[f]^{(j)}(\xi)(x-\xi)^j}{g'(x)} \frac{\mathrm{d}}{\mathrm{d}x} e^{i\omega g(x)} \mathrm{d}x. \end{aligned}$$

Integration by parts gives

$$\begin{aligned} I_g\Big[\rho_s[f]\Big] = & \sum_{j=0}^{r-1} \frac{1}{j!}\rho_s[f]^{(j)}(\xi)\mu_j(\omega;\xi) - \frac{1}{(-i\omega)} \left[\frac{e^{i\omega g(1)}}{g'(1)} \left(\rho_s[f](1) - \sum_{j=0}^{r-1} \frac{1}{j!}\rho_s[f]^{(j)}(\xi)(1-\xi)^j \right) \right. \\ & \left. - \frac{e^{i\omega g(-1)}}{g'(-1)} \left(\rho_s[f](-1) - \sum_{j=0}^{r-1} \frac{1}{j!}\rho_s[f]^{(j)}(\xi)(-1-\xi)^j \right) \right] + \frac{1}{(-i\omega)} I_g\Big[\rho_{s+1}[f]\Big]. \end{aligned}$$

[1] Note that the conclusion in this lemma is different from that of Iserles & Nørsett in [6], Theorem 3.2, which we suggest is flawed.

Inserting into equation (11) proves the Lemma. $\square$

As before, ignoring the remainder term yields the asymptotic method. The asymptotic behaviour of the error in this method depends on the asymptotic properties of of the generalised moments, which in turn are obtained trough the method of stationary phase (as in equation (8).) Thus we get for the asymptotic method(see [6] for details)

$$Q_s^A - I_g[f] \sim \mathcal{O}(\omega^{-s-1/(r+1)}).$$

For an even more general case, in the presence of more than one stationary point, the interval can be partitioned such that each sub interval contains only one stationary point, and then expanding. As before, truncating the expansion after s terms yields the asymptotic method.

Let now ξ be a unique stationary point of order r: $g'(\xi) = 0$ and $g'(x) \neq 0$ for $x \in [-1,1]\backslash\{\xi\}$, $g'(\xi) = \cdots = g^{(r)}(\xi) = 0$, and $g^{(r+1)}(\xi) \neq 0$. The *generalised Filon method* is constructed by choosing nodes $-1 = c_1 < c_2 < \cdots < c_\nu = 1$ such that the stationary point is among the nodes, that is $c_q = \xi$ for some $q \in \{1, 2, \ldots, \nu\}$. Given multiplicities $m_1, m_2, \ldots, m_\nu \geq 1$ corresponding to each node, we let $\tilde{f}$ be the unique Hermite interpolation polynomial of degree $n = \sum_{j=1}^{\nu} m_j - 1$ obtained by interpolating f at the points $\{c_j\}_{j=1}^{\nu}$ with the corresponding multiplicities. The method is now simply

$$Q^F[f] = \int_{-1}^{1} \tilde{f}(x) e^{i\omega g(x)} \mathrm{d}x. \tag{13}$$

The above integral is computed from linear combinations of both moments and generalised moments.

We present another theorem by Iserles and Nørsett[6] regarding the asymptotic error behaviour of the generalised Filon method.

Theorem 2. *Let $m_1, m_\nu \geq s$ and $m_q \geq s(r+1)$. Then*

$$Q_s^F[f] - I_g[f] \sim \mathcal{O}(\omega^{-s-1/(r+1)}).$$

This theorem is, like Theorem 1 proved by expanding $f - \tilde{f}$ and showing that terms up to order s cancel. The method is trivially expanded to cater for several stationary points, possibly of different order.

3 A combined Filon/asymptotic method

Let us for the moment assume that there are no stationary points of g in $[-1,1]$, this assumption will be relaxed later on. A combined Filon/asymptotic method is constructed from the asymptotic expansion with the remainder term (3) by applying a Filon-type method on the remainder term, which is in itself an oscillatory integral. Thus we obtain the method

$$\begin{aligned} Q_{p,s}^{FA}[f] = &-\sum_{m=1}^{s}\frac{1}{(-i\omega)^m}\left[\frac{e^{i\omega g(1)}}{g'(1)}\sigma_{m-1}[f](1)-\frac{e^{i\omega g(-1)}}{g'(-1)}\sigma_{m-1}[f](-1)\right]\\ &+\frac{1}{(-i\omega)^s}Q_p^F\left[\sigma_s[f]\right], \end{aligned} \tag{14}$$

where the $\sigma_m[f]$ are defined as in equation (4). We call this a combined Filon/asymptotic method.

Theorem 3. *Let g be strictly monotonous. For the combined Filon/asymptotic method $Q_{p,s}^{FA}$ constructed with a Filon-type method Q_p^F of asymptotic order p applied to any smooth f it is true that*

$$Q_{p,s}^{FA}[f]-I_g[f]\sim\mathcal{O}(\omega^{-p-s-1}),\quad \omega\to\infty$$

Proof. Writing out the asymptotic expansion of $Q_{p,s}^{FA}[f]-I_g[f]$ gives

$$\begin{aligned} Q_{p,s}^{FA}[f]-I_g[f] &\sim \frac{1}{(-i\omega)^s}Q_p^F[\sigma_s[f](x)]+\sum_{m=s+1}^{\infty}\frac{1}{(i\omega)^m}\left[\frac{e^{i\omega g(1)}}{g'(1)}\sigma_{m-1}[f](1)-\frac{e^{i\omega g(-1)}}{g'(-1)}\sigma_{m-1}[f](-1)\right]\\ &=\frac{1}{(-i\omega)^s}\left(Q_p^F[\sigma_s[f](x)]-\sum_{j=1}^{\infty}\frac{1}{(i\omega)^j}\left[\frac{e^{i\omega g(1)}}{g'(1)}\sigma_{j-1}[\sigma_s[f]](1)-\frac{e^{i\omega g(-1)}}{g'(-1)}\sigma_{j-1}[\sigma_s[f]](-1)\right]\right)\\ &\sim\frac{1}{(-i\omega)^s}\mathcal{O}(\omega^{-p-1})=\mathcal{O}(\omega^{-p-s-1}), \end{aligned}$$

where the last line appears by applying Theorem 1. □

Example 1. *For the simplest, and perhaps most useful case, set $s=1$ and get*

$$Q_{p,1}^{FA}[f]=\frac{1}{i\omega}\left[\frac{e^{i\omega g(1)}}{g'(1)}f(1)-\frac{e^{i\omega g(-1)}}{g'(-1)}f(-1)\right]-\frac{1}{i\omega}Q_p^F\left[\frac{\mathrm{d}}{\mathrm{d}x}\frac{f}{g'}\right], \tag{15}$$

which is a method of asymptotic order $p+1$.

Example 2. *We wish to compute*

$$\int_{-1}^{1}\frac{e^{i\omega x}}{2+x}\mathrm{d}x.$$

Choosing to interpolate $f(x)=1/(2+x)$ and its derivatives at $x=-1$ and $x=1$ will give us a Filon-type method with asymptotic order $p=2$. Interpolating the function value of $\sigma_1(x)=-1/(2+x)^2$ at the two endpoints and approximating the integral as in equation (15) gives the combined Filon/asymptotic scheme which is also of asymptotic order 2. The classical Filon-type method requires four moments to be computed, whereas the Filon/asymptotic scheme

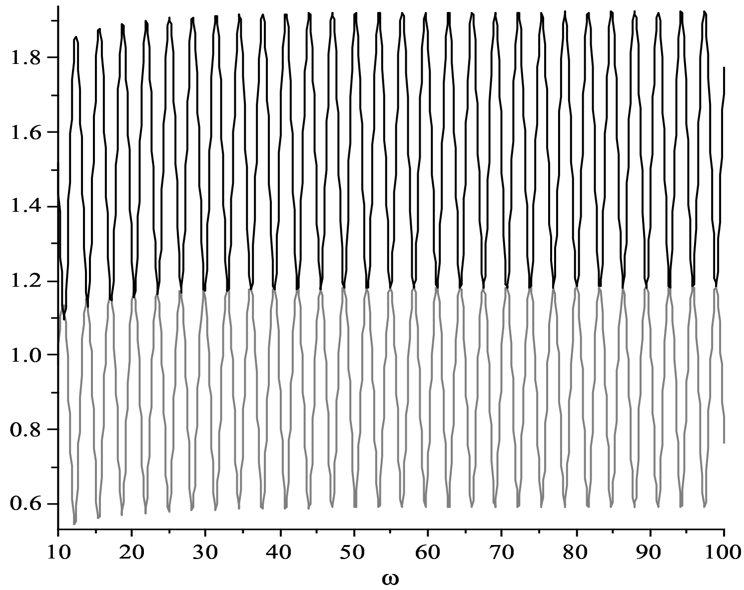


Figure 1: The absolute value of the error for the combined Filon/asymptotic method (black) and the classical Filon-type method (grey), scaled by ω^3.

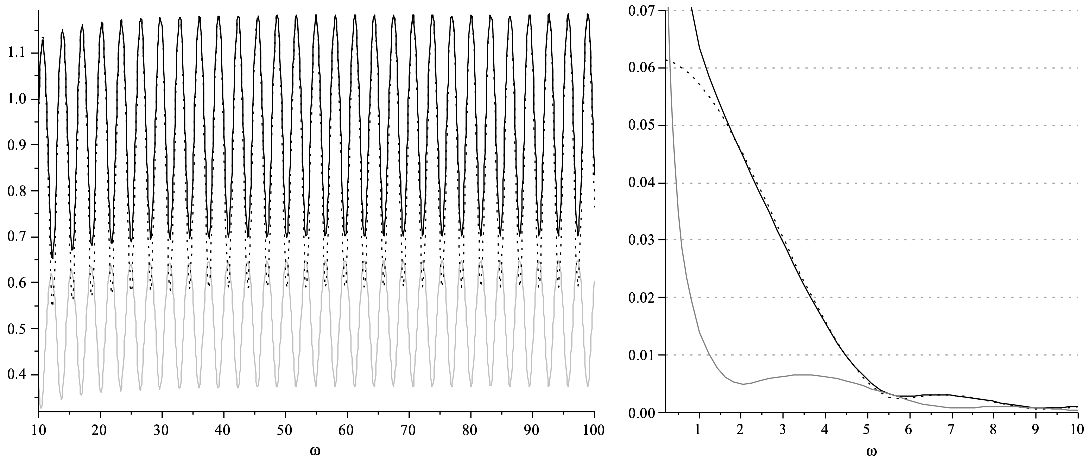
only needs two. We expect this to be at the cost of not that good approximation properties, this is indeed confirmed by experiments, see figure 1. Note that the crests of the curve of one method seems to correspond with the troughs of the other, much like what was pointed out by Iserles & Nørsett in [5]. This behaviour will be explained in section 5.

The key element in a discussion of the efficiency of this method is the need for moments. Recall that a classical asymptotic method needs no moments, but it breaks down for small ω. On the other hand a classical Filon-type method can be made precise also for moderately sized ω, but at the cost of moments. Moreover a Filon-type method needs a minimum of $2p$ moments to obtain asymptotic order p. The combined Filon/asymptotic method is situated between the Filon-type method and the asymptotic method, both in spirit and in terms of requirements. For example can this method obtain any asymptotic order as well as accuracy for moderately sized ω with the use of only two moments, whereas the asymptotic nature of the method ensures that it will break down for $\omega \to 0$(altough not so dramatically as the asymptotic method). The usefulness is here dictated by the cost of calculating moments, as well as the cost of calculating $\sigma_m[f]$ and its derivatives. Moreover, by using the same number of moments, we postulate that a combined Filon/asymptotic method can attain higher accuracy than a classical Filon-type method, this will be discussed in section 5.1. Example 3 shows how a combined Filon/asymptotic method performs better that a classical Filon-type method with approximately the same input data.

Example 3. *Once again we wish to compute*

$$\int_{-1}^{1} \frac{e^{i\omega x}}{2+x} \mathrm{d}x,$$

but this time we include internal nodes. Interpolating $\sigma_1(x) = -1/(2+x)^2$ *at the nodes*



(a) (b)

Figure 2: The absolute value of the error for the combined Filon/asymptotic method with interpolation nodes $[-1, 0, 1]$(black), and $[-1, -1/3, 1/3, 1]$(grey). Error for the classical Filon-type method is plotted as a dotted line. In (a) error is scaled by ω^3 to show asymptotic behaviour, whereas in (b) no scaling is done.

$[-1, 0, 1]$, and $[-1, -1/3, 1/3, 1]$ will result in schemes requiring three and four moments respectively, that means comparable to the classical Filon-type method from example 2. Both methods have asymptotic order 2. Comparing this to the pure Filon-type method shows that the first method has almost exactly the same behaviour as ω increases(this will be explained in section 5), whereas the second one has significantly smaller error, see figure 2. From figure 2 (b) we also see that the combined Filon/asymptotic methods perform well for moderately sized ω, but as ω approaches zero the methods will inevitably fail.

3.1 The combined Filon/asymptotic method with stationary points

Extending the method to cater for stationary point is fairly straightforward given Lemma 1. Assume in the following that ξ is the only stationary point of order r in $[-1, 1]$. This requirement is not crucial, it will just simplify otherwise horrific expressions. Applying the generalised Filon method Q_p^F on the expansion (11) yields the generalised combined Filon/asymptotic method

$$\begin{aligned} Q_{p,s}^{FA}[f] = & \sum_{j=0}^{r-1} \frac{1}{j!} \mu_j(\omega;\xi) \sum_{m=0}^{s-1} \frac{1}{(-i\omega)^m} \rho_m^{(j)}[f](\xi) \\ & - \sum_{m=0}^{s-1} \frac{1}{(-i\omega)^{m+1}} \left[\frac{e^{i\omega g(1)}}{g'(1)} \left(\rho_m[f](1) - \sum_{j=0}^{r-1} \frac{1}{j!} \rho_m[f]^{(j)}(\xi)(1-\xi)^j \right) \right. \\ & \qquad \left. - \frac{e^{i\omega g(-1)}}{g'(-1)} \left(\rho_m[f](-1) - \sum_{j=0}^{r-1} \frac{1}{j!} \rho_m[f]^{(j)}(\xi)(-1-\xi)^j \right) \right] \\ & + \frac{1}{(-i\omega)^s} Q_p^F[\rho_s[f]] \end{aligned} \tag{16}$$

$\rho_m[f]$ are defined as in equation (12). Recall that Q_p^F is constructed by interpolating f in the endpoints and ξ (c_1, c_ν and c_q) with multiplicities m_1, m_ν and m_q respectively. For the proposed method we have the following theorem:

Theorem 4. *Assume $g'(\xi) = \cdots = g^{(r)}(\xi) = 0$, $g^{(r+1)}(\xi) \neq 0$ and $g'(x) \neq 0$ for $x \in [-1,1]\backslash\{\xi\}$. Let Q_p^F be a generalised Filon method where $m_1, m_\nu \geq p$ and $m_q \geq p\,(r+1)$. For the generalised combined Filon/asymptotic method $Q_{p,s}^{FA}$ constructed with Q_p^F, applied to any smooth f it is true that*

$$Q_{p,s}^{FA}[f] - I_g[f] \sim \mathcal{O}(\omega^{-p-s-1/(r+1)}), \quad \omega \to \infty$$

Proof. Completely analogous to the proof of Theorem 3 we get

$$\begin{aligned} Q_{p,s}^{FA}[f] - I_g[f] &\sim \frac{1}{(-i\omega)^s}\left(Q_p^F[\rho_s[f]] - \int_{-1}^{1} \rho_s[f](x)e^{i\omega g(x)}\mathrm{d}x\right) \\ &\sim \frac{1}{(-i\omega)^s}\mathcal{O}(\omega^{-p-1/(r+1)}) = \mathcal{O}(\omega^{-p-s-1/(r+1)}), \end{aligned}$$

where the last line is an application of Theorem 2. □

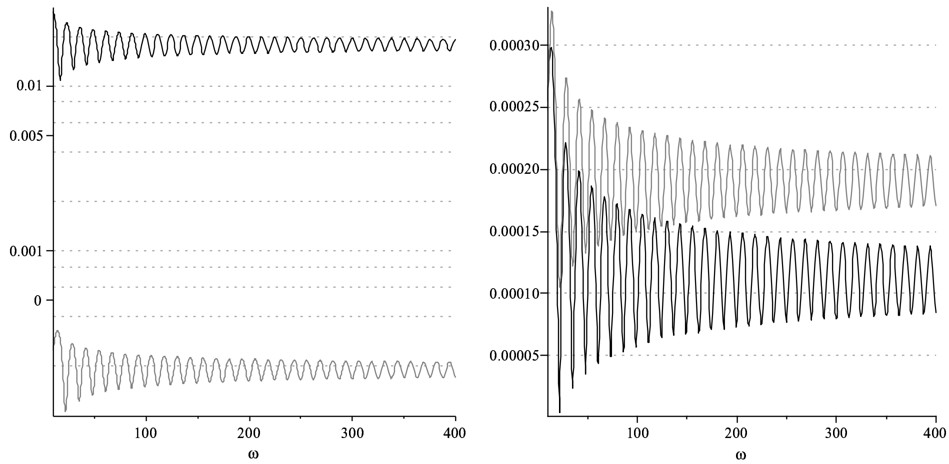
Example 4. *The simplest case is a problem with only one stationary point ξ of order one, expanded with one term(as in equation (9)). The combined Filon/asymptotic method (16) written out is then*

$$\begin{aligned} Q_{p,1}^{FA}[f] =& \mu_0(\omega)f(\xi) + \frac{1}{i\omega}\left(\frac{f(1)-f(\xi)}{g'(1)}e^{i\omega g(1)} - \frac{f(-1)-f(\xi)}{g'(-1)}e^{i\omega g(-1)}\right) \\ &- \frac{1}{i\omega}Q_p^F\left[\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)-f(\xi)}{g'(x)}\right] \end{aligned} \tag{17}$$

Example 5. *The oscillator of the integral*

$$\int_{-1}^{1} e^x e^{i\omega \frac{1}{2}x^2}\mathrm{d}x$$

has an order one stationary point at $x = 0$. We interpolate $\rho_1[f](x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)-f(\xi)}{g'(x)} = \frac{xe^x - e^x + 1}{x^2}$ at the nodes $[-1, 0, 1]$ (using l'Hospital's rule to obtain the value at the stationary point). The interpolant gives a combined Filon/asymptotic scheme on the form of (17). This scheme has a theoretical asymptotic order of $3/2$, which seems to be confirmed by experiments (see figure 3). The proposed scheme needs three moments for the computation whereas a classical Filon-type method requires a total of eight to obtain the same asymptotic order. Figure 3 (a) shows that the proposed method has a much higher asymptotic error constant than the classical Filon-type method, however do we only need to add two interpolation nodes to beat it, see figure 3 (b).



(a) (b)

Figure 3: The absolute value of the error for the combined Filon/asymptotic method(black) with (a) $\mathbf{c} = [-1, 0, 1]$ (logarithmic scale) and (b) $\mathbf{c} = [-1, -1/2, 0, 1/2, 1]$, and a classical Filon-type method(grey), all methods of order 3/2, applied to the problem $\int_{-1}^{1} e^x e^{i\omega \frac{1}{2} x^2} \mathrm{d}x$. All curves are scaled by $\omega^{\frac{5}{2}}$. Logarithmic scale is used in (a) in order to properly represent both curves in the same plot.

4 Extension to the multivariate case

Taking the step into the multivariate case presents us with a whole set of complications. For example will we have to take into account not only stationary points, x s.t $\nabla g(x) = 0$, but also points of resonance, boundary points where ∇g is orthogonal to the boundary. For general smooth boundaries resonance will necessarily be a problem, in this case theory is not yet fully developed. For oscillatory integrals on simplices and polygons we refer to [7] for a theoretical treatment.

We will restrict our treatment of the multivariate case to an example to demonstrate the feasibility of the Filon/asymptotic approach. Our model problem is the oscillatory integral on a square with an affine oscillator:

$$I[f] = \int_{-1}^{1} \int_{-1}^{1} f(x, y) e^{i\omega(\kappa_1 x + \kappa_2 y)} \mathrm{d}y \mathrm{d}x \tag{18}$$

In this case the non-resonance condition becomes simply $\kappa_1, \kappa_2 \neq 0$. Subject to the non-resonance condition the asymptotic behaviour of the integral will, analogous to the univariate case, be determined by information at the corner points.

The simplest possible Filon-type method is obtained by interpolating f in the four corners of the domain(four unknowns), resulting in a method of asymptotic order 2. Stepping up a level in asymptotic order requires us to interpolate f and its gradient at the corners(12 unknowns), resulting in a method of asymptotic order 3.

Applying integration by parts twice, first on the inner integral in (18), yields

$$\begin{aligned}
I[f] =& \frac{1}{(i\omega)^2\kappa_1\kappa_2}\Big[f(1,1)e^{i\omega(\kappa_1+\kappa_2)} - f(-1,1)e^{i\omega(-\kappa_1+\kappa_2)} \\
& - f(1,-1)e^{i\omega(\kappa_1-\kappa_2)} + f(-1,-1)e^{i\omega(-\kappa_1-\kappa_2)}\Big] \\
& - \frac{1}{(i\omega)^2\kappa_1\kappa_2}\Big[\int_{-1}^{1}\frac{\partial}{\partial x}f(x,1)e^{i\omega(\kappa_1 x+\kappa_2)}\mathrm{d}x - \int_{-1}^{1}\frac{\partial}{\partial x}f(x,-1)e^{i\omega(\kappa_1 x-\kappa_2)}\mathrm{d}x\Big] \\
& - \frac{1}{i\omega\kappa_2}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial}{\partial y}f(x,y)e^{i\omega(\kappa_1 x+\kappa_2 y)}\mathrm{d}y\mathrm{d}x.
\end{aligned} \tag{19}$$

This is the first step in an asymptotic expansion of the integral (18), the integral remainders all decay like ω^{-3}. This calculation presents us with several ways of applying Filon quadrature on the remainder term to arrive at a combined Filon/asymptotic method.

- We could apply Filon quadrature on all terms, that is two univariate, and one bivariate integral.
- The univariate integrals can be expanded, and a Filon-type quadrature applied to the bivariate integral.
- The bivariate integral could be expanded further, which would leave us with four univariate integrals on which a Filon-type method could be used.
- Finally, by switching the order of integration before expanding the variables will be permuted.

As we see, the possibilities are virtually endless, and so is the potential complexity of the resulting expressions. Example 6 shows the application of a combined Filon/asymptotic method on a simple 2-D problem.

Example 6. *Expanding the univariate integrals in equation (19) and applying the most basic Filon-type quadrature Q_2^F on the bivariate remainder integral gives us a combined Filon/asymptotic method with asymptotic order* 3.

$$\begin{aligned}
Q^{AF}[f] =& \frac{1}{(i\omega)^2\kappa_1\kappa_2}\Big[f(1,1)e^{i\omega(\kappa_1+\kappa_2)} - f(-1,1)e^{i\omega(-\kappa_1+\kappa_2)} \\
& \qquad - f(1,-1)e^{i\omega(\kappa_1-\kappa_2)} + f(-1,-1)e^{i\omega(-\kappa_1-\kappa_2)}\Big] \\
& - \frac{1}{(i\omega)^3\kappa_1^2\kappa_2}\Big[\frac{\partial}{\partial x}f(1,1)e^{i\omega(\kappa_1+\kappa_2)} - \frac{\partial}{\partial x}f(-1,1)e^{i\omega(-\kappa_1+\kappa_2)} \\
& \qquad - \frac{\partial}{\partial x}f(1,-1)e^{i\omega(\kappa_1-\kappa_2)} + \frac{\partial}{\partial x}f(-1,-1)e^{i\omega(-\kappa_1-\kappa_2)}\Big] \\
& - \frac{1}{i\omega\kappa_2}Q_2^F\Big[\frac{\partial}{\partial y}f(x,y)\Big]
\end{aligned}$$

Applying the method on the problem with $f(x,y) = e^{x+y}$

$$\int_{-1}^{1}\int_{-1}^{1} e^{x+y}e^{i\omega(x+y)}\mathrm{d}y\mathrm{d}x$$

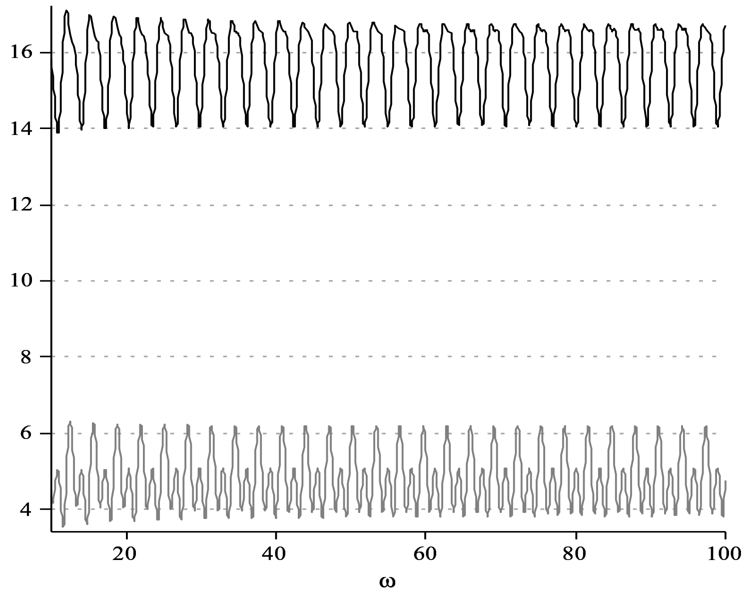


Figure 4: The absolute value of the error for the Filon/asymptotic method(top) and the classical Filon-type method(bottom) applied to the problem $\int_{-1}^{1}\int_{-1}^{1} e^{x+y}e^{i\omega(x+y)}\mathrm{d}y\mathrm{d}x$ scaled by ω^4.

we get

$$Q^{AF}[f] = (e^{2+2i\omega} + e^{-2-2i\omega} - 2)\Big(\frac{1}{\omega^2} - \frac{i}{\omega^3}\Big) - \frac{1}{i\omega}Q_2^F\Big[f\Big]$$

Figure 4 shows how this method compares to the classical Filon-type method Q_3^F obtained by interpolating function values and gradients in all four corners. The asymptotic error constant of the classical Filon-type method is much smaller, but at the cost of 12 *moments, compared to* 4 *for the combined Filon/asymptotic method.*

5 Error estimates

In Example 2 we observed how the troughs in the error plot for a particular Filon/asymptotic method seems to correspond with the peaks of a classical Filon-type method. This is exactly the same observation Iserles and Nørsett made in [5], but then for two different Filon-type methods. The behaviour we have observed can be explained in a similar way.

Assume in the following that $g'(x) \neq 0, \;\; -1 \leq x \leq 1$. From the discussion on the asymptotic order of a Filon-type method and equation (5) it is clear that

$$Q_p^F[f] - I_g[f] \sim \frac{e_p^F[f]}{\omega^{p+1}} + \mathcal{O}(\omega^{-p-2}).$$

$e_p^F[f]$ is basically the next term in the expansion of $f - \tilde{f}$:

$$e_p^F[f] = \frac{e^{i\omega g(1)}}{g'(1)}[\sigma_p[\tilde{f}](1) - \sigma_p[f](1)] - \frac{e^{i\omega g(-1)}}{g'(-1)}[\sigma_p[\tilde{f}](-1) - \sigma_p[f](-1)]. \tag{20}$$

By arguing that $\sigma_p[f] = \frac{f^{(p)}}{(g')^p} +$ a linear combination of $f^{(k)}$, $\;k = 0, \ldots, p-1$, one states that for a Filon-type method the *asymptotic error constant* $|e_p^F|$ can be estimated by

$$\Lambda_-^F[f] \leq |e_p^F[f]| \leq \Lambda_+^F[f],$$

where

$$\Lambda_\pm^F[f] = \left| \frac{|\tilde{f}^{(p)}(1) - f^{(p)}(1)|}{|g'(1)|^{p+1}} \pm \frac{|\tilde{f}^{(p)}(-1) - f^{(p)}(-1)|}{|g'(-1)|^{p+1}} \right|.$$

The exact same reasoning can be used to estimate the asymptotic error constant for a combined Filon/asymptotic method $Q_{p,s}^{AF}$. Keeping in mind that the asymptotic order of this method is $p+s$ we can write

$$Q_{p,s}^{FA}[f] - I_g[f] \sim \frac{e_{p,s}^{FA}[f]}{\omega^{p+s+1}} + \mathcal{O}(\omega^{-p-s-2}),$$

where

$$e_{p,s}^F[f] = \frac{e^{i\omega g(1)}}{g'(1)}[\tilde{\sigma}_s[f]^{(p)}(1) - \sigma_s[f]^{(p)}(1)] - \frac{e^{i\omega g(-1)}}{g'(-1)}[\tilde{\sigma}_s[f]^{(p)}(-1) - \sigma_s[f]^{(p)}(-1)], \qquad (21)$$

giving

$$\Lambda_-^{FA}[f] \leq |e_{p,s}^{FA}[f]| \leq \Lambda_+^{FA}[f]$$

with

$$\Lambda_\pm^{FA}[f] = \left| \frac{|\tilde{\sigma}_s[f]^{(p)}(1) - \sigma_s[f]^{(p)}(1)|}{|g'(1)|^{p+1}} \pm \frac{|\tilde{\sigma}_s[f]^{(p)}(-1) - \sigma_s[f]^{(p)}(-1)|}{|g'(-1)|^{p+1}} \right|.$$

Example 7. *Example 2 concerns the problem $\int_{-1}^{1} \frac{e^{i\omega x}}{2+x} \mathrm{d}x$, whereby applying a Filon-type method we obtain*

$$\tilde{f}(x) = -\frac{1}{9}x^3 + \frac{2}{9}x^2 - \frac{2}{9}x + \frac{4}{9} \qquad \text{and} \qquad [\Lambda_-^F, \Lambda_+^F] = [\frac{16}{27}, \frac{32}{27}].$$

The combined Filon/asymptotic method has

$$\tilde{\sigma}_1[f](x) = \frac{4}{9}x - \frac{5}{9} \qquad \text{and} \qquad [\Lambda_-^{FA}, \Lambda_+^{FA}] = [\frac{32}{27}, \frac{52}{27}].$$

These estimates explain the most significant features of Figure 1. For the schemes in Example 3 we have:

$$\begin{array}{lll} \mathbf{c} = [-1, 0, 1]: & \tilde{\sigma}_1[f](x) = -\frac{11}{36}x^2 + \frac{4}{9}x - \frac{1}{4}, & [\Lambda_-^{FA}, \Lambda_+^{FA}] = [\frac{19}{27}, \frac{32}{27}] \\ \mathbf{c} = [-1, -\frac{1}{3}, \frac{1}{3}, 1]: & \tilde{\sigma}_1[f](x) = \frac{248}{1225}x^3 - \frac{391}{1225}x^2 + \frac{2668}{11025}x - \frac{2606}{11025}, & [\Lambda_-^{FA}, \Lambda_+^{FA}] = [\frac{12416}{33075}, \frac{21472}{33075}] \end{array}$$

These calculations fits well with what has been observed, note in particular how the method with $\mathbf{c} = [-1, 0, 1]$ closely matches the classical Filon-type method.

5.1 Comparing the classical Filon and Filon/asymptotic methods

Now we must address one particular question: Will a combined Filon/asymptotic method get better accuracy than the classical Filon-type method from the same information? For simplicity, consider the case where $g(x) = x$, and also assume derivatives of f are cheaply available. The maximum error for a Filon-type method and a combined Filon/asymptotic method, both of asymptotic order p, as ω becomes large are then

$$\begin{aligned}
\Lambda_+^F[f] &= |\tilde{f}^{(p)}(1) - f^{(p)}(1)| + |\tilde{f}^{(p)}(-1) - f^{(p)}(-1)| \\
\Lambda_+^{FA}[f] &= |\tilde{\sigma}_s[f]^{(p-s)}(1) - \sigma_s[f]^{(p-s)}(1)| + |\tilde{\sigma}_s[f]^{(p-s)}(-1) - \sigma_s[f]^{(p-s)}(-1)| \\
&= |\tilde{\sigma}_s[f]^{(p-s)}(1) - f^{(p)}(1)| + |\tilde{\sigma}_s[f]^{(p-s)}(-1) - f^{(p)}(-1)|
\end{aligned}$$

where $\tilde{\sigma}_s[f]$ is the interpolant of $f^{(s)}$. We see that both methods have an error which is determined by the interpolant's ability to approximate the pth derivative of f at the endpoints. The error constant in the Filon-type method comes from interpolating f and differentiating the interpolant, the combined approach takes s derivatives, interpolates, then differentiates. It seems reasonable that when interpolating, the ability to more freely chose placement of the nodes will also result in a better approximation of the pth derivative. We wish to explore this a bit further.

What can we gain by using $2p$ nodes distributed equidistantly, including endpoints, to approximate the error in a $p-1$ term asymptotic expansion, that is a $Q_{1,p-1}^{AF}$-type method, compared to a Filon-type method of asymptotic order p of minimum complexity Q_p^F? These are two methods that both are of asymptotic order p and use $2p$ moments. Q_p^F requires p data at each endpoint, then it is well known that the error of the Hermite interpolation is

$$\tilde{f}(x) - f(x) = \frac{f^{(2p)}(c_1)}{(2p)!}(x+1)^p(x-1)^p,$$

where $c_1 \in [-1, 1]$. Then from the Rodrigues' formula[1]

$$\tilde{f}^{(p)}(x) - f^{(p)}(x) = \frac{f^{(2p)}(c_1)}{(2p)!}P_p(x)2^p p!,$$

with $P_p(x)$ being the pth Legendre polynomial. As $|P_n(\pm 1)| = 1$ we have

$$\Lambda_+^F[f] = 2^{p+1}p!\frac{|f^{(2p)}(c_1)|}{(2p)!} = |f^{(2p)}(c_1)|\frac{2^{1-p}\sqrt{\pi}}{\Gamma(p+\frac{1}{2})} \tag{22}$$

For the $Q_{1,p-1}^{AF}$-type method, we consider the case with $n+1$ equidistant nodes, including endpoints. We interpolate $\sigma_{p-1}[f]$, and the interpolation error is now:

$$\tilde{\sigma}_{p-1}[f](x) - f^{(p-1)}(x) = \frac{f^{(p-1+n+1)}(c_2)}{(n+1)!}\prod_{i=0}^{n}(x - 1 + i\frac{2}{n}),$$

for $c_2 \in [-1, 1]$. This simplifies to

$$\tilde{\sigma}_{p-1}[f](x) - f^{(p-1)}(x) = \frac{f^{(p+n)}(c_2)}{(n+1)!}\frac{2^{n+1}\Gamma(\frac{n}{2}(x+1))}{n^{n+1}\Gamma(\frac{n}{2}(x-1))}.$$

Differentiating gives

$$\tilde{\sigma}_{p-1}[f]'(x) - f^{(p)}(x) = \frac{f^{(p+n)}(c_2)}{(n+1)!}\frac{2^n}{n^n}\frac{(\Psi(\frac{n}{2}(x+1)+1) - \Psi(\frac{n}{2}(x-1))\Gamma(\frac{n}{2}(x+1)+1)}{\Gamma(\frac{n}{2}(x-1))},$$

with Ψ being the digamma function. The limit of the above expression as x tends to ± 1 can be found with a bit of effort:

$$\lim_{x\to\pm 1}[\tilde{\sigma}_{p-1}[f]'(x) - f^{(p)}(x)] = f^{(p+n)}(c_2)(\pm 1)^n \frac{2^n}{(n+1)n^n}.$$

Now

$$\Lambda_+^{AF}[f] = |f^{(p+n)}(c_2)|\frac{2^{n+1}}{(n+1)n^n} \tag{23}$$

For the case where the two methods use the same moments $n = 2p-1$, and then

$$\Lambda_+^{AF}[f] = |f^{(3p-1)}(c_2)|\frac{2^{2p}}{2p\cdot(2p-1)^{2p-1}}$$

Now we investigate the relative sizes of the two asymptotic error constants

$$\frac{\Lambda_+^{AF}[f]}{\Lambda_+^{F}[f]} = \frac{|f^{(3p-1)}(c_2)|\frac{2^{2p}}{(2p)(2p-1)^{2p-1}}}{|f^{(2p)}(c_1)|\frac{2^{1-p}\sqrt{\pi}}{\Gamma(p+\frac{1}{2})}} = \frac{|f^{(3p-1)}(c_2)|}{|f^{(2p)}(c_1)|}\frac{8^p}{4}\frac{\Gamma(p+1/2)}{\sqrt{\pi}p(2p-1)^{2p-1}}.$$

For $p = 1$, ignoring the derivatives, the ratio is one, and for increasing p it is decreasing. Stirling's formula gives the behaviour for large p.

$$\frac{\Lambda_+^{AF}[f]}{\Lambda_+^{F}[f]} = \frac{|f^{(3p-1)}(c_2)|}{|f^{(2p)}(c_1)|}\frac{8^p}{2\sqrt{2}}\frac{(2p+1)^{p+1}}{p(2p-1)^{2p-1}}e^{-p-1}, \qquad p\to\infty$$

The significance of the above calculations is most easily appreciated through a plot. Figure 5.1 shows that, assuming the derivatives of f are of the same order of magnitude, the combined Filon/asymptotic method will have a smaller error constant when using the same number of moments.

Example 8. *As a final little calculation we once again investigate Example 3 and the close match between the* $\mathbf{c} = [-1, 0, 1]$ *combined Filon/asymptotic method and the classical Filon-type method of asymptotic order* 2. *Assuming derivatives are of order* 1 *equation (22) gives*

$$\Lambda_+^{F}[f] \sim \frac{\sqrt{\pi}}{2\frac{3}{4}\sqrt{\pi}} = \frac{2}{3}$$

Equation (23) gives for the $\mathbf{c} = [-1, 0, 1]$ *combined Filon/asymptotic method*

$$\Lambda_+^{AF}[f] \sim \frac{2^3}{3\cdot 2^2} = \frac{2}{3}$$

This shows that what we observe is really an embodiment of a more general phenomenon regarding the relative strengths of these methods.

We must remark that although the proposed method apparently performs better, it is by no means optimal. The freedom to choose interpolation nodes could be used to minimise the error, placing nodes closer to the boundary would generally be better, but this depends on the size of ω. In the limit $\omega \to \infty$, placing all the nodes at the boundary, increasing the asymptotic order would be best. This makes the whole discussion about asymptotic error constants sligthly artificial.

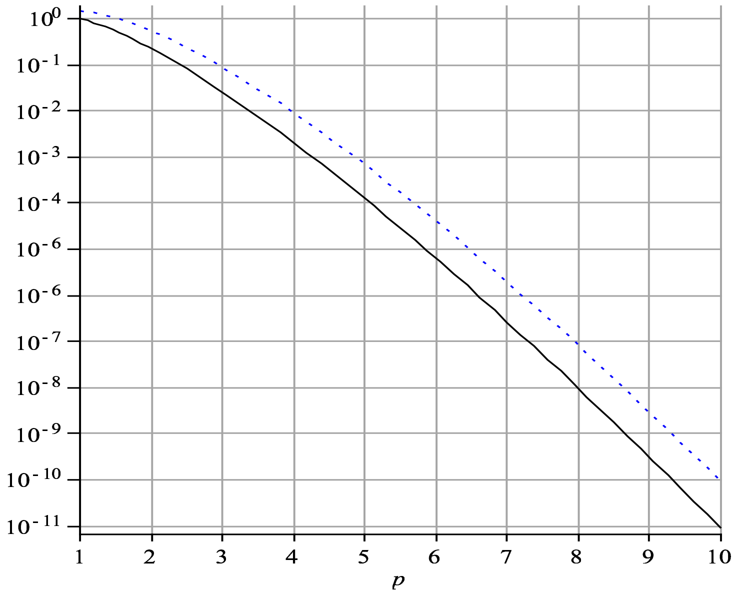


Figure 5: Log-plot of the ratio $\frac{8^p}{4}\frac{\Gamma(p+1/2)}{\sqrt{\pi}p(2p-1)^{2p-1}}$. The dotted line is Stirling's approximation.

6 Conclusion

We have demonstrated the feasibility of combining the asymptotic expansion of highly oscillatory integrals and Filon-type methods. Experiments as well as theoretical calculations show that the combined method can achieve better precision than the classical Filon-type method with more or less the same information. The extra cost of the combined method lies mainly in more complicated expressions, especially for cases with several stationary points or in the multivariate case. In order to make a combined method for more general oscillatory integrals we must have an asymptotic expansion with an oscillatory integral remainder, this might also be a shortcoming of the approach.

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