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A Runge-Kutta Method for Index 1 Stochastic Differential-Algebraic Equations with Scalar Noise

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Abstract

The paper deals with the numerical treatment of stochastic differential-algebraic equations of index one with a scalar driving Wiener process. Therefore, a particularly customized stochastic Runge-Kutta method is introduced. Order conditions for convergence with order 1.0 in the mean-square sense are calculated and coefficients for some schemes are presented. The proposed schemes are stiffly accurate and applicable to nonlinear stochastic differential-algebraic equations. As an advantage they do not require the calculation of any pseudo-inverses or projectors. Further, the mean-square stability of the proposed schemes is analyzed and simulation results are presented bringing out their good performance.

Key words: Stochastic differential-algebraic equation, stochastic Runge-Kutta method, stiffly accurate, mean-square convergence, mean-square stability

1 Introduction

Stochastic differential equations (SDEs) have turned out to be the appropriate instrument to deal with stochastically disturbed ordinary differential equations (ODEs). But in many areas of applications like mechanical multibody systems or network simulation one is not confronted with ODEs but with differential-algebraic equations (DAEs). These are differential equations on manifolds and their stochastic counterparts are called stochastic differential-algebraic equations (SDAEs). One important example, where SDAEs typically arise, is the modelling of electronic circuits that are disturbed by so called electrical noise.

SDAEs are a generalization of both, DAEs and SDEs. While the theory and numerical treatment of DAEs is well-understood (see e.g. [2,6,7]) and a lot of research has been devoted to the field of SDEs (see e.g. [9,10]), the understanding of SDAEs is still in its infancy. However, first results have been obtained in recent years. For example, Schein and Denk consider linear SDAEs and propose a two-step as well as a generalized Euler method for their numerical treatment in [16,17]. This two-step method is restricted to SDAEs with additive noise and therefore attains strong convergence order 1.0 while the generalized Euler method can be applied to linear SDAEs and has strong order 0.5. Moreover, Schein gives existence and uniqueness results for linear SDAEs in [16]. On the other hand, Winkler proves the existence and uniqueness for solutions of nonlinear SDAEs with index 1 in [19] and, together with Römisch, adapts some schemes for SDEs to make them applicable to SDAEs in [12,19]. They propose a drift-implicit Euler, a split-step backward Euler and a trapezoidal scheme which attain strong order 0.5 and a drift-implicit Milstein scheme of strong order 1.0. Furthermore, Sickenberger, Weinmüller and Winkler propose linear two-step Maruyama schemes for the numerical treatment of SDAEs with small noise in [18]. The proposed multistep Maruyama schemes in general attain at most strong order 0.5, however in case of small noise some higher orders of convergence are possible for a restricted range of stepsizes.

In the present paper we introduce stochastic Runge-Kutta (SRK) methods with convergence order 1.0 in the mean-square sense for SDAEs of index 1 with scalar noise. In contrast to the schemes proposed in [16,17], our SRK methods can also be applied to nonlinear SDAEs and are not restricted to the additive noise case. Compared to the Euler schemes, the trapezoidial rule and the multistep Maruyama schemes in [12,18,19] the SRK method proposed in the following attains a higher order of convergence. Further, the split-step backward Euler scheme and the Milstein scheme are not easy to implement because they need the explicit calculation of projectors, pseudo-inverses or derivatives. As an additional advantage, the new SRK method is derivative-free and does not require the calculation of any projectors or pseudo-inverses.

The paper is organized as follows: Firstly, we give a brief introduction to SDAEs of index 1. In Section 2 we propose a new SRK method and calculate order conditions for the coefficients of the SRK method assuring convergence for SDEs with order 1.0 in the mean-square sense. Then, we show how to modify an SRK scheme designed for SDEs such that it can be directly applied to SDAEs of index 1, and we present some stiffly accurate SRK schemes for order 1.0. Furthermore, a stability analysis is carried out for the proposed schemes w.r.t. mean-square stability in Section 4 and the corresponding domains of stability are determined. Finally, we present some simulation results that confirm the theoretical order of convergence for the proposed SRK schemes. The results are summarized by a short conclusion.

In the following, let $(\Omega, \mathcal{F}, P)$ be a complete probability space and let $(\mathcal{F}_t)_{t\geq 0}$ denote a filtration which fulfills the usual conditions [9]. We consider a d-dimensional index 1 SDAE system in integral form which can be written as

$$M \cdot X_t = M \cdot X_{t_0} + \int_{t_0}^{t} f(s, X_s)\,\mathrm{d}s + \int_{t_0}^{t} g(s, X_s)\,\mathrm{d}W_s \tag{1.1}$$

for $t \in \mathcal{I} = [t_0, T]$. Here, M is a constant $d \times d$ matrix which can be singular, $(W_t)_{t\geq 0}$ denotes a one-dimensional Wiener process which is adapted w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ and $X = (X_t)_{t\in\mathcal{I}}$ denotes the d-dimensional solution process. Further, the functions $f, g : \mathcal{I} \times \mathbb{R}^d \to \mathbb{R}^d$ are assumed to be Borel-measurable with $f \in C^{1,1}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d)$ and for the initial value we assume $X_{t_0} \in L^2(\Omega)$ and that X_{t_0} is independent of the Wiener process $(W_t)_{t\geq t_0}$. For ease of notation, we also make use of the short-hand notation for SDAE (1.1)

$$M \cdot \mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}W_t \tag{1.2}$$

with consistent inital value X_{t_0}. If the matrix M is non-singular, premultiplying by M^{-1} transforms (1.2) to a classical SDE. However, if M is singular, we have a system of SDAEs. To be more specific: If M is singular, there exist nonsingular matrices P and Q such that

$$PMQ = \begin{bmatrix} I_D & 0 \\ 0 & 0 \end{bmatrix}. \tag{1.3}$$

Premultiplying (1.2) by P and using the transformed variables $Q^{-1}X_t = (X_t^D, X_t^A)$ we obtain the system

$$\begin{aligned} dX_t^D &= f^D(t, X_t^D, X_t^A)dt + g^D(t, X_t^D, X_t^A)dW_t \\ 0 &= f^A(t, X_t^D, X_t^A)dt + g^A(t, X_t^D, X_t^A)dW_t \end{aligned} \tag{1.4}$$

consisting of differential equations and algebraic constraints. Here, X_t^D and X_t^A denote the differential and algebraic components of X_t respectively. Then, the following definition is given in [19]:

Definition 1.1 *The SDAE* (1.2) *is said to be an index 1 SDAE if*

(i) the noise sources do not appear in the constraints, and
(ii) the constraints are globally uniquely solvable for the algebraic variables.

Thus, the definition of index 1 SDAEs implies that $g^A \equiv 0$ and the remaining algebraic constraints $f^A(t, X_t^D, X_t^A) = 0$ can be solved with respect to X_t^A, i.e. there exists a function F^A such that $X_t^A = F^A(t, X_t^D)$. Inserting this into the differential part of (1.4), we obtain an SDE in X_t^D, which is sometimes referred to as the state space form (SSF):

$$\begin{aligned} dX_t^D &= f^D(t, X_t^D, F^A(t, X_t^D))dt + g^D(t, X_t^D, F^A(t, X_t^D))dW_t \\ X_t^A &= F^A(t, X_t^D) \end{aligned} \tag{1.5}$$

In Definition 1.1, the second condition is guaranteed, if the inverse of the Jacobian $f_z^A(t, y, z)$ is uniformely bounded w.r.t. t, y and z. If in addition f and g fulfill a global Lipschitz condition w.r.t. the state variable x and if g is continuous in t, then there exists a pathwise unique solution process X of SDAE (1.1) (see Theorem 4 in [19]).

Our next aim is to construct numerical schemes which give identical solutions when directly applied to the SDAE (1.1) and to the state space form (1.5). Therefore, we firstly introduce a new SRK method attaining order 1.0 for SDEs in the following section.

2 A stochastic Runge-Kutta method for SDEs

In this section, we consider a d-dimensional SDE system with a scalar driving Wiener process given by

$$\mathrm{d}Y_t = f(t, Y_t)\,\mathrm{d}t + g(t, Y_t)\,\mathrm{d}W_t \tag{2.1}$$

for $t \in \mathcal{I}$ with some initial value $Y_{t_0} \in L^2(\Omega)$. Let a discretization $\mathcal{I}_h = \{t_0, t_1, \dots, t_N\}$ with $t_0 \leq t_1 \leq \dots \leq t_N = T$ of the time interval $\mathcal{I}$ with step sizes $h_n = t_{n+1} - t_n$ and maximum step size h be given. Then, we say that a time discrete approximation process $(y_t)_{t \in \mathcal{I}_h}$ converges in the mean square sense with order p to the solution Y of SDE (2.1) at time T if there exists a constant $C > 0$, not depending on h, and some $\delta_0 > 0$ such that for $h \in]0, \delta_0]$

$$(\mathrm{E}(\|Y_T - y_T\|^2))^{1/2} \leq C h^p. \tag{2.2}$$

We consider the following s-stages stochastic Runge-Kutta method for SDE (2.1) defined by $y_0 = Y_{t_0}$ and

$$\begin{aligned} y_{n+1} = y_n &+ \sum_{i=1}^{s} \alpha_i h_n\, f(t_n + c_i\, h_n, H_i) \\ &+ \sum_{i=1}^{s} \left(\beta_i^{(1)} I_{(1),n} + \beta_i^{(2)} \frac{I_{(1,1),n}}{\sqrt{h_n}} + \beta_i^{(3)} \sqrt{h_n} \right) g(t_n + c_i\, h_n, H_i) \end{aligned} \tag{2.3}$$

for $n = 0, \ldots, N-1$, making use of the notation $y_n = y_{t_n}$ and with stages

$$H_i = y_n + \sum_{j=1}^{s} A_{ij} h_n \, f(t_n + c_j \, h_n, H_j) + \sum_{j=1}^{s} \left(B_{ij}^{(1)} I_{(1),n} + B_{ij}^{(2)} \frac{I_{(1,1),n}}{\sqrt{h_n}} + B_{ij}^{(3)} \sqrt{h_n} \right) g(t_n + c_j \, h_n, H_j) \tag{2.4}$$

for $i = 1, \ldots, s$. For some independent $\mathcal{N}(0,1)$-distributed random variables ξ_n, the random variables $I_{(1),n}$ and the iterated stochastic integrals $I_{(1,1),n}$ can be calculated by

$$I_{(1),n} = \sqrt{h_n} \cdot \xi_n \qquad \text{and} \qquad I_{(1,1),n} = \tfrac{1}{2}(I_{(1),n}^2 - h_n). \tag{2.5}$$

The coefficients of the SRK method (2.3) are presented by an extended Butcher tableau:

$$\begin{array}{c|c|c|c|c} c & A & B^{(1)} & B^{(2)} & B^{(3)} \\ \hline & \alpha^T & {\beta^{(1)}}^T & {\beta^{(2)}}^T & {\beta^{(3)}}^T \end{array} \tag{2.6}$$

Now, by applying the colored rooted tree theory for Itô SDEs given in [4,13,14], order conditions for the coefficients of the SRK method (2.3) can be easily calculated, making use of the vector $e = (1, \ldots, 1)^T \in \mathbb{R}^s$:

Theorem 2.1 *Let $f, g \in C^{1,3}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d)$. If the coefficients of the stochastic Runge–Kutta method (2.3) fulfill the equations*

1. $\alpha^T e = 1$
2. ${\beta^{(1)}}^T e = 1$
3. ${\beta^{(2)}}^T e = 0$
4. ${\beta^{(3)}}^T e = 0$
5. ${\beta^{(1)}}^T B^{(1)} e = \frac{\lambda}{2}$
6. ${\beta^{(3)}}^T B^{(3)} e = -\frac{\lambda}{2}$
7. ${\beta^{(2)}}^T B^{(3)} e + {\beta^{(3)}}^T B^{(2)} e = 1 - \lambda$
8. $\alpha^T B^{(3)} e = 0$
9. ${\beta^{(1)}}^T B^{(3)} e + {\beta^{(3)}}^T B^{(1)} e = 0$
10. ${\beta^{(2)}}^T B^{(2)} e = 0$
11. ${\beta^{(1)}}^T B^{(2)} e + {\beta^{(2)}}^T B^{(1)} e = 0$
12. ${\beta^{(3)}}^T A e = 0$

13. $$2{\beta^{(1)}}^T (B^{(1)}e)(B^{(2)}e) + 2{\beta^{(1)}}^T (B^{(1)}e)(B^{(3)}e) + {\beta^{(2)}}^T (B^{(1)}e)^2 + {\beta^{(2)}}^T (B^{(2)}e)^2 + {\beta^{(2)}}^T (B^{(2)}e)(B^{(3)}e) + {\beta^{(3)}}^T (B^{(1)}e)^2 + \frac{1}{2}{\beta^{(3)}}^T (B^{(2)}e)^2 + {\beta^{(3)}}^T (B^{(3)}e)^2 = 0$$
14. $${\beta^{(1)}}^T (B^{(1)}(B^{(2)}e)) + {\beta^{(1)}}^T (B^{(2)}(B^{(1)}e)) + {\beta^{(1)}}^T (B^{(1)}(B^{(3)}e)) + {\beta^{(1)}}^T (B^{(3)}(B^{(1)}e)) + {\beta^{(2)}}^T (B^{(1)}(B^{(1)}e)) + {\beta^{(2)}}^T (B^{(2)}(B^{(2)}e)) + \frac{1}{2}{\beta^{(2)}}^T (B^{(2)}(B^{(3)}e)) + \frac{1}{2}{\beta^{(2)}}^T (B^{(3)}(B^{(2)}e)) + {\beta^{(3)}}^T (B^{(1)}(B^{(1)}e))$$

$$+\frac{1}{2}\beta^{(3)^T}(B^{(2)}(B^{(2)}e))+\beta^{(3)^T}(B^{(3)}(B^{(3)}e))=0$$

for some $\lambda \in \mathbb{R}$ and if $c = Ae$, then the stochastic Runge–Kutta method (2.3) attains order 1.0 for the strong approximation of the solution of the Itô SDE (2.1) with scalar noise.

Proof. The SRK method (2.3) is covered by the general class of SRK methods proposed in [13]. Therefore, Proposition 5.2 in [13] based on the colored rooted tree analysis can be applied, which directly results in the order conditions given in Theorem 2.1. □

Remark 2.2 *Let $f, g \in C^{1,2}(\mathcal{I} \times \mathbb{R}^d, \mathbb{R}^d)$. Then, Conditions 1.–4. together with the condition $\beta^{(1)^T}B^{(1)}e + \frac{1}{2}\beta^{(2)^T}B^{(2)}e + \beta^{(3)^T}B^{(3)}e = 0$ are sufficient for an order 0.5 strong SRK method* (2.3).

Example 2.3 *The well-known stochastic trapezoidal rule [10] satisfies the order 0.5 conditions. The method is given by the following Butcher tableau:*

$$\begin{array}{c|cc|cc|cc|cc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
 & \frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0
\end{array}$$

Remark 2.4 *From the order conditions of Theorem 2.1 one can choose $B_{ij}^{(2)} = 0$ and $\beta_i^{(2)} = 0$ for $i, j = 1, \ldots, s$ in the case of $\lambda = 1$. Thus, the SRK method* (2.3) *can be significantly simplified.*

3 Stiffly accurate SRK methods applied to SDAEs

The SRK method (2.3) is called stiffly accurate if $y_{n+1} = H_s$, or equivalent, if $\alpha_i = A_{si}$ and $\beta_i^{(\nu)} = B_{si}^{(\nu)}$ for $i = 1, 2, \ldots, s$ and $\nu = 1, 2, 3$. Stiffly accurate SRK methods with a nonsingular coefficient matrix $A = (A_{ij})$ can be applied to the index 1 SDAE (1.2) with singular matrix M as follows

$$\begin{aligned}
M \cdot H_i = M \cdot y_n &+ \sum_{j=1}^{s} A_{ij} h_n \, f(t_n + c_j h_n, H_j) \\
&+ \sum_{j=1}^{s} \left(B_{ij}^{(1)} I_{(1),n} + B_{ij}^{(2)} \frac{I_{(1,1),n}}{\sqrt{h_n}} + B_{ij}^{(3)} \sqrt{h_n} \right) g(t_n + c_j h_n, H_j), \\
y_{n+1} = H_s &
\end{aligned} \tag{3.1}$$

for $i = 1, \ldots, s$ and $n = 0, 1, \ldots, N-1$, cf. [11]. Moreover, the order conditions given in Theorem 2.1 apply without modifications. This can be proved by the following argument: The linear transformations used to transfer (1.2) to (1.4) can be applied to the numerical solution as well, resulting in

$$\begin{aligned}
H_i^D &= y_n^D + \sum_{j=1}^{s} A_{ij} h_n \, f^D(t_n + c_j h_n, H_j^D, H_j^A) \\
&\qquad + \sum_{j=1}^{s} \left(B_{ij}^{(1)} I_{(1),n} + B_{ij}^{(2)} \frac{I_{(1,1),n}}{\sqrt{h_n}} + B_{ij}^{(3)} \sqrt{h_n} \right) g^D(t_n + c_j h_n, H_j^D, H_j^A), \qquad \text{(3.2a)} \\
0 &= \sum_{j=1}^{s} A_{ij} h_n \, f^A(t_n + c_j h_n, H_j^D, H_j^A), \qquad \text{(3.2b)} \\
y_{n+1}^D &= H_s^D, \qquad \text{(3.2c)} \\
y_{n+1}^A &= H_s^A. \qquad \text{(3.2d)}
\end{aligned}$$

If A is nonsingular, then (3.2b) can be solved with respect to H_j^A due to the index 1 condition. Thus $H_j^A = F^A(t_n + c_j h_n, H_j^D)$. Inserting this into (3.2a) gives exactly the same result as if the SRK method had been applied to the SSF (1.5) directly. Thus, the SRK method (3.1) attains the same strong order p for SDAE (1.1) as it possessed for SDE (2.1).

The requirement that A has to be nonsingular may be relaxed: If the first stage is explicit, i.e. if $H_1 = y_n$, it is sufficient to require a nonsingular submatrix $(A_{ij})_{i,j=2}^{s}$. The trapezoidal rule of Example 2.3 is an example of this.

When implicit methods are applied, care has to be taken to ensure the regularity of the solution of the nonlinear equations (3.1). In this paper, we restrict ourself to diagonal implicit methods, in which the coefficient matrices A and $B^{(3)}$ are lower triangular, while $B^{(1)}$ and $B^{(2)}$ are strictly lower triangular. Within these restrictions, from the order conditions given by Theorem 2.1, we have constructed five different order 1.0 SRK methods, presented in Table 1. The first (RK1W1) is a first order extension of the trapezoidal rule. Because of the FSAL property ($H_{1,n+1} = H_{3,n}$) this method is efficiently of only two stages. The next three (RK1W2–4) are extensions of the deterministic second order L-stable method of Alexander [1], two of them drift-implicit, and one drift-diffusion implicit. We like to draw attention to the second, which does not depend on the stochastic variable $I_{(1,1),n}$. For comparison, we finally present a second drift-diffusion-implicit method, RK1W5.

$$
\begin{array}{c|ccc|ccc|ccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
\hline
 & \frac{1}{2} & 0 & \frac{1}{2} & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0
\end{array}
$$

RK1W1: Reduces to the trapezoidal rule in the deterministic case.

$$
\begin{array}{c|ccc|ccc|ccc|ccc}
1-\frac{\sqrt{2}}{2} & 1-\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\frac{\sqrt{2}}{2} & 0 & 1-\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & \frac{\sqrt{2}}{2} & 0 & 1-\frac{\sqrt{2}}{2} & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
\hline
 & \frac{\sqrt{2}}{2} & 0 & 1-\frac{\sqrt{2}}{2} & 1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0
\end{array}
$$

RK1W2: Reduces to the Alexander order 2 method in the deterministic case.

$$
\begin{array}{c|ccc|ccc|ccc|ccc}
1-\frac{\sqrt{2}}{2} & 1-\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1-\frac{\sqrt{2}}{2} & 0 & 1-\frac{\sqrt{2}}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\
1 & \frac{\sqrt{2}}{2} & 0 & 1-\frac{\sqrt{2}}{2} & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
\hline
 & \frac{\sqrt{2}}{2} & 0 & 1-\frac{\sqrt{2}}{2} & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0
\end{array}
$$

RK1W3: As RK1W2, but with no use of $I_{(1,1),n}$.

$$
\begin{array}{c|ccc|ccc|ccc|ccc}
1-\frac{\sqrt{2}}{2} & 1-\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1-\frac{\sqrt{2}}{2} & 0 & 1-\frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 1-\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\
\hline
 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & 1-\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0
\end{array}
$$

RK1W4: As RK1W2 and RK1W3, additionally implicit in the diffusion.

$$
\begin{array}{c|ccc|ccc|ccc|ccc}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & 0 \\
1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\
\hline
 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0
\end{array}
$$

RK1W5: Alternative diffusion-implicit method.

Table 1
Stiffly accurate SRK-methods of order 1 in 3 stages.

4 Mean–square stability analysis

The stability analysis for SDEs is based on the following scalar linear test equation with multiplicative noise,

$$\mathrm{d}Y_t = \lambda\, Y_t \,\mathrm{d}t + \mu\, Y_t \,\mathrm{d}W_t\,, \quad Y_{t_0} = y_0 \in \mathbb{R} \setminus \{0\}\,, \tag{4.1}$$

for $t \geq t_0$ and with some constants $\lambda, \mu \in \mathbb{C}$. In order to analyze the stability of the proposed SRK method (2.3), we apply a stiffly accurate version of the method to test problem (4.1).
In this paper, we restrict our studies to the so–called mean–square stability (MS–stability) which denotes the analysis w.r.t. the second moment of the solution process of SDE (4.1) and the corresponding approximation process, respectively. See [3,5,8,10,15] for further details. The solution of SDE (4.1) is said to be (asymptotically) MS–stable if

$$\lim_{t \to \infty} \mathrm{E}(|Y_t|^2) = 0 \quad \Leftrightarrow \quad 2\,\Re(\lambda) + |\mu|^2 < 0 \tag{4.2}$$

holds for the coefficients $\lambda, \mu \in \mathbb{C}$. We point out that for $\mu = 0$ the stability condition (4.2) reduces to the well known deterministic stability condition $\Re(\lambda) < 0$.

We are now looking for conditions such that a numerical method applied to SDE (4.1) yields numerically stable solutions. As a consequence of the left hand side of the equivalence in (4.2), we say that a method is numerically MS–stable if the approximations y_n satisfy $\lim_{n \to \infty} \mathrm{E}\left(|y_n|^2\right) = 0$. Applying the numerical method to the linear test equation (4.1), we obtain

$$y_{n+1} = R_n(\hat{h}, k)\, y_n\,, \tag{4.3}$$

where $R_n(\hat{h}, k)$ denotes the stability function with the parametrization $\hat{h} = \lambda\, h$ and $k = \mu\sqrt{h}$ [5,8]. Then, calculating the mean–square norm of equation (4.3), we obviously yield MS–stability, if $\hat{R}(\hat{h}, k) := \mathrm{E}(|R_n(\hat{h}, k)|^2) < 1$. The set $\mathcal{R}_{MS} = \{(\hat{h}, k) \in \mathbb{C}^2 : \hat{R}(\hat{h}, k) < 1\} \subset \mathbb{C}^2$ is then called the domain of MS–stability of the method. Especially, $\mathcal{R}_{MS}$ is called region of stability in the case of $(\hat{h}, k) \in \mathbb{R}^2$ [5]. Now, the numerical method is said to be A–stable if the domain of stability of the test equation (4.1) is a subset of $\mathcal{R}_{MS}$. Since the domain of stability for $\lambda, \mu \in \mathbb{C}$ is not easy to visualize, we restrict our attention to figures presenting the region of stability with $\lambda, \mu \in \mathbb{R}$ in the $\hat{h}$–k^2–plane. Then, for fixed values of λ and μ, the set $\{(\lambda\, h, \mu^2\, h) \subset \mathbb{R}^2 : h > 0\}$ is a straight ray starting at the origin and going through the point (λ, μ^2). Clearly, varying the step size h corresponds to moving along this ray. For $\lambda, \mu \in \mathbb{R}$, the region of MS–stability for SDE (4.1) reduces to the area of the

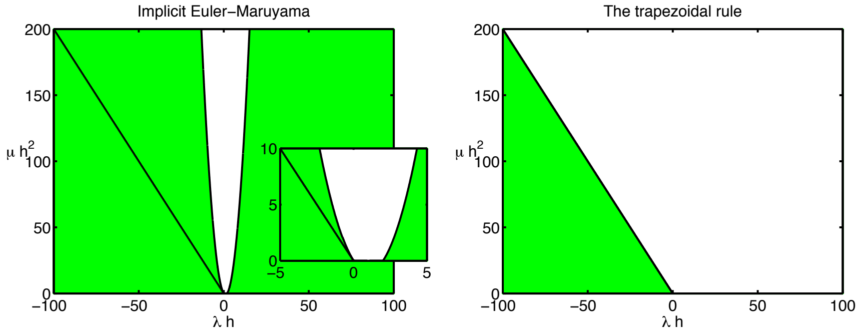

Fig. 1. Mean–square stability region for implicit Euler method and trapezoidal rule from Example 2.3 in the left and right figure, respectively.

$\hat{h}$–k^2–plane with the $\hat{h}$–axis as the lower bound and $k^2 < -2\hat{h}$ as the upper bound for $\hat{h} < 0$.

Next, we will find an expression for the stability function $R_n(\hat{h}, k)$ for an s-stage method: Let $H = (H_1, \ldots, H_s)^T$ and $\mathbf{1}_s = (1, \ldots, 1)^T \in \mathbb{R}^s$. Then (2.4) applied to (4.1) with equidistant step size $h = h_n$ becomes

$$H = \mathbf{1}_s y_n + h\lambda A H + \mu\Big(I_{(1),n} B^{(1)} + \frac{I_{(1,1),n}}{\sqrt{h}} B^{(2)} + \sqrt{h} B^{(3)}\Big) H.$$

Together with (2.5) and the parametrization $\hat{h} = \lambda h$ and $k = \mu\sqrt{h}$ this can be reformulated to

$$H = \Big(I_s - \hat{h}A - k\Big(\xi_n B^{(1)} + \frac{1}{2}(\xi_n^2 - 1) B^{(2)} + B^{(3)}\Big)\Big)^{-1} \mathbf{1}_s y_n.$$

Since the methods are stiffly accurate, that is $y_{n+1} = H_s$, the stability function becomes

$$R_n(\hat{h}, k) = \varepsilon_s^T \Big(I_s - \hat{h}A - k\Big(\xi_n B^{(1)} + \frac{1}{2}(\xi_n^2 - 1) B^{(2)} + B^{(3)}\Big)\Big)^{-1} \mathbf{1}_s \tag{4.4}$$

where $\varepsilon_s^T = (0, \ldots, 0, 1) \in \mathbb{R}^s$.

Based on this we give the stability function for 3–stages diagonally implicit SRK methods (2.3) with $A_{ij} = B_{ij}^{(3)} = 0$ for $j > i$ and $B_{ij}^{(1)} = B_{ij}^{(2)} = 0$ for $j \geq i$ in the Appendix. As a result, the considered SRK method is MS–stable for the linear test equation (4.1) on the domain

$$\begin{aligned} \mathcal{R}_{MS} = \{(\hat{h}, k) \in \mathbb{C}^2 :& \Gamma^2 + 2\Gamma\Sigma_2 + 6\Gamma\Sigma_4 + \Sigma_1^2 + 6\Sigma_1\Sigma_3 + 3\Sigma_2^2 \\ & + 30\Sigma_2\Sigma_4 + 15\Sigma_3^2 + 105\Sigma_4^2 < 1\}\,. \end{aligned} \tag{4.5}$$

with Γ and Σ_i, $i = 1, \ldots, 4$, given in the Appendix.

In the following, we present the regions of stability for the implicit Euler method [12,19] and the trapezoidal rule from Example 2.3, both representing

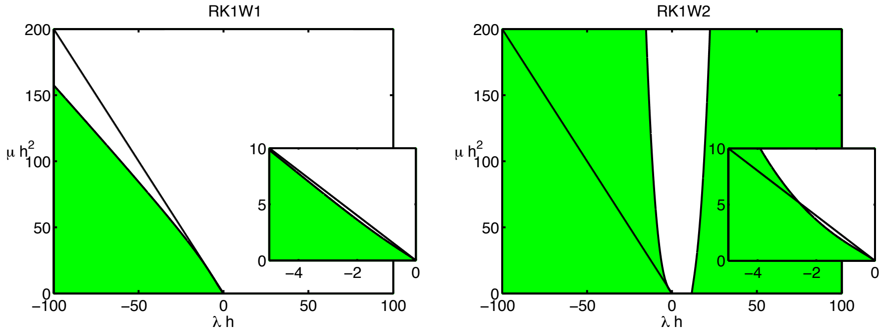

Fig. 2. Mean–square stability region for RK1W1 and RK1W2 in the left and right figure, respectively.

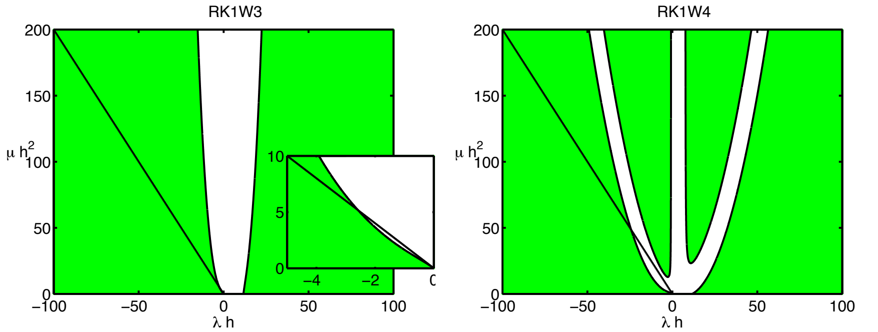

Fig. 3. Mean–square stability region for RK1W3 and RK1W4 in the left and right figure, respectively.

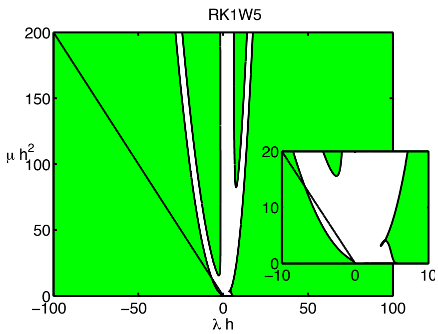

Fig. 4. Mean–square stability region for RK1W5.

order 0.5 methods, and for the proposed order 1.0 SRK methods RK1W1–RK1W5 in Figures 1–4, respectively. All regions of stability for the numerical method under consideration were plotted using the software Matlab. As mentioned before, the region of MS–stability for the solution of the test equation (4.1) is given by the area beneath the linear slope $k^2 = -2\hat{h}$. The coloured areas mark the region of numerical MS–stability for the corresponding method. It is a well–known result, that the implicit Euler method as well as the trapezoidal rule are A–stable methods [8]. However, the proposed order 1.0 methods do not have this eligible property, even though the methods RK1W2 and RK1W3 get close to A–stability when stability considerations are restricted to the real plane (cf. zoomed parts of the stability regions). The construction of A–stable order 1.0 methods may be subject to further research.

5 Simulation results

In this section, we examine the mean square error of the introduced SRK methods versus the used step sizes. Therefore, we compare the performance of these methods to that of the strong order 0.5 implicit Euler (IEu) and the trapezoidal rule (Trapez) from Example 2.3 when applied to some test equations. In the following, for each step size h under consideration we simulate 2000 trajectories with each of the considered schemes, in order to estimate the mean square error at $T = 1/16$, respectively. Then, the errors are plotted against the step sizes into a $\log_2$–$\log_2$–diagram. The absolute value of the slope of each of the resulting lines in the diagram gives the convergence order of each particular scheme.

5.1 Test example 1

We consider a test equation which is linear in the state variable given by

$$M \cdot \mathrm{d}X_t = (B \cdot X_t + s(t))\,\mathrm{d}t + G \cdot X_t\,\mathrm{d}W_t \tag{5.1}$$

with some consistent initial value $X_{t_0} \in \mathbb{R}^d$, where the matrix $G \in \mathbb{R}^{d \times d}$ has the special form $G = BV\left(\begin{smallmatrix} \tilde{B} & 0 \\ 0 & 0 \end{smallmatrix}\right)V^{-1}$ with $\tilde{B} \in \mathbb{R}^{\tilde{d} \times \tilde{d}}$, $\tilde{d} < d$. $M \in \mathbb{R}^{d \times d}$ is a singular and $B, V \in \mathbb{R}^{d \times d}$ are regular matrices, see also [11]. Further, we require that $s \in C^1(\mathcal{I}, \mathbb{R}^d)$. In the index 1 case, SDAE (5.1) can be decomposed into differential and algebraical equations

$$\mathrm{d}X_t^D = (L^{-1}X_t^D + L^{-1}s_1(t))\,\mathrm{d}t + L^{-1}\tilde{B}X_t^D\,\mathrm{d}W_t \tag{5.2}$$

$$0 = X_t^A + s_2(t) \tag{5.3}$$

where the transformations $V^{-1}B^{-1}MV = \left(\begin{smallmatrix} L & 0 \\ 0 & 0 \end{smallmatrix}\right)$ with $L \in \mathbb{R}^{\tilde{d} \times \tilde{d}}$ regular, $X_t = V(X_t^D, X_t^A)^T$ and $s(t) = BV(s_1(t), s_2(t))^T$ are used. If the matrices L^{-1} and $L^{-1}\tilde{B}$ commute, then we get the following analytical expression for the solution of SDE (5.2) (see [10]):

$$X_t^D = \Phi_{t,t_0}\left(X_{t_0}^D + \int_{t_0}^{t} \Phi_{u,t_0}^{-1} L^{-1} s_1(u)\,\mathrm{d}u\right) \tag{5.4}$$

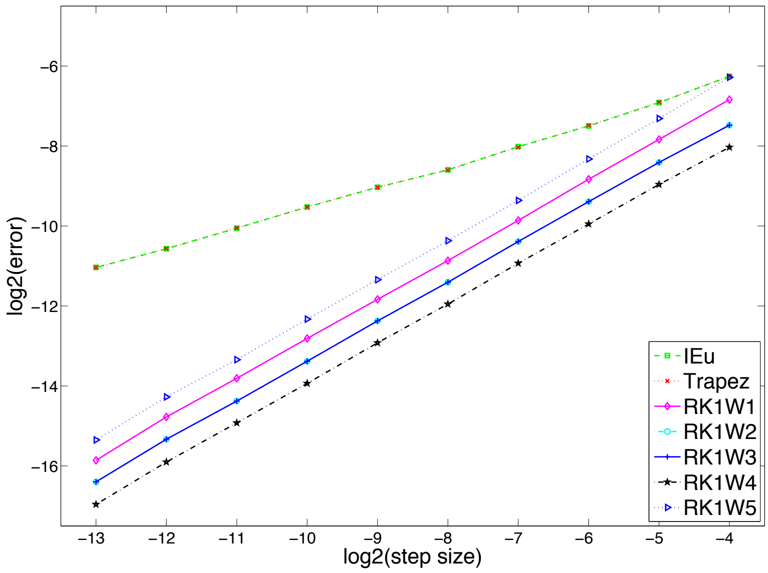

Fig. 5. Mean–square error vs. step size for test example 1.

with the fundamental solution

$$\Phi_{t,t_0} = \exp\left(\left(L^{-1} - \frac{1}{2}(L^{-1}\tilde{B})(L^{-1}\tilde{B})\right)(t - t_0) + L^{-1}\tilde{B}(W_t - W_{t_0})\right) \quad (5.5)$$

and with a consistent initial value $(X_{t_0}^D, X_{t_0}^A)^T = V^{-1}X_{t_0}$. Furthermore, it is obvious that the algebraical equations (5.3) can be solved for the algebraic variables X_t^A:

$$X_t^A = -s_2(t). \quad (5.6)$$

As a concrete example for our simulations we consider the 4-dimensional SDAE (5.1) where

$$M = \begin{pmatrix} -\frac{1}{10} & -\frac{12}{5} & \frac{9}{10} & \frac{1}{10} \\ \frac{3}{2} & 3 & 0 & -\frac{3}{2} \\ -\frac{3}{2} & -\frac{15}{2} & \frac{9}{2} & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Furthermore, the matrices $\tilde{B}$ and L are given by

$$\tilde{B} = \begin{pmatrix} -\frac{1}{8} & \frac{1}{15} & \frac{1}{8} \\ \frac{1}{30} & -\frac{13}{80} & 0 \\ \frac{1}{120} & -\frac{1}{45} & -\frac{59}{240} \end{pmatrix}, \quad L = \begin{pmatrix} \frac{3}{2} & 3 & 0 \\ 3 & -3 & \frac{9}{2} \\ -\frac{12}{15} & \frac{3}{2} & -\frac{9}{10} \end{pmatrix},$$

and we have the nonlinear function $s(t) = BV(0,0,0,\sin(t))^T$ and the initial value $X_{t_0} = V(\frac{1}{5}, \frac{1}{2}, \frac{7}{10}, 0)^T$ for $t_0 = 0$. For the simulation results presented in Figure 5 we considered the integration interval $[0, \frac{1}{16}]$ and step sizes $h = 2^{-4}, 2^{-5}, \ldots, 2^{-13}$. The convergence order 0.5 of the implicit Euler (IEu) method and the trapezoidal rule (Trapez) as well as the order 1.0 of the SRK methods (RK1W1–RK1W5) are clearly revealed. As expected, the SRK methods perform better than the other 0.5 schemes. For this example, the schemes RK1W2 and RK1W3 yield similar errors. This may be due to the fact that the drift dominates the solution and that both schemes have the same coefficient matrix A. Further, the scheme RK1W4, being L-stable for ODEs and implicit in $B^{(3)}$, produces the best results.

5.2 Test example 2

Secondly, we consider a nonlinear 2–dimensional test example of the form

$$M \cdot \mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + g(t, X_t)\,\mathrm{d}W_t \tag{5.7}$$

with the singular matrix

$$M = \begin{pmatrix} db^2 - cab & dba - ca^2 \\ cb^2 + dab & cba + da^2 \end{pmatrix}$$

using the abbreviations $a = \sin(\alpha)$, $b = \cos(\alpha)$, $c = \sin(\beta)$, $d = \cos(\beta)$, where $\alpha, \beta \in [0, 2\pi[$. Let $X_t^{[1]}$ and $X_t^{[2]}$ be the components of the solution process X_t. With some constant $r \in \mathbb{R}$, the drift and diffusion function are given by

$$f(X_t) = \begin{pmatrix} r^2(db - ca) & -da - cb \\ r^2(cb + da) & db - ca \end{pmatrix} \begin{pmatrix} (bX_t^{[1]} + aX_t^{[2]})(bX_t^{[2]} - aX_t^{[1]})^2 \\ (bX_t^{[2]} - aX_t^{[1]})^2 - (bX_t^{[1]} + aX_t^{[2]})^2 - 1 \end{pmatrix}$$

and

$$g(X_t) = r(-aX_t^{[1]} + bX_t^{[2]})^2 \begin{pmatrix} db - ca \\ cb + da \end{pmatrix}.$$

Note that the value of r influences the intensity of the noise and part of the drift function. For a given consistent initial value $X_0 = (X_0^{[1]}, X_0^{[2]})^T$ and with

$$U_t = \tan(rW_t + \arctan(bX_0^{[1]} + aX_0^{[2]}))$$

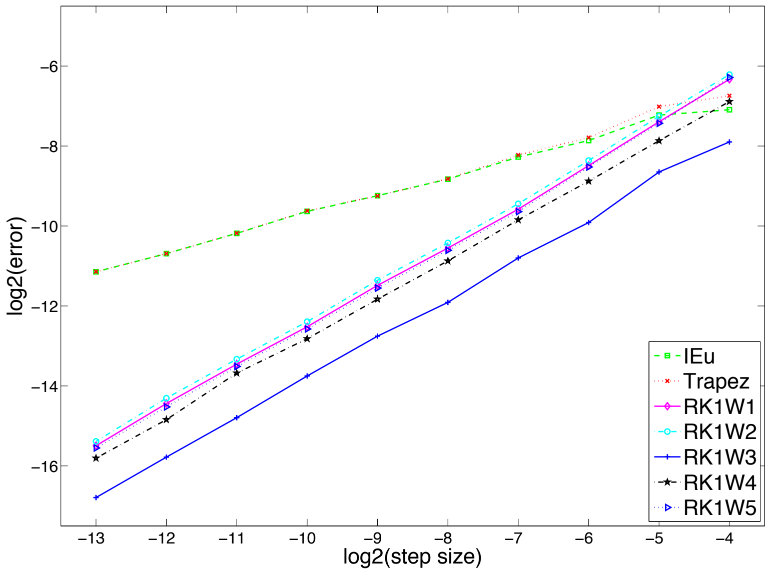

Fig. 6. Mean–square error vs. step size for test example 2.

the corresponding solution can be calculated by

$$X_t = \begin{pmatrix} U_t \\ \sqrt{1+U_t^2} \end{pmatrix}$$

for $t \geq 0$. For the following simulation results we chose the initial value $X_0 = (-a, b)^T$ and the parameters $\alpha = \frac{3}{5}\pi$, $\beta = \frac{9}{5}\pi$, $r = 0.8$ as well as the integration interval $[0, \frac{1}{16}]$. We applied the step sizes $h = 2^{-4}, \dots, 2^{-13}$ for the results presented in Figure 6. Again, the implicit Euler scheme and the trapezoidal rule as well as the considered SRK methods show their expected convergence behaviour. For this nonlinear example, RK1W3 with $B^{(2)} \equiv 0$ performs best compared to the other methods.

6 Conclusion

In the present paper, we propose some new order 1.0 stochastic Runge-Kutta methods for the approximation of the solution of index 1 SDAEs with scalar noise. The new methods combine the following advantageous properties compared to the schemes in the literature: First, they do not require the calculation of pseudo-inverses or other projectors and they are derivative-free which makes them easy to implement. Second, there is no restriction to any structure of the SDAE or any special type of noise. As a third advantage, the methods additionally attain the higher convergence order 1.0. The investigation of their mean–square stability revealed their good stability properties and their application to linear and nonlinear test examples confirmed the good performance for either type of test equation.

There are numerous interesting topics for future research, such as the complete analysis of the solution space of the order conditions. This includes the search for optimal schemes with respect to minimal error constants and good stability properties. Furthermore, the good performance of the schemes should be confirmed by their application to further test equations. Since SDAEs driven by multi-dimensional noise arise in applications as well, the derivation of methods dealing with this case are of special interest. The generalization of the proposed SRK methods to these problems is work in progress.

7 Appendix

The stability function for the considered SRK method (2.3) is calculated as

$$R_n(\hat{h},k) = \Gamma + \Sigma_1 \xi_n + \Sigma_2 \xi_n^2 + \Sigma_3 \xi_n^3 + \Sigma_4 \xi_n^4 \tag{7.1}$$

with some independent normally distributed random variables $\xi_n \sim \mathcal{N}(0,1)$ for $n = 0, 1, \ldots, N$ and

$$\begin{aligned}
\Gamma = & \frac{1}{(1 - A_{11}\hat{h} - B_{11}^{(3)}k)(1 - A_{22}\hat{h} - B_{22}^{(3)}k)(1 - A_{33}\hat{h} - B_{33}^{(3)}k)} \Big(1 + A_{31}\hat{h} \\
& + A_{32}\hat{h} + A_{21}A_{32}\hat{h}^2 - B_{11}^{(3)}k - B_{22}^{(3)}k - \tfrac{1}{2}B_{31}^{(2)}k + B_{31}^{(3)}k - \tfrac{1}{2}B_{32}^{(2)}k \\
& + B_{32}^{(3)}k - A_{32}B_{11}^{(3)}\hat{h}k - \tfrac{1}{2}A_{32}B_{21}^{(2)}\hat{h}k + A_{32}B_{21}^{(3)}\hat{h}k - A_{31}B_{22}^{(3)}\hat{h}k \\
& - \frac{1}{2}A_{21}B_{32}^{(2)}\hat{h}k + A_{21}B_{32}^{(3)}\hat{h}k + B_{11}^{(3)}B_{22}^{(3)}k^2 + \tfrac{1}{2}B_{22}^{(3)}B_{31}^{(2)}k^2 - B_{22}^{(3)}B_{31}^{(3)}k^2 \\
& + \tfrac{1}{2}B_{11}^{(3)}B_{32}^{(2)}k^2 + \frac{1}{4}B_{21}^{(2)}B_{32}^{(2)}k^2 - \tfrac{1}{2}B_{21}^{(3)}B_{32}^{(2)}k^2 - B_{11}^{(3)}B_{32}^{(3)}k^2 - \tfrac{1}{2}B_{21}^{(2)}B_{32}^{(3)}k^2 \\
& + B_{21}^{(3)}B_{32}^{(3)}k^2 - A_{22}\hat{h}(1 + A_{31}\hat{h} - B_{11}^{(3)}k - \tfrac{1}{2}B_{31}^{(2)}k + B_{31}^{(3)}k) \\
& - A_{11}\hat{h}(1 - A_{22}\hat{h} + A_{32}\hat{h} - B_{22}^{(3)}k - \tfrac{1}{2}B_{32}^{(2)}k + B_{32}^{(3)}k) \Big) ,
\end{aligned} \tag{7.2}$$

$$\begin{aligned}
\Sigma_1 = & \frac{1}{(1 - A_{11}\hat{h} - B_{11}^{(3)}k)(1 - A_{22}\hat{h} - B_{22}^{(3)}k)(1 - A_{33}\hat{h} - B_{33}^{(3)}k)} \Big(B_{31}^{(1)}k \\
& + B_{32}^{(1)}k + A_{32}B_{21}^{(1)}\hat{h}k + A_{21}B_{32}^{(1)}\hat{h}k - B_{22}^{(3)}B_{31}^{(1)}k^2 - B_{11}^{(3)}B_{32}^{(1)}k^2 \\
& - \tfrac{1}{2}B_{21}^{(2)}B_{32}^{(1)}k^2 + B_{21}^{(3)}B_{32}^{(1)}k^2 - \tfrac{1}{2}B_{21}^{(1)}B_{32}^{(2)}k^2 + B_{21}^{(1)}B_{32}^{(3)}k^2 \\
& - A_{22}B_{31}^{(1)}\hat{h}k - A_{11}B_{32}^{(1)}\hat{h}k \Big) ,
\end{aligned} \tag{7.3}$$

$$\Sigma_2 = \frac{1}{2(1-A_{11}\hat{h}-B_{11}^{(3)}k)(1-A_{22}\hat{h}-B_{22}^{(3)}k)(1-A_{33}\hat{h}-B_{33}^{(3)}k)} \Big(B_{31}^{(2)}k + B_{32}^{(2)}k + A_{32}B_{21}^{(2)}\hat{h}k + A_{21}B_{32}^{(2)}\hat{h}k - B_{22}^{(3)}B_{31}^{(2)}k^2 + 2B_{21}^{(1)}B_{32}^{(1)}k^2 - B_{11}^{(3)}B_{32}^{(2)}k^2 - B_{21}^{(2)}B_{32}^{(2)}k^2 + B_{21}^{(3)}B_{32}^{(2)}k^2 + B_{21}^{(2)}B_{32}^{(3)}k^2 - A_{22}B_{31}^{(2)}\hat{h}k - A_{11}B_{32}^{(2)}\hat{h}k \Big) , \tag{7.4}$$

$$\Sigma_3 = \frac{B_{21}^{(2)}B_{32}^{(1)}k^2 + B_{21}^{(1)}B_{32}^{(2)}k^2}{2(1-A_{11}\hat{h}-B_{11}^{(3)}k)(1-A_{22}\hat{h}-B_{22}^{(3)}k)(1-A_{33}\hat{h}-B_{33}^{(3)}k)} , \tag{7.5}$$

and with

$$\Sigma_4 = \frac{B_{21}^{(2)}B_{32}^{(2)}k^2}{4(1-A_{11}\hat{h}-B_{11}^{(3)}k)(1-A_{22}\hat{h}-B_{22}^{(3)}k)(1-A_{33}\hat{h}-B_{33}^{(3)}k)} . \tag{7.6}$$

Then, we calculate the second moment of the stability function in order to obtain the MS–stability function as

$$\hat{R}(\hat{h},k) = \Gamma^2 + 2\Gamma\Sigma_2 + 6\Gamma\Sigma_4 + \Sigma_1^2 + 6\Sigma_1\Sigma_3 + 3\Sigma_2^2 + 30\Sigma_2\Sigma_4 + 15\Sigma_3^2 + 105\Sigma_4^2 . \tag{7.7}$$

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