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Approximation of Parametric Derivatives by the Empirical Interpolation Method

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Abstract

We introduce a general *a priori* convergence result for the approximation of parametric derivatives of parametrized functions. We show, with rather general assumptions on the particular approximation scheme, that the approximations of parametric derivatives of a given parametrized function are convergent provided that the approximation to the function itself is convergent. We present numerical results with one particular method for the approximation of parametrized functions — the Empirical Interpolation Method — to illustrate the general theory.

1 Introduction

We consider in this paper the approximation of *parametrized* functions, i.e., functions that in addition to spatial variables depend on one or several scalar parameters. In particular, we are concerned with the approximation of *parametric derivatives* of such functions, i.e., derivatives of parametrized functions with respect to the parameters. We develop a new convergence theory that demonstrates — with rather general assumptions on the particular approximation scheme — that the approximations of parametric derivatives of a given parametrized function are convergent provided that the approximation to the function itself is convergent.

The Empirical Interpolation Method (EIM), introduced in [1, 5], is an interpolation method specifically constructed for the approximation of parametrized functions.[1] The main focus of this paper is the EIM approximation of parametric

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[1]In particular, the EIM serves to construct parametrically affine approximations of parameter dependent non-affine or non-linear differential operators within the Reduced Basis (RB) framework for parametric reduced order modelling of partial differential equations [10]. An affine representation (or approximation) of the operator allows an efficient "offline-online" computational decoupling, which in turn is a crucial ingredient in the RB computational framework. We refer to [4, 5] for the application of the EIM for RB approximations.

derivatives of parametrized functions. The new convergence theory is developed with the EIM in mind, and is discussed and applied within the context of the EIM.

However, our new theoretical results here also apply to rather general approximation schemes *other* than the EIM; in particular, we may consider both projection-based and interpolation-based approximation. The main limitation of the theory is related to regularity assumptions in space and parameter on the parametrized function.

The results in this paper have several useful implications. First, if the EIM is employed for evaluation of an objective function subject to optimization with respect to a set of parameters, our theory shows that we may accurately compute the parametric Jacobian without expensive generation of additional EIM spaces, or alternatively finite difference Jacobian approximations. Second, the rigorous *a posteriori* bounds for the error in the EIM approximation recently introduced in [3] require computation of the EIM approximation of parametric derivatives at a finite number of points in the parameter domain; smaller EIM errors associated with these derivatives imply sharper EIM error bounds. This second point in particular motivates our work here.

The remainder of the paper is organized as follows. First, in Section 2 we introduce notation and recall some results from polynomial approximation theory. Next, in Section 3, we present the new general *a priori* convergence result. Then, in Section 4 we review the EIM and apply the new convergence theory in this particular context. Subsequently, in Section 5, we demonstrate the theory within the context of the EIM through numerical results. Finally, in Section 6 we provide some concluding remarks.

2 Preliminaries

2.1 Notation

We denote by $\Omega \subset \mathbb{R}^d$ the spatial domain ($d = 1, 2, 3$); a particular point $x \in \Omega$ shall be denoted by $x = (x_{(1)}, \ldots, x_{(d)})$. We denote by $\mathcal{D} \subset \mathbb{R}^P$ the parameter domain ($P \geq 1$); a particular parameter value $\mu \in \mathcal{D}$ shall be denoted by $\mu = (\mu_{(1)}, \ldots, \mu_{(P)})$.

We introduce a sufficiently smooth function $\mathcal{F} : \Omega \times \mathcal{D} \to \mathbb{R}$. We suppose that $\mathcal{F}(\cdot; \mu) \in L^\infty(\Omega)$ for all $\mu \in \mathcal{D}$, and, for purposes of our theoretical arguments later, that $\mathcal{F}(x; \cdot) \in C^2(\mathcal{D})$ for all $x \in \Omega$. Here, $L^\infty(\Omega) = \{v : \operatorname{ess\,sup}_{x \in \Omega} |v(x)| < \infty\}$ and $C^s(O)$ denotes the space of functions with continuous s-order derivatives over a domain O. We then introduce a multi-index of dimension P,

$$\beta = (\beta_1, \ldots, \beta_P), \tag{1}$$

where the entries β_i, $1 \leq i \leq P$, are non-negative integers. We define for any multi-index β the parametric derivatives of $\mathcal{F}$,

$$\mathcal{F}^{(\beta)} = \frac{\partial^{|\beta|} \mathcal{F}}{\partial \mu_{(1)}^{\beta_1} \cdots \partial \mu_{(P)}^{\beta_P}}, \tag{2}$$

where

$$|\beta| = \sum_{i=1}^{P} \beta_i \tag{3}$$

is the length of β and hence the differential order. We denote the set of all distinct multi-indices β of dimension P of length p by $\mathcal{M}_p^P$.

For our theoretical arguments in Section 3 we shall write $\mathcal{D}$ as the tensor product $\mathcal{D} = \mathcal{D}_{(1)} \times \cdots \times \mathcal{D}_{(P)}$, where $\mathcal{D}_{(i)} \subset \mathbb{R}$, $1 \leq i \leq P$. We shall further consider any particular parameter dimension $\mathcal{S} \equiv \mathcal{D}_j$, $1 \leq j \leq P$, and assume without loss of generality[2] that $\mathcal{S} = [-1, 1]$. In this case we fix the $P - 1$ parameter values $\mu_{(i)} \in \mathcal{D}_{(i)}$, $1 \leq i \leq P$, $i \neq j$, and we introduce the function $\mathcal{J}_{\beta,j} : \Omega \times \mathcal{S} \to \mathbb{R}$ defined for $x \in \Omega$ and $\kappa \in \mathcal{S}$ by

$$\mathcal{J}_{\beta,j}(x;\kappa) \equiv \mathcal{F}^{(\beta)}\big(x; (\mu_{(1)}, \ldots, \mu_{(j-1)}, \kappa, \mu_{(j+1)}, \ldots, \mu_{(P)})\big). \tag{4}$$

2.2 Polynomial Approximation

In this section we recall some results from polynomial interpolation theory. We first describe a general interpolation framework for which we state three hypotheses. These hypotheses are the key ingredients in the proof of our new convergence theory in Section 3.

Let $\Gamma = [-1, 1]$, and let $f : \Gamma \to \mathbb{R}$ be a sufficiently smooth function. We introduce $N + 1$ distinct interpolation nodes $y_{N,i} \in \Gamma$, $0 \leq i \leq N$, and $N + 1$ characteristic functions $\chi_{N,i}$, $0 \leq i \leq N$, that satisfy $\chi_{N,i}(y_{N,j}) = \delta_{i,j}$, $0 \leq i, j \leq N$. We finally introduce an interpolation operator I_N defined by $I_N f = \sum_{i=0}^{N} f(y_{N,i})\chi_{N,i}$. We may now formally state our three hypotheses.

Hypothesis 1. *The error in the derivative of the interpolant $I_N f$ satisfies*

$$|f'(x) - (I_N f)'(x)| \leq G_f(N), \quad \forall x \in \Gamma, \tag{5}$$

where for a given f the function $G_f : \mathbb{N} \to (0, \infty)$ with $G_f(N) \to 0$ as $N \to \infty$.

Hypothesis 2. *The characteristic functions $\chi_{N,i}$, $0 \leq i \leq N$, satisfy*

$$\sum_{i=0}^{N} |\chi'_{N,i}(x)| \leq D(N), \quad \forall x \in \Gamma, \tag{6}$$

where the function $D : \mathbb{N} \to (0, \infty)$ is fixed (for a given interpolation scheme) with $D(N) \to \infty$ as $N \to \infty$.

Hypothesis 3. *Let $\epsilon \in \mathbb{R}^+$, and consider the equation*

$$G_f(N) = D(N)\epsilon \tag{7}$$

for the unknown N as $\epsilon \to 0$. Equation (7) has a solution $N = N(\epsilon) \geq 0$ that satisfies

$$\epsilon D\big(N(\epsilon)\big) \to 0 \tag{8}$$

as $\epsilon \to 0$.

[2] We may always transform our parameter dependent function such that the parameters reside in the hypercube $[-1, 1]^P$.

We next consider several interpolation schemes and in each case confirm the corresponding instantiations of our hypotheses under suitable regularity conditions. First, we assume $f \in C^2(\Gamma)$ and consider piecewise linear interpolation over equidistant interpolation nodes $y_{N,i} = (2i/N - 1) \in \Gamma$, $0 \le i \le N$. In this case the characteristic functions $\chi_{N,i}$ are continuous and piecewise linear "hat functions" with support only on the interval $[y_{N,0}, y_{N,1}]$ for $i = 0$, only on the interval $[y_{N,i-1}, y_{N,i+1}]$ for $1 \le i \le N-1$, and only on the interval $[y_{N,N-1}, y_{N,N}]$ for $i = N$. For piecewise linear interpolation Hypothesis 1 and Hypothesis 2 obtain for

$$G_f(N) = 2N^{-1}\|f''\|_{L^\infty(\Gamma)}, \tag{9}$$

$$D(N) = N, \tag{10}$$

respectively. In this case (6) in Hypothesis 2 obtains with equality. We include the proofs in Appendix A.1. It is straightforward to demonstrate Hypothesis 3: we note that

$$N^{-1} = N\epsilon \tag{11}$$

has the solution $N(\epsilon) = \epsilon^{-1/2}$ and that $\epsilon^{-1/2}\epsilon \to 0$ as $\epsilon \to 0$.

Next, we assume $f \in C^3(\Gamma)$ and consider piecewise quadratic interpolation over equidistant interpolation nodes $y_{N,i} = (2i/N - 1) \in \Gamma$, $0 \le i \le N$. We assume that N is even such that we may divide Γ into $N/2$ intervals $[y_{N,i}, y_{N,i+2}]$, for $i = 0, 2, 4, \ldots, N-2$. The characteristic functions are for $x \in [y_{N,i}, y_{N,i+2}]$ then given as

$$\chi_{N,i}(x) = \frac{(x - y_{N,i+1})(x - y_{N,i+2})}{2h^2}, \tag{12}$$

$$\chi_{N,i+1}(x) = \frac{(x - y_{N,i})(x - y_{N,i+2})}{-h^2}, \tag{13}$$

$$\chi_{N,i+2}(x) = \frac{(x - y_{N,i})(x - y_{N,i+1})}{2h^2}, \tag{14}$$

for $i = 0, 2, 4, \ldots, N-2$, where $h = 2/N = y_{N,j+1} - y_{N,j}$, $0 \le j \le N-1$. For piecewise quadratic interpolation Hypothesis 1 and Hypothesis 2 obtain for

$$G_f(N) = \text{const} \cdot N^{-2}\|f'''\|_{L^\infty(\Gamma)}, \tag{15}$$

$$D(N) = \frac{5}{2}N, \tag{16}$$

respectively. We include the proofs in Appendix A.2. It is straightforward to demonstrate Hypothesis 3: we note that

$$N^{-2} = N\epsilon \tag{17}$$

has the solution $N(\epsilon) = \epsilon^{-1/3}$ and that $\epsilon^{-1/3}\epsilon \to 0$ as $\epsilon \to 0$.

Finally, we assume that f is analytic in Γ and consider standard Chebyshev interpolation over the usual Chebyshev-nodes $y_{N,i} = -\cos(i\pi/N)$, $0 \le i \le N$. The characteristic functions are in this case the Lagrange polynomials $\chi_{N,i} \in \mathbb{P}_N(\Gamma)$ that satisfy $\chi_{N,i}(y_{N,j}) = \delta_{ij}$, $0 \le i, j \le N$. For Chebyshev interpolation Hypothesis 1 and Hypothesis 2 obtain for

$$G_f(N) = c_f N e^{-N\log(\rho_f)}, \tag{18}$$

$$D(N) = N^2, \tag{19}$$

respectively, where $c_f > 0$ and $\rho_f > 1$ depend only on f. In this case (6) in Hypothesis 2 obtains with equality. We refer to Reddy and Weideman [8] for a proof of (18) and to Rivlin [9, pp. 119–121] for a proof of (19). We finally demonstrate Hypothesis 3: we let $\eta = \log(\rho_f) > 0$ and we note that the transcendental equation

$$Ne^{-N\eta} = N^2\epsilon. \tag{20}$$

admits the solution

$$N(\epsilon) = \frac{1}{\eta}\mathcal{W}\Big(\frac{\eta}{\epsilon}\Big), \tag{21}$$

where $\mathcal{W}$ denotes the Lambert$W$ function(s) defined by $\xi = \mathcal{W}(\xi)e^{\mathcal{W}(\xi)}$ for any $\xi \in \mathbb{C}$. As $\xi \to \infty$, $\xi \in \mathbb{R}$, it can be shown [2] that $\mathcal{W}(\xi) < \log(\xi)$. Thus, as $\epsilon \to 0$, we obtain

$$N(\epsilon) < \frac{1}{\eta}\log\Big(\frac{\eta}{\epsilon}\Big) = \frac{1}{\eta}\Big(\log(\eta) + \log(1/\epsilon)\Big) \le A\log(1/\epsilon) \tag{22}$$

for some sufficiently large constant A. We now consider the product $\epsilon(N(\epsilon))^2$ as $\epsilon \to 0$. By application of L'Hôpital's rule twice (Eqs. (25) and (27) below) we obtain

$$\begin{aligned}
\lim_{\epsilon\to 0}\epsilon(N(\epsilon))^2 &\le A^2\lim_{\epsilon\to 0}\epsilon(\log(1/\epsilon))^2 && (23)\\
&= A^2\lim_{\epsilon\to 0}\frac{\big(\log(\epsilon)\big)^2}{1/\epsilon} && (24)\\
&= A^2\lim_{\epsilon\to 0}\frac{2\log(\epsilon)/\epsilon}{-1/\epsilon^2} && (25)\\
&= 2A^2\lim_{\epsilon\to 0}\frac{\log(\epsilon)}{-1/\epsilon} && (26)\\
&= 2A^2\lim_{\epsilon\to 0}\frac{1/\epsilon}{1/\epsilon^2} && (27)\\
&= 2A^2\lim_{\epsilon\to 0}\epsilon = 0. && (28)
\end{aligned}$$

Hypothesis 3 thus holds.

3 A General Convergence Result

We introduce an approximation space $W_M \equiv W_M(\Omega)$ of finite dimension M. For any $\mu \in \mathcal{D}$, our approximation to the function $\mathcal{F}(\cdot;\mu) : \Omega \to \mathbb{R}$ shall reside in W_M; the particular approximation procedure invoked is not relevant for our theoretical results in this section. We show here that if, for any $\mu \in \mathcal{D}$, the error in the best $L^\infty(\Omega)$ approximation to $\mathcal{F}(\cdot;\mu)$ in W_M goes to zero as $M \to \infty$, then, for any multi-index β, $|\beta| \ge 0$, the error in the best $L^\infty(\Omega)$ approximation to $\mathcal{F}^{(\beta)}(\cdot;\mu)$ in W_M also goes to zero as $M \to \infty$. Of course, only modest M are of interest in practice: the computational cost associated with the approximation is M-dependent. However, our theoretical results in this section provide some promise that we may in practice invoke the "original" approximation space and approximation procedure also for the approximation of parametric derivatives.

We introduce, for any fixed $p \geq 0$ and any $M \geq 1$,

$$e_M^p \equiv \max_{\beta \in \mathcal{M}_p^P} \max_{\mu \in \mathcal{D}} \inf_{w \in W_M} \|\mathcal{F}^{(\beta)}(\cdot;\mu) - w\|_{L^\infty(\Omega)}. \tag{29}$$

We then recall the definition of $\mathcal{J}_{\beta,j}$ from (4), and state

Proposition 1. *Let p be a fixed non-negative integer. Assume that Hypotheses 1, 2, and 3 hold for $f = \mathcal{J}_{\beta,j}(x;\cdot)$, $1 \leq j \leq P$, for all $x \in \Omega$, and for all $\beta \in \mathcal{M}_p^P$. In this case, if $e_M^p \to 0$ as $M \to \infty$, then*

$$e_M^{p+1} \to 0 \tag{30}$$

as $M \to \infty$.

Proof. For each $x \in \Omega$, we introduce the interpolant $\mathcal{J}_{N,\beta,j}(x;\cdot) \equiv I_N \mathcal{J}_{\beta,j}(x;\cdot) \in \mathbb{P}_N(\mathcal{S})$ given by

$$\mathcal{J}_{N,\beta,j}(x;\cdot) \equiv I_N \mathcal{J}_{\beta,j}(x;\cdot) = \sum_{i=0}^{N} \mathcal{J}_{\beta,j}(x;y_{N,i})\chi_{N,i}(\cdot); \tag{31}$$

recall that here, $\chi_{N,i} : \mathcal{S} \to \mathbb{R}$, $0 \leq i \leq N$, are characteristic functions that satisfy $\chi_{N,i}(y_{N,j}) = \delta_{i,j}$, $0 \leq i,j \leq N$.

Let $'$ denote differentiation with respect to the variable κ in (4). For each $x \in \Omega$ we consider an approximation to $\mathcal{J}'_{\beta,j}(x;\cdot)$ which we write as $\sum_{i=0}^{N} \chi'_{N,i} w_i(x)$, where $w_i \in W_M$, $1 \leq i \leq N$. We note that $\sum_{i=0}^{N} \chi'_{N,i}(\kappa) w_i \in W_M$ for all $\kappa \in \mathcal{S}$ (we define the w_i shortly; however we note that in general $\mathcal{J}_{\beta,j}(\cdot;y_{N,i}) \notin W_M$, $1 \leq i \leq N$, are not valid choices). For the error in this approximation we note by the triangle inequality that (for any $w_i \in W_M$, $1 \leq i \leq N$)

$$\begin{aligned}
\Big\|\mathcal{J}'_{\beta,j} - \sum_{i=0}^{N} \chi'_{N,i} w_i\Big\|_{L^\infty(\Omega\times\mathcal{S})} &= \Big\|\mathcal{J}'_{N,\beta,j} - \sum_{i=0}^{N} \chi'_{N,i} w_i + \mathcal{J}'_{\beta,j} - \mathcal{J}'_{N,\beta,j}\Big\|_{L^\infty(\Omega\times\mathcal{S})} \\
&\leq \Big\|\mathcal{J}'_{N,\beta,j} - \sum_{i=0}^{N} \chi'_{N,i} w_i\Big\|_{L^\infty(\Omega\times\mathcal{S})} + \Big\|\mathcal{J}'_{\beta,j} - \mathcal{J}'_{N,\beta,j}\Big\|_{L^\infty(\Omega\times\mathcal{S})}.
\end{aligned} \tag{32}$$

Here, $\mathcal{J}'_{N,\beta,j} \equiv (\mathcal{J}_{N,\beta,j})' = \sum_{i=0}^{N} \mathcal{J}_{\beta,j}(\cdot;y_{N,i})\chi'_{N,i}(\cdot)$.

In our approximation, we use as coefficient functions $\chi'_{N,i}$ (and not, for example, $\chi_{N,i}$). With this choice and the definition of $\mathcal{J}'_{\beta,j}$, we may relate the error in our approximation to the error in the approximation of $\mathcal{J}_{\beta,j}$, which is our ultimate goal. For the first term on the right hand side of (32) we first invoke (31), then the triangle inequality, and finally Hypothesis 2 to obtain

$$\begin{aligned}
\Big\|\mathcal{J}'_{N,\beta,j} - \sum_{i=0}^{N} \chi'_{N,i} w_i\Big\|_{L^\infty(\Omega\times\mathcal{S})} &= \Big\|\sum_{i=0}^{N} (\mathcal{J}_{\beta,j}(\cdot;y_{N,i}) - w_i)\chi'_{N,i}\Big\|_{L^\infty(\Omega\times\mathcal{S})} \\
&\leq \Big\|\sum_{i=0}^{N} |\chi'_{N,i}||\mathcal{J}_{\beta,j}(\cdot;y_{N,i}) - w_i|\Big\|_{L^\infty(\Omega\times\mathcal{S})} \\
&\leq \Big\|\max_{0\leq i\leq N} |\mathcal{J}_{\beta,j}(\cdot;y_{N,i}) - w_i| \sum_{j=0}^{N} |\chi'_{N,j}|\Big\|_{L^\infty(\Omega\times\mathcal{S})} \\
&\leq D(N) \max_{0\leq i\leq N} \|\mathcal{J}_{\beta,j}(\cdot;y_{N,i}) - w_i\|_{L^\infty(\Omega)}.
\end{aligned} \tag{33}$$

Next, for any $\kappa \in \mathcal{S}$ we introduce the functions

$$w^*_{\beta,j}(\cdot;\kappa) \equiv \arg \inf_{w \in W_M} \|\mathcal{J}_{\beta,j}(\cdot;\kappa) - w\|_{L^\infty(\Omega)}. \tag{34}$$

We then consider (33) for $w_i = w^*_{\beta,j}(\cdot; y_{N,i})$ and note that

$$\begin{aligned}
&\Big\|\mathcal{J}'_{N,\beta,j} - \sum_{i=0}^{N} \chi'_{N,i} w^*_{\beta,j}(\cdot; y_{N,i})\Big\|_{L^\infty(\Omega\times\mathcal{S})} \\
&\qquad \le D(N) \max_{0\le i\le N} \|\mathcal{J}_{\beta,j}(\cdot; y_{N,i}) - w^*_{\beta,j}(\cdot; y_{N,i})\|_{L^\infty(\Omega)} \\
&\qquad \le D(N) \max_{\kappa\in\mathcal{S}} \|\mathcal{J}_{\beta,j}(\cdot;\kappa) - w^*_{\beta,j}(\cdot;\kappa)\|_{L^\infty(\Omega)} \\
&\qquad = D(N) \max_{\kappa\in\mathcal{S}} \inf_{w\in W_M} \|\mathcal{J}_{\beta,j}(\cdot;\kappa) - w\|_{L^\infty(\Omega)} \le D(N) e^p_M,
\end{aligned} \tag{35}$$

where the last step follows from the definition of e^p_M in (29).

For the second term on the right hand side of (32) we invoke Hypothesis 1 for $f = \tilde{f}_{\beta,j} \equiv \mathcal{J}_{\beta,j}(\tilde{x}_{\beta,j};\cdot)$ to obtain

$$\|\mathcal{J}'_{\beta,j} - \mathcal{J}'_{N,\beta,j}\|_{L^\infty(\Omega\times\mathcal{S})} \le G_{\tilde{f}_{\beta,j}}(N); \tag{36}$$

here $\tilde{x}_{\beta,j} \in \Omega$ is the particular point in Ω such that for given β and j, $\tilde{f}_{\beta,j}$ yields the "worst" behavior of the right-hand-side.

We now combine (32) for $w_i = w^*_{\beta,j}(\cdot; y_{N,i})$ with (35) and (36) to obtain

$$\Big\|\mathcal{J}'_{\beta,j} - \sum_{i=0}^{N} \chi'_{N,i} w^*_{\beta,j}(\cdot; y_{N,i})\Big\|_{L^\infty(\Omega\times\mathcal{S})} \le G_{\tilde{f}_{\beta,j}}(N) + D(N) e^p_M. \tag{37}$$

We then introduce $\beta^+_j = \beta + e_j$ where e_j is the canonical unit vector with the j'th entry equal to unity; we recall that β has length $|\beta| = p$ and hence β^+_j has length $|\beta^+_j| = p+1$. We note that the multi-index β, the parameter values $\mu_{(i)} \in \mathcal{D}_{(i)}$, $1 \le i \le P$, $i \ne j$, as well as the dimension j, were chosen arbitrarily above. We may thus conclude

$$\begin{aligned}
&\max_{\beta\in\mathcal{M}^P_p} \max_{1\le j\le P} \max_{\mu\in\mathcal{D}} \Big\|\mathcal{F}^{(\beta^+_j)}(\cdot;\mu) - \sum_{i=0}^{N} \chi'_{N,i}(\mu_{(j)}) w^*_{\beta,j}(\cdot; y_{N,i})\Big\|_{L^\infty(\Omega)} \\
&\qquad\qquad \le G_{\hat{f}}(N) + D(N) e^p_M
\end{aligned} \tag{38}$$

(recall above we wrote $\kappa = \mu_{(j)}$ for each fixed j); here, $\hat{f} = \mathcal{J}_{\tilde{\beta},\tilde{j}}(\tilde{x}_{\tilde{\beta},\tilde{j}};\cdot)$, where $1 \le \tilde{j} \le P$ and $\tilde{\beta} \in \mathcal{M}^P_p$ are the particular indices that yield the "worst" behavior of the right-hand-side.

We note that $\sum_{i=0}^{N} \chi'_{N,i}(\mu_{(j)}) w^*_{\beta,j}(\cdot; y_{N,i})$ is a particular member of W_M for any $\beta \in \mathcal{M}^P_p$, any $\mu_{(j)} \in \mathcal{D}_{(j)}$, and any $1 \le j \le P$. We thus obtain

$$e^{p+1}_M = \max_{\beta\in\mathcal{M}^P_{p+1}} \max_{\mu\in\mathcal{D}} \inf_{w\in W_M} \|\mathcal{F}^{(\beta)}(\cdot;\mu) - w\|_{L^\infty(\Omega)} \le G_{\hat{f}}(N) + D(N) e^p_M. \tag{39}$$

The final step is to bound the right-hand side of (39) in terms of e^p_M. To this end we note that we may choose N freely. In particular we may choose N

as the minimizer $N = N_{\min}(e_M^p) > 0$ of the right hand side of (39); however for simplicity we shall make a different choice for N. Let $N_{\rm bal}(e_M^p)$ denote the value of N that balances the two terms on the right hand side of (39); by Hypothesis 3 $N_{\rm bal}(e_M^p)$ exists for sufficiently small e_M^p. We then choose $N = N_{\rm bal}(e_M^p)$ in (39) to obtain

$$e_M^{p+1} \leq 2D(N_{\rm bal}(e_M^p))e_M^p, \tag{40}$$

where $e_M^{p+1} \to 0$ as $e_M^p \to 0$ by Hypothesis 3. $\square$

We now provide three lemmas. The first lemma quantifies the convergence in Proposition 1 in the case that $\mathcal{F}(x,\cdot) \in C^2(\mathcal{D})$ for all $x \in \Omega$.

Lemma 1. *Assume $\mathcal{F}(x,\cdot) \in C^2(\mathcal{D})$ for any $x \in \Omega$. If for any fixed $p \geq 0$ $e_M^p \to 0$ as $M \to \infty$, then there is a constant $C_{p+1} > 0$ such that*

$$e_M^{p+1} \leq C_{p+1}\sqrt{e_M^p} \tag{41}$$

as $M \to \infty$.

Proof. In this case we may invoke piecewise linear interpolation as our interpolation system in the proof of Proposition 1. By (9) and (10) we obtain $N_{\rm bal}(e_M^p) = (2\|\hat{f}''\|_{L^\infty(\Gamma)}/e_M^p)^{1/2}$ and hence (40) for $D(N) = N$ becomes $e_M^{p+1} \leq 2(2\|\hat{f}''\|_{L^\infty(\Gamma)}/e_M^p)^{1/2}e_M^p$. The result follows for $C_{p+1} = 2(2\|\hat{f}''\|_{L^\infty(\Gamma)})^{1/2}$. $\square$

The next lemma quantifies the convergence in Proposition 1 in the case that $\mathcal{F}(x,\cdot) \in C^3(\mathcal{D})$ for all $x \in \Omega$.

Lemma 2. *Assume $\mathcal{F}(x,\cdot) \in C^3(\mathcal{D})$ for any $x \in \Omega$. If for any fixed $p \geq 0$ $e_M^p \to 0$ as $M \to \infty$, then there is a constant $C_{p+1} > 0$ such that*

$$e_M^{p+1} \leq C_{p+1}(e_M^p)^{2/3} \tag{42}$$

as $M \to \infty$.

Proof. In this case we may invoke piecewise quadratic interpolation as our interpolation system in the proof of Proposition 1. By (15) and (16) we obtain, for a positive constant $\tilde{c}$, $N_{\rm bal}(e_M^p) = \tilde{c}(\|\hat{f}'''\|_{L^\infty(\Gamma)}/e_M^p)^{1/3}$ and hence (40) for $D(N) = 5N/2$ becomes $e_M^{p+1} \leq 5\tilde{c}(\|\hat{f}'''\|_{L^\infty(\Gamma)}/e_M^p))^{1/3}e_M^p$. The result follows for $C_{p+1} = 5\tilde{c}\|\hat{f}'''\|_{L^\infty(\Gamma)}^{1/3}$. $\square$

We make the following remark concerning Lemma 1 and Lemma 2 in the case of algebraic convergence.

Remark 1. *Let $|\beta| = p$, and assume that $\mathcal{F}^{(\beta)}(x,\cdot) \in C^{q_p}(\mathcal{D})$, $q_p > 0$, for all $x \in \Omega$. Assume that $e_M^p \sim M^{-r_p}$, $r_p > 0$, as $M \to \infty$; here the convergence rate r_p typically depends on the regularity q_p. For $q_p = 2$ we may invoke Lemma 1 to obtain*

$$e_M^{p+1} \leq C_{p+1}(e_M^p)^{1/2} \sim M^{-r_p/2} \sim M^{1-r_p+(r_p/2-1)} \sim M^{1+(r_p/2-1)}e_M^p. \tag{43}$$

Similarly, for $q_p = 3$ we may invoke Lemma 2 to obtain

$$e_M^{p+1} \leq C_{p+1}(e_M^p)^{2/3} \sim M^{-2r_p/3} \sim M^{1-r_p+(r_p/3-1)} \sim M^{1+(r_p/3-1)}e_M^p. \tag{44}$$

More generally, with higher-regularity versions of Lemma 1 and Lemma 2, we expect for any $q_p > 0$ that

$$e_M^{p+1} \leq C_{p+1}(e_M^p)^{1-1/q_p} \sim M^{-r_p(1-1/q_p)} \sim M^{1-r_p+(r_p/q_p-1)} \sim M^{1+(r_p/q_p-1)}e_M^p. \tag{45}$$

for any $q_p > 0$. We shall comment on these estimates further in our discussion of numerical results in Section 5.

The third lemma quantifies the convergence in Proposition 1 in the case that $\mathcal{F}(x,\cdot)$ is analytic over $\mathcal{D}$.

Lemma 3. *Assume $\mathcal{F}(x,\cdot) : \mathcal{D} \to \mathbb{R}$ is analytic over $\mathcal{D}$ for any $x \in \Omega$. If for any fixed $p \geq 0$ $e_M^p \to 0$ as $M \to \infty$, then there is a constant $C_{p+1} > 0$ such that*

$$e_M^{p+1} \leq C_{p+1}(\log(e_M^p))^2 e_M^p \tag{46}$$

as $M \to \infty$. In particular, if for some p

$$e_M^p \sim M^\sigma e^{-\gamma M} \tag{47}$$

as $M \to \infty$, where σ is a non-negative constant and γ is a positive constant, then there is a constant C_{p+1} such that

$$e_M^{p+1} \leq C_{p+1}M^{\sigma+2}e^{-\gamma M} \tag{48}$$

as $M \to \infty$.

Proof. In this case we may invoke Chebyshev interpolation as our interpolation system in the proof of Proposition 1. By (18), (19), and (22) we obtain $N_{\mathrm{bal}}(e_M^p) < \hat{c}\log(1/e_M^p)$ for a sufficiently large constant $\hat{c}$. Hence, with $D(N) = N^2$ and (40), we obtain $e_M^{p+1} \leq 2\hat{c}^2(\log(1/e_M^p))^2 e_M^p$. The result (46) follows for $C_{p+1} = 2\hat{c}^2$ since $(\log(1/e_M^P))^2 = (\log(e_M^P))^2$. The result (48) follows under the additional assumption (47) since in this case there is a constant B such that

$$N_{\mathrm{bal}}(e_M^p) \leq B\log\Big(\frac{1}{M^\sigma e^{-\gamma M}}\Big) = B(-\sigma\log M + \gamma M) < B\gamma M, \tag{49}$$

and $D(N_{\mathrm{bal}}(e_M^p)) \leq (B\gamma M)^2$. □

Remark 2. *Note that, in Lemma 3, we can not obtain an* explicit *expression for the convergence rate of derivatives of order larger than $p+1$ (by for example an induction argument) since the result* (48) *is not sharp; an asymptotic* lower bound *for e_M^{p+1} is required to explicitly bound $N_{\mathrm{bal}}(e_M^{p+1})$ as $M \to \infty$. Hence, we invoke an exact asymptotic relation in the assumption* (47) *in order to bound the convergence of the "next" derivative approximation based on the "current" derivative approximation.*

We also note that if the bound (48) *were sharp, we could invoke the argument recursively to obtain an estimate of the form $e_M^p \sim M^{\sigma+2p}e^{-\gamma M}$.*

4 The Empirical Interpolation Method

In this section we first recall the empirical interpolation method (EIM) [1, 5, 6] and then consider the convergence theory of the previous section applied to the EIM. The EIM approximation space is spanned by precomputed snapshots of a parameter dependent "generating function" for judicuosly chosen parameter values from a predefined parameter domain. Given any new parameter value in this parameter domain, we can construct an approximation to the generating function at this new parameter value — or in fact an approximation to any function defined over the same spatial domain — as a linear combination of the EIM basis functions. The particular linear combination is determined through interpolation at judiciuosly chosen points in the spatial domain. For parametrically smooth functions, the EIM approximation to the generating function yields rapid, typically exponential, convergence.

4.1 Procedure

We introduce the generating function $\mathcal{G} : \Omega \times \mathcal{D} \to \mathbb{R}$ such that for all $\mu \in \mathcal{D}$, $\mathcal{G}(\cdot;\mu) \in L^\infty(\Omega)$. We introduce a training set $\Xi_{\mathrm{train}} \subset \mathcal{D}$ of finite cardinality $|\Xi_{\mathrm{train}}|$ which shall serve as our computational surrogate for $\mathcal{D}$. We also introduce a triangulation $\mathcal{T}_{\mathcal{N}}(\Omega)$ of Ω with $\mathcal{N}$ vertices over which we shall in practice, for any $\mu \in \mathcal{D}$, realize $\mathcal{G}(\cdot;\mu)$ as a piecewise linear function.

Now, for $1 \le M \le M_{\max} < \infty$, we define the EIM approximation space $W_M^{\mathcal{G}}$ and the EIM interpolation nodes $T_M^{\mathcal{G}}$ associated with $\mathcal{G}$; here, $M_{\max}$ is a specified maximum EIM appproximation space dimension. We first choose (randomly, say) an initial parameter value $\mu_1 \in \mathcal{D}$; we then determine the first EIM interpolation node as $t_1 = \arg\sup_{x\in\Omega} |\mathcal{G}(x;\mu_1)|$; we next define the first EIM basis function as $q_1 = \mathcal{G}(\cdot;\mu_1)/\mathcal{G}(t_1;\mu_1)$. We can then, for $M = 1$, define $W_M^{\mathcal{G}} = \mathrm{span}\{q_1\}$ and $T_M^{\mathcal{G}} = \{t_1\}$. We also define a nodal value matrix B^1 with (a single) element $B^1_{1,1} = q_1(t_1) = 1$.

Next, for $2 \le M \le M_{\max}$, we first compute the empirical interpolation of $\mathcal{G}(\cdot;\mu)$ for all $\mu \in \Xi_{\mathrm{train}}$: we solve the linear system

$$\sum_{j=1}^{M-1} \phi_j^{M-1}(\mu) B_{i,j}^{M-1} = \mathcal{G}(t_i;\mu), \quad 1 \le i \le M-1, \tag{50}$$

and compute the empirical interpolation $\mathcal{G}_{M-1}(\cdot;\mu) \in W_{M-1}^{\mathcal{G}}$ as

$$\mathcal{G}_{M-1}(\cdot;\mu) = \sum_{i=1}^{M-1} \phi_i^{M-1}(\mu) q_i, \tag{51}$$

for all $\mu \in \Xi_{\mathrm{train}}$. We then choose the next parameter $\mu_M \in \mathcal{D}$ as the maximizer of the EIM interpolation error over the training set,

$$\mu_M = \arg\max_{\mu\in\Xi_{\mathrm{train}}} \|\mathcal{G}_{M-1}(\cdot;\mu) - \mathcal{G}(\cdot;\mu)\|_{L^\infty(\Omega)}; \tag{52}$$

note that thanks to our piecewise linear realization of $\mathcal{G}(\cdot;\mu)$, the norm evaluation is a simple comparison of function values at the $\mathcal{N}$ vertices of $\mathcal{T}_{\mathcal{N}}(\Omega)$.

We now choose the next EIM interpolation node as the point in Ω at which the EIM error associated with $\mathcal{G}_{M-1}(\mu_M)$ is largest,

$$t_M = \arg\sup_{x\in\Omega} |\mathcal{G}_{M-1}(x;\mu_M) - \mathcal{G}(x;\mu_M)|. \tag{53}$$

The next EIM basis function is then

$$q_M = \frac{\mathcal{G}_{M-1}(\cdot;\mu_M) - \mathcal{G}(\cdot;\mu_M)}{\mathcal{G}_{M-1}(t_M;\mu_M) - \mathcal{G}(t_M;\mu_M)}. \tag{54}$$

We finally enrich the EIM space: $W_M^{\mathcal{G}} = \mathrm{span}\{q_1,\dots,q_M\}$; expand the set of nodes: $T_M^{\mathcal{G}} = \{t_1,\dots,t_M\}$; and expand the nodal value matrix: $B_{i,j}^M = q_j(t_i)$, $1 \le i,j \le M$.

Now, given any function $\mathcal{F} : \Omega \times \mathcal{D} \to \mathbb{R}$ (in particular, we shall consider $\mathcal{F} = \mathcal{G}^{(\beta)}$), we define for any $\mu \in \mathcal{D}$ and for $1 \le M \le M_{\max}$ the empirical interpolation of $\mathcal{F}(\cdot;\mu)$ in the space $W_M^{\mathcal{G}}$ (the space generated by $\mathcal{G}$) as

$$\mathcal{F}_M^{\mathcal{G}}(\cdot;\mu) = \sum_{i=1}^{M} \phi_i^M(\mu) q_i, \tag{55}$$

where the coefficients $\phi_i^M(\mu)$, $1 \le i \le M$, solve the linear system

$$\sum_{j=1}^{M} \phi_j^M(\mu) B_{i,j}^M = \mathcal{F}(t_i;\mu), \quad 1 \le i \le M. \tag{56}$$

We note that by construction the matrices $B^M \in \mathbb{R}^{M\times M}$, $1 \le M \le M_{\max}$, are lower triangular: by (50), $\mathcal{G}_{M-1}(t_j;\mu_M) = \mathcal{G}(t_j;\mu_M)$ for $j < M$. As a result, computation of the EIM coefficients ϕ_j^M, $1 \le j \le M$, in (56) and (50) are $\mathcal{O}(M^2)$ operations. We emphasize that the computational cost associated with the EIM approximation (55)–(56) (after snapshot precomputation), is independent of the number $\mathcal{N}$ of vertices in the triangulation $\mathcal{T}_{\mathcal{N}}(\Omega)$. We may thus choose $\mathcal{N}$ conservatively.

We next note that, for any multi-index β,

$$(\mathcal{F}_M^{\mathcal{G}})^{(\beta)} = \Big(\sum_{i=1}^{M} \phi_i^M(\mu) q_i\Big)^{(\beta)} = \sum_{i=1}^{M} \varphi_i^M(\mu) q_i, \tag{57}$$

where $\varphi_i^M(\mu) = (\phi_i^M)^{(\beta)}(\mu)$, $1 \le i \le M$, solve the linear system (recall that the matrix B^M is μ-independent)

$$\sum_{j=1}^{M} \varphi_j^M(\mu) B_{i,j}^M = \mathcal{F}^{(\beta)}(t_i;\mu), \quad 1 \le i \le M. \tag{58}$$

Hence,

$$(\mathcal{F}_M^{\mathcal{G}})^{(\beta)} = (\mathcal{F}^{(\beta)})_M^{\mathcal{G}}, \tag{59}$$

that is, the parametric derivative of the approximation is equivalent to the approximation of the parametric derivative. We note that this equivalence holds since we invoke the same approximation space $W_M^{\mathcal{G}}$ for both EIM approximations $\mathcal{F}_M^{\mathcal{G}}$ and $(\mathcal{F}^{(\beta)})_M^{\mathcal{G}}$.

4.2 Convergence theory applied to the EIM

We introduce the Lebesgue constants [7]

$$\Lambda_M = \sup_{x\in\Omega} \sum_{i=1}^{M} |V_i^M(x)|, \quad 1 \le M \le M_{\max}, \tag{60}$$

where $V_i^M \in W_M^{\mathcal{G}}$ are the characteristic functions associated with $W_M^{\mathcal{G}}$ and $T_M^{\mathcal{G}}$: $V_i^M(t_j) = \delta_{ij}$, $1 \le i, j \le M$, where δ is the Kroenecker delta symbol. It can be proven [1, 5] that the EIM error satisfies

$$\|\mathcal{F}(\cdot;\mu) - \mathcal{F}_M^{\mathcal{G}}(\cdot;\mu)\|_{L^\infty(\Omega)} \le (1+\Lambda_M) \inf_{w\in W_M^{\mathcal{G}}} \|\mathcal{F}(\cdot;\mu) - w\|_{L^\infty(\Omega)}, \tag{61}$$

for $1 \le M \le M_{\max}$. It can furthermore be proven that $\Lambda_M < 2^M - 1$; however, in actual practice the growth of Λ_M is much slower, as we shall observe below (see also results in [1, 5, 6]).

Our theory of Section 3 considers the convergence in the best approximation error. In the following remark we apply Lemma 3 within the context of the EIM.

Remark 3. *It can be shown [1, 5] that the error in the EIM derivative approximation satisfies*

$$\begin{aligned}\|\mathcal{F}^{(\beta)}(\cdot;\mu) - (\mathcal{F}^{(\beta)})_M^{\mathcal{G}}(\cdot;\mu)\|_{L^\infty(\Omega)}& \\ \le (1+\Lambda_M) \inf_{w\in W_M^{\mathcal{G}}} \|\mathcal{F}^{(\beta)}(\cdot;\mu) - w\|_{L^\infty(\Omega)},&\end{aligned} \tag{62}$$

for any $\mu \in \mathcal{D}$ and any multi-index β. Assume that the best approximation error

$$e_M^p = \max_{\mu\in\mathcal{D}} \inf_{w\in W_M^{\mathcal{G}}} \|\mathcal{F}^{(\beta)}(\cdot;\mu) - w\|_{L^\infty(\Omega)} \to 0 \tag{63}$$

as $M \to \infty$ for all $\mu \in \mathcal{D}$ and any multi-index β such that $|\beta| = p$ is a non-negative integer. We may then conclude from Lemma 3 and (62) *that*

$$\begin{aligned}\max_{\mu\in\mathcal{D}} \|\mathcal{F}^{(\beta')}(\cdot;\mu) - (\mathcal{F}^{(\beta')})_M^{\mathcal{G}}(\cdot;\mu)\|_{L^\infty(\Omega)} &\le (1+\Lambda_M) e_M^{p+1} \\ &\le (1+\Lambda_M) C_{p+1} (\log(e_M^p))^2 e_M^p,\end{aligned} \tag{64}$$

for any multi-index β' such that $|\beta'| = p+1$.

The term $e_M^p \to 0$ as $M \to \infty$ by assumption and thus $e_M^{p+1} \to 0$ as $M \to \infty$ by Proposition 1. Hence, the convergence of the EIM derivative approximation associated with derivatives of order $p+1$ depends on the growth of the Lebesgue constant; precisely, we must require

$$\Lambda_M e_M^{p+1} \to 0 \tag{65}$$

as $M \to \infty$. We recall that the Lebesgue constant typically grows only modestly and thus we expect in practice convergence of the EIM derivative approximation.

Clearly, if the EIM approximation associated with derivatives of order p converges, then the best approximation associated with derivatives of order p converges as well. Hence covergence of the EIM approximation associated with derivatives of order p implies convergence of the EIM approximation associated with derivatives of order $p+1$ provided the Lebesgue constant grows sufficiently modestly.

We expect that Remark 1, Remark 2, and Remark 3 may be applied (non-rigorously) to EIM convergence if the growth of the Lebesgue constant is modest, since then the convergence rates associated with the best approximation and EIM approximation can not be very different.

For any $p \geq 0$ we introduce the maximum EIM error over $\mathcal{E} \subseteq \mathcal{D}$

$$\epsilon^p_{M,\max}(\mathcal{E}) \equiv \max_{\mu \in \mathcal{E}} \max_{\beta \in \mathcal{M}^P_p} \|\mathcal{F}^{(\beta)}(\cdot;\mu) - (\mathcal{F}^{(\beta)})^{\mathcal{G}}_M(\cdot;\mu)\|_{L^\infty(\Omega)} \tag{66}$$

for $|\beta| = p$ and $1 \leq M \leq M_{\max}$. We also introduce a function $R_M : \mathcal{D} \to \mathbb{R}$ such that

$$\|\mathcal{F}^{(\beta)}(\cdot;\mu) - (\mathcal{F}^{(\beta)})^{\mathcal{G}}_M(\cdot;\mu)\|_{L^\infty(\Omega)} = R_M(\mu) \inf_{w \in W^{\mathcal{G}}_M} \|\mathcal{F}^{(\beta)}(\cdot;\mu) - w\|_{L^\infty(\Omega)} \tag{67}$$

for $1 \leq M \leq M_{\max}$. We note that by (62) $1 \leq R_M(\mu) \leq 1 + \Lambda_M$ for all $\mu \in \mathcal{D}$. With (66) and (67) we then obtain, for any $p \geq 0$,

$$\begin{aligned}\epsilon^p_{M,\max}(\mathcal{E}) &= \max_{\mu \in \mathcal{E}} \max_{\beta \in \mathcal{M}^P_p} \|\mathcal{F}^{(\beta)}(\cdot;\mu) - (\mathcal{F}^{(\beta)})^{\mathcal{G}}_M(\cdot;\mu)\|_{L^\infty(\Omega)} \\ &= \max_{\mu \in \mathcal{E}} \max_{\beta \in \mathcal{M}^P_p} \Big(R_M(\mu) \inf_{w \in W^{\mathcal{G}}_M} \|\mathcal{F}^{(\beta)}(\cdot;\mu) - w\|_{L^\infty(\Omega)} \Big) = R_M(\hat{\mu}_p) e^p_M \end{aligned} \tag{68}$$

for a particular $\hat{\mu}_p \in \mathcal{E}$. We now introduce the EIM error *degradation factor*

$$\rho_{M,p}(\mathcal{E}) \equiv \frac{\epsilon^p_{M,\max}(\mathcal{E})}{\epsilon^0_{M,\max}(\mathcal{E})}, \tag{69}$$

and note that

$$\rho_{M,p}(\mathcal{E}) = \frac{R_M(\hat{\mu}_p) e^p_M}{R_M(\hat{\mu}_0) e^0_M} \leq (1 + \Lambda_M) \frac{e^p_M}{e^0_M}. \tag{70}$$

We make two observations. First, the EIM error degradation factor is similar (for fixed p as a function of M) to the best approximation error degradation factor $\rho^*_{M,p} \equiv e^p_M / e^0_M$ if the Lebesgue constant grows slowly. Second, if the ratio $R_M(\hat{\mu}_p)/R_M(\hat{\mu}_0) \sim 1$ as $M \to \infty$, then $\rho_{M,p}(\mathcal{E})$ will be similar to $\rho^*_{M,p}$ independent of the Lebesgue constant.

In our discussion of each of our numerical examples in the next section we plausibly assume that the Lebesgue constant grows only modestly, and in particular that $\rho_{M,p}(\mathcal{E})$ is similar to $\rho^*_{M,p}$. We confirm this assumption with explicit calculation of the Lebesgue constant.

The following remark is particularly relevant in our subsequent discussion of the sharpness of our theoretical results.

Remark 4. *Assume that the Lebesgue constant grows slowly and thus that the convergence rate associated with the EIM approximation is similar to the convergence rate associated with the best approximation. Consider the case of exponential convergence and assume that the bound provided by Lemma 3 is sharp. If* $\epsilon^0_{M,\max}(\mathcal{D}) \sim M^\sigma e^{-\gamma M}$ *for* $\sigma, \gamma > 0$, *we expect that* $\epsilon^p_{M,\max}(\mathcal{D}) \sim M^{\sigma+2p} e^{-\gamma M}$, *and thus an EIM error degradation factor* $\rho_{M,p} \sim (M^{2p})$. *As we shall observe shortly for our numerical results this estimate for the degradation factor is not quite sharp.*

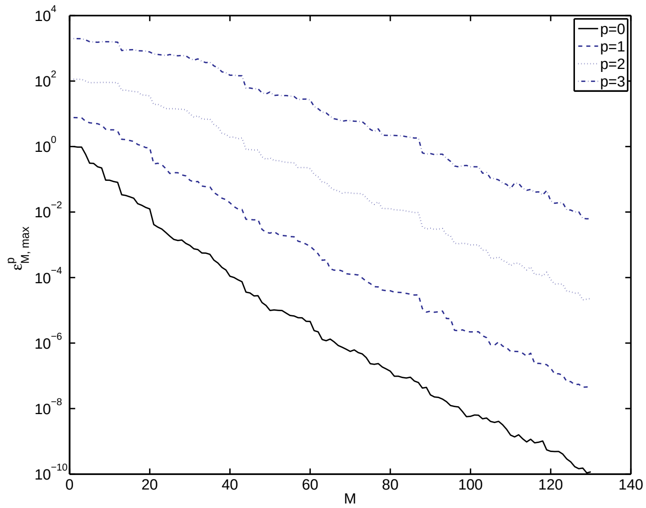

Figure 1: The maximum EIM error over the test set $\epsilon^p_{M,\max}(\Xi_{\text{test}})$ for $0 \leq p \leq 3$ for Example 1.

We may obtain an expression for the EIM error degradation factor also in the case of algebraic convergence. However, the relation between the regularity of the function (q_p in Remark 1) and the convergence (r_p in Remark 1) is not a priori *known for the EIM (or best) approximation. We thus save the discussion of the EIM error degradation factor in the case of algebraic convergence for our numerical results section, in which we compute the relation between q_p and r_p* a posteriori*.*

5 Numerical Results

5.1 Example 1: Parametrically smooth Gaussian surface

We introduce the spatial domain $\Omega = [0,1]^2$ and the parameter domain $\mathcal{D} = [0.4, 0.6]^2$. We consider the 2D Gaussian $\mathcal{F} : \Omega \times \mathcal{D} \to \mathbb{R}$ defined by

$$\mathcal{F}(x;\mu) = \exp\left(\frac{-(x_{(1)} - \mu_{(1)})^2 - (x_{(2)} - \mu_{(2)})^2}{2\sigma^2}\right) \tag{71}$$

for $x \in \Omega$, $\mu \in \mathcal{D}$, and $\sigma \equiv 0.1$. This function is thus parametrized by the location of the maximum of the Gaussian surface. We note that for all $x \in \Omega$ the function $\mathcal{F}(x;\cdot) \in C^\infty(\mathcal{D})$; we may thus invoke Lemma 3.

We introduce a triangulation $\mathcal{T}_\mathcal{N}(\Omega)$ with $\mathcal{N} = 2601$ vertices; we introduce an equi-distant training set "grid" $\Xi_{\text{train}} \subset \mathcal{D}$ of size $|\Xi_{\text{train}}| = 900 = 30 \times 30$. We then pursue the EIM with $\mathcal{G} = \mathcal{F}$ for $M_{\max} = 130$.

We now introduce a uniformly distributed random test set $\Xi_{\text{test}} \subset \mathcal{D}$ of size 1000. In Figure 1 we show the maximum interpolation errors $\epsilon^p_{M,\max}(\Xi_{\text{test}})$ for $p = 0,1,2,3$; the convergence is exponential (note the lin-log scaling of the axes). We note that for large M, the rate of convergence associated with

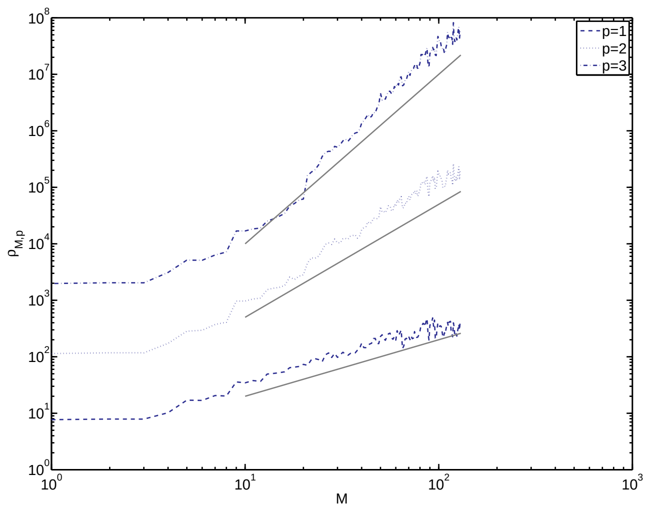

Figure 2: EIM error degradation factors $\rho_{M,p}(\Xi_{\mathrm{test}})$, $p = 1, 2, 3$, for Example 1. The shorter solid gray lines represent exact rates M^p.

the derivatives ($p > 1$) is close to the rate of convergence associated with the generating function ($p = 0$).

In Figure 2 we show the EIM error degradation factors $\rho_{M,p}(\Xi_{\mathrm{test}})$ for $p = 1, 2, 3$ as functions of M. We observe that the degradation factors behave approximately as M^p: there is an M^p degradation of the convergence associated with the derivative approximation for $p > 0$ compared to the convergence associated with the original function.

From Remark 4 we recall that we would have expected $\rho_{M,p}(\Xi_{\mathrm{test}}) \sim M^{2p}$ if our theoretical result (48) were sharp. Since in practice we observe $\rho_{M,p}(\Xi_{\mathrm{test}}) \sim M^p$, we conclude that the result (48) is not in general sharp. We also note that the factor M^2 in (48) originates from the *sharp* result (19); hence with our present strategy for the proof of Proposition 1 it is not clear how to sharpen (48). However, we note that our theory captures the correct qualitative behavior: a degradation by an algebraic factor for the derivative approximation.

Finally, in Figure 3, we report the Lebesgue constant Λ_M. We note that the growth of the Lebesgue constant is only modest. The EIM derivative approximation is thus close to the best $L^\infty(\Omega)$ approximation in the space $W_M^{\mathcal{F}}$.

5.2 Example 2: A parametrically singular function

We introduce the spatial domain $\Omega = [-1, 1]$ and the parameter domain $\mathcal{D} = [-1, 1]$. We consider the function $\mathcal{F} : \Omega \times \mathcal{D} \to \mathbb{R}$ defined by

$$\mathcal{F}(x; \mu) = |x - \mu|^5 \tag{72}$$

for $x \in \Omega$ and $\mu \in \mathcal{D}$. The function thus has a singularity at $x = \mu$ for any $\mu \in \mathcal{D}$. For any $x \in \Omega$ we have $\mathcal{F}(x; \cdot) \in C^4(\mathcal{D})$. More generally, for any $x \in \Omega$ and $p = 0, 1, 2, 3$, we have $\mathcal{F}^{(p)}(x; \cdot) \in C^{q_p}(\mathcal{D})$ for $q_p = 4 - p$.

We introduce a triangulation $\mathcal{T}_{\mathcal{N}}(\Omega)$ with $\mathcal{N} = 1000$ vertices; we introduce an equi-distant training set "grid" $\Xi_{\mathrm{train}} \subset \mathcal{D}$ of size $|\Xi_{\mathrm{train}}| = 1000$. We then

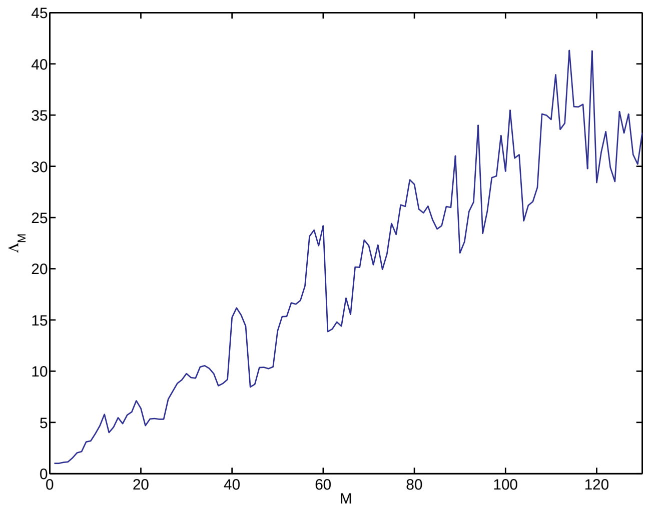

Figure 3: The Lebesgue constant Λ_M for Example 1.

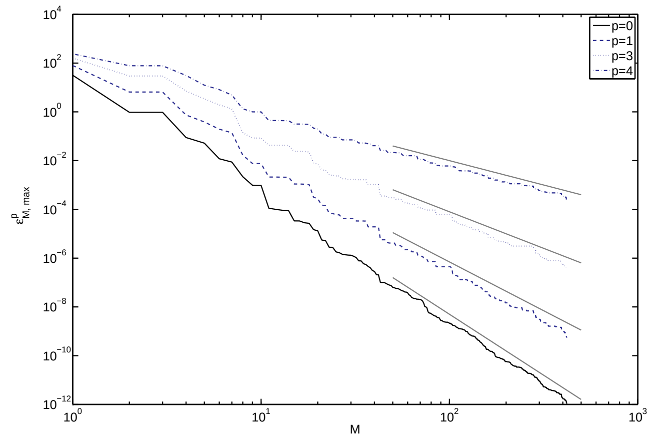

Figure 4: The maximum EIM error over the test set $\epsilon^p_{M,\max}(\Xi_{\text{test}})$ for $0 \le p \le 3$ for Example 2. The shorter gray lines represent exact rates M^{-5+p}.

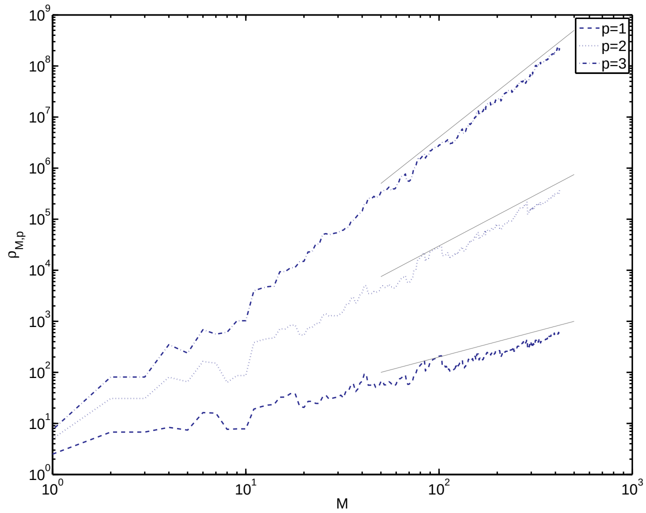

Figure 5: EIM error degradation factors $\rho_{M,p}(\Xi_{\text{test}})$, $p = 1, 2, 3$, for Example 2. The shorter solid gray lines represent exact rates M^p.

pursue the EIM with $\mathcal{G} = \mathcal{F}$ for $M_{\max} = 420$.

We now introduce a uniformly distributed random test set $\Xi_{\text{test}} \subset \mathcal{D}$ of size 1000. In Figure 4 we show the maximum interpolation errors $\epsilon^p_{M,\max}(\Xi_{\text{test}})$ for $p = 0, 1, 2, 3$; we observe for the convergence $\epsilon^p_{M,\max}(\Xi_{\text{test}}) \sim M^{-5+p}$. These results suggest that, in general, if $\mathcal{F}^{(p)} \in C^{q_p}(\mathcal{D})$, then $e^p_M \sim M^{-q_p-1}$, which corresponds to $r_p = q_p + 1$ in Remark 1.

In Figure 5 we show the EIM error degradation factors $\rho_{M,p}(\Xi_{\text{test}})$ for $p = 1, 2, 3$ as functions of M. As for Example 1, we note that $\rho_{M,p}(\Xi_{\text{test}}) \sim M^p$ (of course this factor may in this case be interpreted directly from Figure 4).

With $r_p = q_p + 1$ in Remark 1, the estimate (45) in Remark 1 becomes

$$e^{p+1}_M \leq M^{1+\frac{1}{q_p}} e^p_M = M^{1+\frac{1}{4-p}} e^p_M. \tag{73}$$

If this is a sharp estimate, we expect for our example with $q_p = 4 - p$

$$e^1_M \sim M^{1+\frac{1}{4}} e^0_M, \tag{74}$$

$$e^2_M \sim M^{1+\frac{1}{3}} e^1_M \sim M^{2+\frac{1}{3}+\frac{1}{4}} e^0_M, \tag{75}$$

$$e^3_M \sim M^{1+\frac{1}{2}} e^2_M \sim M^{3+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}} e^0_M. \tag{76}$$

From these estimates we may expect EIM error degradation factors

$$\rho_{M,p}(\Xi_{\text{train}}) \sim M^{p+\sum_{j=0}^{p-1} \frac{1}{4-j}}, \quad p = 1, 2, 3. \tag{77}$$

However, from our computations we see that this is not the case in practice: our results show $\rho_{M,p}(\Xi_{\text{test}}) \sim M^p$. We thus conclude that our theoretical results in Lemma 1 and Lemma 2 (and higher order versions of these as indicated in Remark 1) are not sharp. The bounds predict $e^{p+1}_M \leq C_{p+1} M^{1+1/q_p} e^p_M$ for $\mathcal{F}^{(p)}(x; \cdot) \in C^{q_p}(\mathcal{D})$: a non-optimality of a factor M^{1/q_p}. We note that for

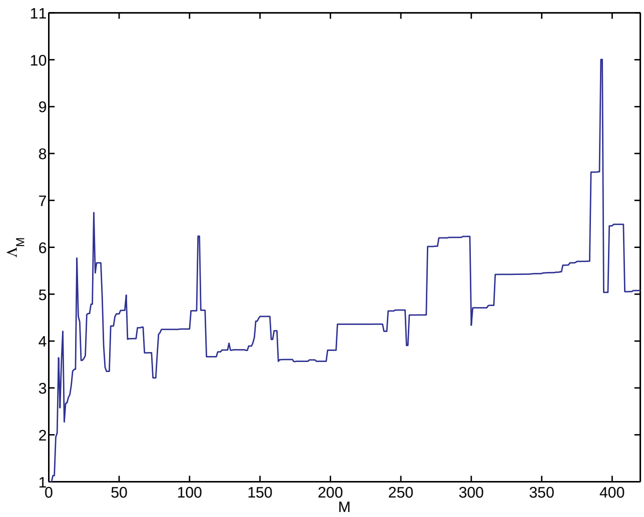

Figure 6: The Lebesgue constant Λ_M for Example 2.

functions with high regularity — large q_p — the sharpness of the bounds will improve since $e_M^{p+1} \leq M^{1+1/q_p} e_M^p \to M e_M^p$ as $q_p \to \infty$.

Finally, in Figure 6 we report the Lebesgue constant Λ_M; any growth of the Lebesgue constant is hardly present. The EIM derivative approximation is thus close to the best $L^\infty(\Omega)$ approximation in the space $W_M^{\mathcal{F}}$.

6 Concluding remarks

We have introduced new *a priori* convergence theory for the approximation of parametric derivatives by a general approximation scheme. In particular, we have focused on approximation by the EIM both in our discussion and for our numerical results. The results suggest that the EIM may be invoked in practice for the approximation of parametric derivatives without construction of additional EIM spaces with the parametric derivatives as generating functions, or alternatively enrichment of the original space with parametric derivatives.

There are several opportunities for improvements of the theory. First, our numerical results suggest that it should be possible to sharpen the theoretical bounds. We note in our numerical results an EIM error degradation factor M^p for the convergence associated with the approximation of p'th order derivatives for both parametrically analytic and parametrically non-analytic functions. In contrast, our theory and remarks predict a degradation factor M^{2p} for parametrically analytic functions, and a degradation factor $M^{p+\sum_{j=0}^{p-1} \frac{1}{s-j}}$ for parametrically non-analytic functions when the original function resides in $C^s(\mathcal{D})$ (but not in $C^{s+\alpha}(\mathcal{D})$ for arbitrarily small $\alpha > 0$).

Second, we would like to extend the validity of the theory to other (e.g. Sobolev) norms; in this case we may for example consider reduced basis [10] approximations to parametric derivatives of solutions to partial differential equations.

A Proofs for Hypotheses 1 and 2

A.1 Piecewise linear interpolation

We consider piecewise linear interpolation over the equidistant interpolation nodes $y_{N,i} = (2i/N-1) \in \Gamma = [-1,1]$, $0 \leq i \leq N$. In this case the characteristic functions $\chi_{N,i}$ are continuous and piecewise linear "hat functions" with support only on the interval $[y_{N,0}, y_{N,1}]$ for $i = 0$, on $[y_{N,i-1}, y_{N,i+1}]$ for $1 \leq i \leq N-1$, and on $[y_{N,N-1}, y_{N,N}]$ for $i = N$.

We recall the results (9) and (10) from Section 2.2. Let $f : \Gamma \to \mathbb{R}$ with $f \in C^2(\Gamma)$. We then have, for any $x \in \Gamma$,

$$|f'(x) - (I_N f)'(x)| \leq 2N^{-1} \|f''\|_{L^\infty(\Gamma)} \tag{78}$$

as $N \to \infty$. Further, for all $x \in \Gamma$, the characteristic functions $\chi_{N,i}$, $0 \leq i \leq N$, satisfy

$$\sum_{i=0}^{N} |\chi'_{N,i}(x)| = N. \tag{79}$$

We first demonstrate (78) (and hence (9)). For $x \in [y_{N,i}, y_{N,i+1}]$, $0 \leq i \leq N-1$, we have

$$(I_N f)'(x) = \frac{1}{h}\big(f(y_{N,i+1}) - f(y_{N,i})\big), \tag{80}$$

where $h = 2/N$. We next write $f(y_{N,i})$ and $f(y_{N,i+1})$ as Taylor series around x as

$$f(y_{N,i}) = \sum_{j=0}^{1} \frac{f^{(j)}(x)}{j!}(y_{N,i} - x)^j + \int_x^{y_{N,i}} f''(t)(y_{N,i} - t)\, dt, \tag{81}$$

$$f(y_{N,i+1}) = \sum_{j=0}^{1} \frac{f^{(j)}(x)}{j!}(y_{N,i+1} - x)^j + \int_x^{y_{N,i+1}} f''(t)(y_{N,i+1} - t)\, dt, \tag{82}$$

which we then insert in the expression (80) for $(I_N f)'$ to obtain

$$\begin{aligned} (I_N f)'(x) - f'(x) &= \frac{1}{h}\int_x^{y_{N,i+1}} f''(t)(y_{N,i+1} - t)\, dt - \frac{1}{h}\int_x^{y_{N,i}} f''(t)(y_{N,i} - t)\, dt \\ &\leq \frac{1}{h}\|f''\|_{L^\infty(\Gamma)} \max_{x \in [y_{N,i}, y_{N,i+1}]} \big(|y_{N,i+1} - x|^2 + |y_{N,i} - x|^2\big) \\ &\leq h\|f''\|_{L^\infty(\Gamma)} = 2N^{-1}\|f''\|_{L^\infty(\Gamma)}. \end{aligned} \tag{83}$$

We next demonstrate (79) (and hence (10)). It suffices to consider $x \in [y_{N,i}, y_{N,i+1}]$ for $0 \leq i \leq N-1$. On $[y_{N,i}, y_{N,i+1}]$ only $|\chi'_{N,i}(x)|$ and $|\chi'_{N,i+1}(x)|$ contribute to the sum; furthermore we have $|\chi'_{N,i}(x)| = |\chi'_{N,i+1}(x)| = 1/h = N/2$, from where the result (79) follows.

A.2 Piecewise quadratic interpolation

We consider piecewise quadratic interpolation over equidistant interpolation nodes $y_{N,i} = (2i/N - 1) \in \Gamma$, $0 \leq i \leq N$. We consider N equal such that we may divide Γ into $N/2$ intervals $[y_{N,i}, y_{N,i+2}]$, for $i = 0, 2, 4, \ldots, N-2$. The characteristic functions $\chi_{N,i}$ are for $x \in [y_{N,i}, y_{N,i+2}]$ given as

$$\chi_{N,i}(x) = \frac{(x - y_{N,i+1})(x - y_{N,i+2})}{2h^2}, \tag{84}$$

$$\chi_{N,i+1}(x) = \frac{(x - y_{N,i})(x - y_{N,i+2})}{-h^2}, \tag{85}$$

$$\chi_{N,i+2}(x) = \frac{(x - y_{N,i})(x - y_{N,i+1})}{2h^2}, \tag{86}$$

for $i = 0, 2, 4, \ldots, N$.

We recall the results (15) and (16) from Section 2.2. Let $f : \Gamma \to \mathbb{R}$ with $f \in C^3(\Gamma)$. We then have, for any $x \in \Gamma$,

$$|f'(x) - (I_N f)'(x)| = \mathcal{O}(N^{-2}) \tag{87}$$

as $N \to \infty$. Further, for all $x \in \Gamma$, the characteristic functions $\chi_{N,i}$, $0 \le i \le N$, satisfy

$$\sum_{i=0}^{N} |\chi'_{N,i}(x)| = \frac{5}{2} N. \tag{88}$$

We first demonstrate (87). It suffices to consider the interpolant $I_N f(x)$ for $x \in [y_{N,i}, y_{N,i+2}]$, in which case

$$I_N f(x) = f(y_{N,i})\chi_{N,i}(x) + f(y_{N,i+1})\chi_{N,i+1}(x) + f(y_{N,i+2})\chi_{N,i+2}(x). \tag{89}$$

Insertion of (84)–(86) and differentiation yields

$$\begin{aligned}(I_N f)'(x) = \frac{1}{2h^2}\Big(& f(y_{N,i})(2x - y_{N,i+1} - y_{N,i+2}) \\ & - 2f(y_{N,i+1})(2x - y_{N,i} - y_{N,i+2}) + f(y_{N,i+2})(2x - y_{N,i} - y_{N,i+1})\Big).\end{aligned} \tag{90}$$

We next write $f(y_{N,i})$, $f(y_{N,i+1})$, and $f(y_{N,i+2})$ as Taylor series around x as

$$f(y_{N,i}) = \sum_{j=0}^{3} \frac{f^{(j)}(x)}{j!}(y_{N,i} - x)^j + \mathcal{O}(h^4), \tag{91}$$

$$f(y_{N,i+1}) = \sum_{j=0}^{3} \frac{f^{(j)}(x)}{j!}(y_{N,i+1} - x)^j + \mathcal{O}(h^4), \tag{92}$$

$$f(y_{N,i+2}) = \sum_{j=0}^{3} \frac{f^{(j)}(x)}{j!}(y_{N,i+2} - x)^j + \mathcal{O}(h^4), \tag{93}$$

where $h = 2/N = y_{N,j+1} - y_{N,j}$, $0 \le j \le N-1$. We may then insert the expressions (91)–(93) into (90) to obtain

$$(I_N f)'(x) = f'(x) + \mathcal{O}(h^2). \tag{94}$$

(For $j = 0$ and $j = 2$ the terms on the right-hand-side of (90) cancel. For $j = 1$ we obtain $f'(x)$ and for $j = 3$ we obtain $\mathcal{O}(h^2)$.)

We next demonstrate (88). It suffices to consider $x \in \Gamma_i \equiv [y_{N,i}, y_{N,i+2}]$. On Γ_i only $\chi'_{N,i}(x)$, $\chi'_{N,i+1}(x)$, and $\chi'_{N,i+2}(x)$ contribute to the sum. With $h = 2/N = y_{j+1} - y_j$, $0 \le j \le N-1$, we have

$$\max_{x\in\Gamma_i} |\chi'_{N,i}(x)| = \frac{N^2}{8} \max_{x\in\Gamma_i} |2x - y_{N,i+1} - y_{N,i+2}| = \frac{3}{4}N, \tag{95}$$

$$\max_{x\in\Gamma_i} |\chi'_{N,i+1}(x)| = \frac{N^2}{4} \max_{x\in\Gamma_i} |2x - y_{N,i} - y_{N,i+2}| = N, \tag{96}$$

$$\max_{x\in\Gamma_i} |\chi'_{N,i+2}(x)| = \frac{N^2}{8} \max_{x\in\Gamma_i} |2x - y_{N,i} - y_{N,i+1}| = \frac{3}{4}N. \tag{97}$$

The result then follows.

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