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by

Bawfeh Kingsley Kometa

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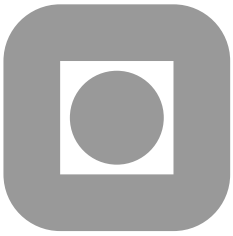

NORWEGIAN UNIVERSITY OF
SCIENCE AND TECHNOLOGY
TRONDHEIM, NORWAY

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Address: Department of Mathematical Sciences, Norwegian University of Science and Technology, N-7491 Trondheim, Norway.

On discontinuous Galerkin methods and commutator-free exponential integrators for advection problems

Bawfeh Kingsley Kometa

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Discontinuous Galerkin (DG) finite-element methods are well-known to be suitable for solving convection-dominated convection diffusion problems. High order Runge-Kutta methods such as the RKDG and SSP methods have been developed and tested to work quite well for convection-dominated problems(see e.g., Cockburn & Shu (2001), Gottlieb *et al.* (2001,2009)). An issue of concern remains the strong CFL restrictions on the discretization parameters. Restelli *et al.* (2006) proposed combining the DG methods with the semi-Lagrangian methods in what they called *semi-Lagragian discontinuous Galerkin* (SLDG) methods. Proposed implementations of the SLDG methods however suffer from limited spatial accuracy as opposed to the RKDG methods. We hereby propose a method in the framework of Lie-group exponential integrators (see Cellodoni *et al.* [6, 5]), that uses a modified version of the SLDG method as a building block for computing compositions of convection flows and maintain the good properties of the DG formulations.

1 Introduction

We study advection problems of the form

$$u_t + \mathbf{V} \cdot \nabla u = 0, \tag{1.1}$$

where $u = u(\mathbf{x}, t) \in \mathbb{R}$ is an unknown scalar field dependent on space and time variables $\mathbf{x}$ and t respectively, while (in a more general setting) $\mathbf{V} = \mathbf{V}(u, \mathbf{x}, t) \in \mathbb{R}^d$, $d = 1, 2$ or 3, represents a given advection velocity. The equation is treated over a bounded uniform domain $(\mathbf{x}, t) \in \Omega \times (0, T) \subset \mathbb{R}^d \times \mathbb{R}$ with suitably prescribed initial and boundary conditions on u. We denote by subscript-t the partial derivative with respect to time; $\nabla\cdot$ is the divergence operator with respect to $\mathbf{x}$. Equation (1.1) is said to be in *advective or Lagrangian form*. Throughout the rest of the paper we assume that the advection velocity $\mathbf{V}$ is divergence-free (i.e. $\nabla \cdot \mathbf{V} = 0$). Under this assumption (1.1) becomes equivalent to the *conservative form*

$$u_t + \nabla \cdot (\mathbf{V}u) = 0. \tag{1.2}$$

The form (1.2) is suitable for formulating discontinuous Galerkin methods, while (1.1) allows for the use of traditional semi-Lagrangian methods.

Discontinuous Galerkin (DG) finite-element methods are well-known to be suitable for solving convection-dominated convection diffusion problems. This is due to their ability to admit solution profiles with jump or contact discontinuities. High order methods such as the Runge-Kutta discontinuous Galerkind (RKDG) and the strong stability-preserving (SSP) methods have been developed and tested to work quite well for the treatment of hyperbolic conservation laws (see e.g.,[8, 12, 11]). However the CFL restriction on the discretization parameters is still an issue of concern especially for methods with high temporal order. Nevertheless, the high spatial accuracy and high level of parallelism of these methods remain attractive from a numerical point of view. Semi-Lagrangian (SL) methods on the other hand are well known to be very efficient and accurate (see for example [15, 9] and references therein). The SLDG methods of Restelli *et al.* [19] aim at combining the good properties of both the SL and DG methods. Such methods were found useful for applications in nonhydrostatic atmospheric modelling. Implementations of the SLDG methods however suffer from low spatial accuracy as opposed to RKDG methods.

In this paper we study the relations between commutator-free Lie-group exponential integrators (CF) of Celledoni *et al.* [6] and the semi-Lagrangian discontinuous Galerkin methods (SLDG) presented in [19] for advection problems. We reformulate the CF methods using the SLDG method as a building block for computing the pure convection flows. When compared to RKDG methods of high temporal order, the new formlation allow for the use of larger Courant numbers. The spatial discretization is based on high order Gauss-Lobatto-Legendre (GLL) polynomial approximations. The SLDG component is thus modified to maintain this high spatial accuracy. Numerical experiments are presented both for 1D and 2D advection problems to demonstrate the performance of the new methods.

2 Discontinuous Galerkin methods for hyperbolic conservation laws

Following the recipes of Cockburn and Shu [8] for the spatial discretization of (1.2) one obtains a semi-discrete system of ODEs.

For the sake of completeness we shall briefly describe the discontinuous Galerkin (DG) method for solving numerically a scalar hyperbolic conservation law

$$u_t + \nabla \cdot f(u) = 0, \quad \mathbf{x} \in \Omega, \quad t > 0, \tag{2.1}$$

with prescribed boundary conditions and initial data u_0. We observe that $f(u) := \mathbf{V}u$, in relation to (1.2).

2.1 Weak formulation on a broken Sobolev space

Consider a finite element discretization of (2.1) in which Ω is subdivided into N subdomains (or "elements") denoted by $K := \Omega_j$, $j = 1, \ldots, N$. Further assume that the functions u and $f(u) = (f \circ u)(\mathbf{x})$ are smooth on Ω. We multiply (2.1) by a test function v and integrate by parts over a subdomain K to obtain

$$\int_K u_t v, \, d\mathbf{x} - \int_K f(u) \cdot \nabla v, \, d\mathbf{x} + \int_{\partial K} f(u) \cdot \mathbf{n}_K v \, ds = 0, \tag{2.2}$$

where $\mathbf{n}_K$ is the outward unit normal on the subdomain boundary ∂K. The goal, however, is to allow for functions u that admit jump discontinuities at the element boundaries ∂K, and have a certain amount of regularity within the interior of each element. The space

described by such functions is refered to as the *'broken' Sobolev* space in the DG literature (see for example [16, 2, 13]).

Now let u be a function in a broken Sobolev space $H^1 := H^1(\Omega)$ (functions with H_1 regularity in each element K). Note that for such u the flux function $f(u)$ is well-defined on Ω *except* (possibly) at the element boundaries, where u may have a jump discontinuity. To remove this ambiguity the restriction of $f(u)$ on the nodal points is replaced by a function $\hat{f}(u)$, also known as the *numerical flux function*. The function $\hat{f}(u)$ can be seen as a well-defined (single-valued) approximation to $f(u)$. It is defined such that

$$\hat{f}(u)\big|_{\partial K} := \hat{f}(u^-, u^+), \quad \text{and} \quad \hat{f}(u)\big|_K := f(u), \tag{2.3}$$

where for each $s \in \partial K$, $u^+(s)$ denotes the value of the trace of u on K and $u^-(s)$ denotes the value of the trace of u on the neighbouring element K^- having $s \in \partial K^-$. The function $\hat{f}(\cdot,\cdot)$ can be defined in several ways. For example, in the case of locally conservative schemes, it can represent a Godunov, Lax-Friedrichs, Ösher-Enquist, or an upwind flux function (see [8]). Two important requirements for the definition of $\hat{f}(\cdot,\cdot)$ include

(a) Consistency: $\hat{f}(v, v) = f(v)$, for any v;

(b) Monotonicity: $\hat{f}(v,\cdot)$ is non-decreasing, while $\hat{f}(\cdot, v)$ is non-increasing (a necessary requirement for constructing monotone schemes).

Now summing (2.2) over all elements and introducing the numerical flux function at element boundaries, we get the following *weak formulation*:
Find $u \in H^1 := H^1(\Omega)$ such that, for $t > 0$,

$$\sum_K \int_K (u_t v - f(u)\cdot\nabla v),\, d\mathbf{x} + \sum_K \int_{\partial K} f(u)\cdot\mathbf{n}_K v\, ds = 0, \quad \text{for all } v \in H^1. \tag{2.4}$$

2.2 The discrete weak formulation on a piece-wise polynomial space

Let $\mathcal{T} := \mathcal{T}_h$ denote a triangulation of the domain Ω (i.e., the set of all elements of Ω) with a mesh parameter given by $h = \max_{K\in\mathcal{T}} diam(K)$ (or $h = \max_{K\in\mathcal{T}} |K|$ in 1D). A simple discrete finite dimensional subspace of the broken H^1 space is given by

$$V_h^p := \{v \in L^2 : v\big|_K \in \mathbb{P}^p(K), \forall K \in \mathcal{T}_h\},$$

i.e. V_h^p consist of piecewise polynomials of degree p, possibly discontinuous across sub-domain boundaries. Unless, not obvious, we shall henceforth surpress the writing of the superscript p in V_h^p.

An approximation for u in the discrete subspace V_h is sought for via the discrete weak formulation (related to (2.4)), namely: Find $u_h \in V_h$ such that, for $t > 0$,

$$\sum_K \int_K (u_{h,t} v - f(u_h)\cdot\nabla v),\, d\mathbf{x} + \sum_K \int_{\partial K} f(u_h)\cdot\mathbf{n}_K v\, ds = 0, \quad \text{for all } v \in V_h, \tag{2.5}$$

where $u_{h,t} := \frac{\partial u_h}{\partial t}$.

Basis functions for V_h can be chosen as functions $\varphi_K^m = \psi_K^m \cdot \chi_K$, $m = 0, \ldots, p$, $K \in \mathcal{T}_h$ where χ_K denotes the characteristic function on K, and ψ_K^m, $m = 0, \ldots, p$ represent the basis functions for $\mathbb{P}^p(\Omega_j)$. For example, affine map of Lagrange interpolation polynomials based on Gauss-Legendre (GL) nodes in a reference domain $\hat{\Omega}$.

The resulting semi-discrete system assembled over all elements, would give rise to a system of ODEs of the form[1]

$$\dot{y} = C(y)y + \hat{r}(y), \tag{2.6}$$

where $y \in \mathbb{R}^{\mathcal{N}}$ ($\mathcal{N}$ = computational degrees of freedom), and $C(y)$ is a matrix-valued. Meanwhile $\hat{r}(y)$ is a vector representing interelement contributions resulting from the numerical flux terms in (2.5), and $C(y)y$ represents elemental contributions. The representation in (2.6) is obtainable, for example, in the semi-discretization of the Burgers equation.

Suitable and popular time integration schemes for solving such ODE system are the TVD methods of Cockburn and Shu [20, 8], also known as the *RKDG* method. Applying the Lie-group commutator-free exponential integrators (CF) [6] for such ODE system is not immediate, due to the presence of an extra flux term. We therefore interprete the CF methods within the DG framework in a way that permits the evaluation of flows ("exponentials") of convecting vector fields in a manner similar to the SLDG schemes [19].

3 Semi-Lagrangian DG schemes and Commutator-free exponential integrators

Our goal in this section is to derive a suitable Semi-Lagrangian scheme in the DG setting for numerically computing the flows (exponentials) of frozen vector fields in CF integrators [6].

3.1 The semi-Lagrangian discontinous Galerkin (SLDG) methods

Suppose $\mathcal{T}_h$ is a triangulation of Ω into elements K, and let V_h denote the corresponding DG FEM space on $\mathcal{T}_h$. Then the SLDG method for approximating the solution of (1.1) (or equivalently (1.2)) over one time step $[t_n, t_{n+1}]$, with stepsize $\Delta t := t_{n+1} - t_n$, follows the discrete weak formulation:
Given initial data $u_h^n \in V_h$, find $u_h^{n+1} \in V_h$ such that for each $K \in \mathcal{T}_h$

$$\begin{aligned}\int_K u_h^{n+1} v \; d\mathbf{x} = \int_K u_h^n v \; d\mathbf{x} + \int_0^{\Delta t} \int_K [E_\tau^n u_h^n] \tilde{\mathbf{V}} \cdot \nabla v \; d\mathbf{x} d\tau \\ - \int_0^{\Delta t} \int_{\partial K} [E_\tau^n u_h^n] \tilde{\mathbf{V}} \cdot \mathbf{n}_K v \, ds d\tau, \quad \text{for all } v \in V_h,\end{aligned} \tag{3.1}$$

where $[E_\tau^n u_h^n](\mathbf{x})$ denotes the time *evolution operator* of u_h from time level t_n to time level $t_n + \tau$, defined such that

$$u_h(\mathbf{x}, t_n + \tau) = [E_\tau^n u_h^n](\mathbf{x}) := u_h(\boldsymbol{\chi}(t_n + \tau), t_n),$$

$\boldsymbol{\chi}$ being the solution of the ODE

$$\frac{d\boldsymbol{\chi}(t)}{dt} = \tilde{\mathbf{V}}(\boldsymbol{\chi}(t)), \quad t \in (t_n, t_n + \tau), \quad \text{given } \boldsymbol{\chi}(t_n) = \mathbf{x}. \tag{3.2}$$

$\tilde{\mathbf{V}}$ is an approximation of $\mathbf{V}$ that ensures the *numerical flux conservation*[2] across interelement boundaries. In order to achieve this, the authors of [19] proposed choosing $\tilde{\mathbf{V}}$ as the projection of $\mathbf{V}$ on Raviart-Thomas FEM elements of lowest order. This however limits

[1] Here we have, for simplicity, suppressed the dependence of the solution $y = y(t)$ on mesh parameter h.
[2] numerical fluxes are single-valued on interlement boundaries

the spatial accuracy of the method. For the time integrals they used a quadrature with intermediate time nodes given such that $\tau \notin \{0, \Delta t\}$, e.g., the midpoint rule. This way any further ambiguity with evaluating the flux at interelement boundaries is avoided (at least for nonzero advecting velocity). A suitable numerical flux limiter is also used to ensure the monotonicity or positivity of the overall scheme.

The SLDG has been recently developed by Qiu and Shu [17] with essential modifications being the *exact* implementation of the evolution operator $E_\tau^n u_h^n$. Also the Divergence Theorem is used to evaluate the last two integrals in (3.1), and only Gauss quadratures and nodes are used (as opposed to Raviart-Thomas elements). The SLDG methods in [17] are further extended for the numerical treatment of the Vlasov-Poisson equations.

Inspired by pioneering work on DG methods for linear advection problems based on the *upwind principle* [18, 14] we make the following *modifications* to the SLDG (3.1): Find $u_h^{n+1} \in V_h$ such that for each element K,

$$\begin{aligned} \int_K u_h^{n+1} v \; d\mathbf{x} = \int_K u_h^n v \; d\mathbf{x} + \int_0^{\Delta t} \int_K [E_\tau^n u_h^n] \mathbf{V} \cdot \nabla v, \, d\mathbf{x} d\tau \\ - \int_0^{\Delta t} \int_{\partial K} \widehat{[E_\tau^n u_h^n]} \mathbf{V} \cdot \mathbf{n}_K v \, ds d\tau, \quad \text{for all } v \in V_h, \end{aligned} \tag{3.3}$$

where in $\widehat{(\cdot)}$ we consider the *upwind* values of u_h at element boundaries. The upwinding principle is also proposed in [17]. Other approximations for the numerical fluxes includes those mentioned in Section 2.1 and are generally acceptable. However, for the sake of simplicity, we use the upwinding principle. Also this choice, as opposed to low order Raviart-Thomas approximations, does not destroy the high spatial accuracy obtainable via the DG methods.

3.2 Commutator-free Lie group method

Semi-Lagrangian commutator-free Lie group methods (or exponial integrators) are known to be very accurate for approximating linear pure convection problems and have good performance in convection-dominated problems [3, 4, 5]. We intend to use the SLDG scheme (3.3) as a building block for computing exponentials in the commutator-free methods.

Now suppose $u_h^n \in V_h$. Then for $\mathbf{x} \in K$, we have the following algorithm:

Algorithm .1. *Commutator-free method.*

$w^0 = u_h^n(\mathbf{x})$
for $i = 1 : s$ **do**
  $U_i = \exp\left(\sum_{j=1}^s \alpha_{ij}^J F_j\right) \cdots \exp\left(\sum_{j=1}^s \alpha_{ij}^1 F_j\right) w^0$
  $F_i = \Delta t F_{U_i} = \Delta t \mathbf{V}_i \cdot \nabla$
end for
$u_h^{n+1} = \exp\left(\sum_{j=1}^s \beta_j^J F_j\right) \cdots \exp\left(\sum_{j=1}^s \beta_j^1 F_j\right) w^0$

where $\mathbf{V}_i$ denotes the value of $\mathbf{V}$ at the intermediate time $t_n^i = t_n + c_i \Delta t$. Meanwhile F_i represent vector fields frozen at stage values U_i, and $\{\alpha_{ij}^l, \beta_j^l\}$, $i, j = 1, \ldots, s$, $l = 1, \ldots, J$ are coefficients of the CF method, typically constructed out of s-stage Runge-Kutta methods with coefficients $\{a_{ij}, b_i, c_i\}$, $i, j = 1, \ldots, s$ (see [6] for details).

We write [3] $\hat{F}_{il} = \sum_{j=1}^s \alpha_{ij}^l F_j$, $i = 1, \ldots, s+1$, choosing $\alpha_{s+1,j}^l := \beta_j^l$. Then on each

[3] For an explicit method, this sum depends on previously computed frozen vector fields F_j, $j < i$.

element K the flow

$$w_h^1 = \exp\left(\hat{F}_{il}\right) w_h^0$$

would be approximated via the SLDG scheme (3.3) as follows

$$\begin{aligned}\frac{1}{\Delta t}\int_K \left(w_h^1 - w_h^0\right) v \; d\mathbf{x} = \sum_{j=1}^{s} \alpha_{ij}^l \int_K [E_{\tau_j}^n w_h^0]\mathbf{V}_j \cdot \nabla v \; d\mathbf{x} \\ - \alpha_{ij}^l \int_{\partial K} \widehat{[E_{\tau_j}^n w_h^0]}\mathbf{V}_j \cdot \mathbf{n}_K v(s)\, ds, \quad \text{for all } v \in V_h,\end{aligned} \tag{3.4}$$

where $\tau_j = c_j \Delta t$. The formula (3.4) is adapted from (3.3) by replacing the integral over the time interval with the quadrature rule dictated by the CF method.

Generally in the DG as well as the SLDG methods a *flux-limiter* is being introduced at each time step to preserve the monotonicity or positivity of the solution. This has not been exploited in our numerical examples, since the problems considered are linear, and the solution and advecting velocities are fairly smooth. For more discussion on the use of flux-limiters in the context of DG methods we refer to [8, 21].

4 Numerical results

4.1 Pure advection in 1D

To show the stability of the new SLDG methods over RKDG methods, we consider the constant advection of the Gaussian cone in 1D described by

$$u_t + a u_x = 0, \quad x \in \Omega := (-1, 1), \quad t > 0, \tag{4.1}$$

where $a = 2$, and the initial data and exact solution are given by the equation

$$u(x,t) = \exp(-(x - x_0 - at)^2/2\lambda^2),$$

with $\lambda = \frac{1}{8}$, $x_0 = 0$. We use periodic boundary conditions. The spatial discretization is carried out using the discontinuous Galerkin method with $N_e = 10$ elements and Gauss-Lobatto-Legendre (GLL) quadratures of polynomial degree p. Integration is done upto final time $T = 1$, using n_{steps} time steps corresponding to a Courant number C_r (see [10]). We compare the performance of the SLDG method with RKDG method at various Courant numbers (C_r). For the RKDG method we have used the Godunov-type flux approximation. Both methods are constructed out of the third order RK method described by the Butcher tableau (see e.g. [8, 21])

$$\begin{array}{c|ccc} 0 & & & \\ 1 & 1 & & \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & \\ \hline & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{array}.$$

All errors are measured in an element-wise relative L_2-norm, that is,

$$\|u - u_{exact}\|_{L_2(\Omega)} = \sqrt{\frac{\sum_K \int_K (u - u_{exact})^2}{\sum_K \int_K u_{exact}^2}}$$

The results shown in Table 1 clearly reveals that the SLDG methods perform better than the RKDG methods at higher Courant numbers. The RKDG methods become unstable at Courant numbers higher than 1. The SLDG method on the other hand loses some accuracy at lower Courant numbers. This is as a result of accumulation of the SL interpolation error as we take many time steps. A remedy to this is to us higher order interpolation. In table 2 we have again compared the two methods at a fixed Courant number $C_r \approx 0.25$ but different values of polynomial degree p. Both methods exhibit spectral (spatial) order of convergence. Notice that since this Courant number is small, the SLDG only begins to out-perform the RKDG at larger values of p (in this case $p >= 10$), where SL interpolation errors are minimal.

Also shown in Figure 1 are results obtained with CF time integrators (CF122 and CF233) based on second and third order explicit Runge-Kutta methods (respectively) taken from [1], and the Euler method (CF111). The CF122 method is based on the midpoint rule. Integration is done upto a final time $T = 1$, and in each case the number of time steps used is $N_{steps} = 400$ which corresponds to a Courant number $C_r \approx 0.5$. We notice the the CF111 method leads to numerical damping (error $= 1.50 \times 10^{-1}$), but the second and third order methods CF122 and CF233 yield accurate results (error $= 2.00 \times 10^{-3}$ and 8.38×10^{-6} respectively). The CF111, CF122 and CF233 are constructed out of explicit Runge-Kutta methods with Butcher tableaus given respectively as follows

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array} \qquad \begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array} \qquad \begin{array}{c|ccc} 0 & & & \\ \gamma & \gamma & & \\ 1-\gamma & \gamma-1 & 2(1-\gamma) & \\ \hline & 0 & \frac{1}{2} & \frac{1}{2} \end{array},$$

where $\gamma = (3+\sqrt{3})/6$ (see [1]).

4.2 Pure advection in 2D

We now consider the 2D test problem of [7], which involves the advection of a passive tracer in a nondivergent deformational flow. The equation is given by

$$u_t + \mathbf{V} \cdot \nabla u = 0, \quad \text{in} \quad \Omega = (0,1) \times (0,1), \tag{4.2}$$

where $u(\mathbf{x}, t)$ represents the scalar field variable and (with $\mathbf{x} := (x_1, x_2)$) the velocity field is given as

$$\mathbf{V}(\mathbf{x}, t) = \begin{bmatrix} \sin^2(\pi\, x_1) \sin(2\pi\, x_2) \cos(\pi\, t/5) \\ -\sin^2(\pi\, x_2) \sin(2\pi\, x_1) \cos(\pi\, t/5) \end{bmatrix}.$$

The boundary condition is homogeneous Dirichlet and the initial data is

$$u(\mathbf{x}, 0) = \frac{1 + \cos(\pi\, r)}{2}, \quad \text{with} \quad r := r(\mathbf{x}) = \min\left(1, 4\sqrt{\left(x_1 - \frac{1}{4}\right)^2 + \left(x_2 - \frac{1}{4}\right)^2}\right).$$

We solve this problem using the CF233 method. The spatial domain is sub-divided into $N_e = 10 \times 10 = 100$ elements. The fourth order explicit Runge-Kutta method is used to solve the accurately the characteristic equation (3.2). The initial cone is swirled around in the anti-clockwise direction, being deformed in the process, until it comes to a stationary position at time $t = 2.5$. It is then driven in the reverse direction so that at time $t = 5$ the initial cone is being re-formed. Therefore the exact solution at time $t = 5$ coincides with the initial data. The results are shown in Figure 2.

Table 1: Results for the 1D advection problem for the RKDG and SLDG methods, using various Courant numbers C_r; $p = 8, N_e = 10, T = 1$.

$\mathbf{C_r}$	$\mathbf{n_{steps}}$	$\mathbf{\Delta t}$	**Relative error ($\mathbf{L_2}$)**	
			RKDG	**SLDG**
0.10	1996	0.0005	1.1699×10^{-6}	2.5513×10^{-5}
0.25	799	0.0013	1.3736×10^{-5}	1.7444×10^{-5}
0.50	400	0.0025	1.0924×10^{-4}	5.9188×10^{-6}
0.75	267	0.0037	3.6679×10^{-4}	4.0203×10^{-6}
1.00	200	0.0050	1.3536×10^{-3}	1.1646×10^{-5}
1.25	160	0.0063	∞	7.3864×10^{-6}
1.49	134	0.0075	∞	3.2610×10^{-4}

Table 2: Results for the 1D advection problem for the RKDG and SLDG methods, using various polynomial degrees p; Courant numbers $C_r \approx 0.25, N_e = 10, T = 1$.

$\mathbf{p}$	$\mathbf{C_r}$	$\mathbf{n_{steps}}$	$\mathbf{\Delta t}$	**Relative error ($\mathbf{L_2}$)**	
				RKDG	**SLDG**
1	0.25	40	0.0250	7.3567×10^{-1}	1.1407×10^{0}
2	0.25	80	0.0125	2.0851×10^{-1}	4.1497×10^{-1}
4	0.23	250	0.0040	5.1525×10^{-3}	3.8406×10^{-2}
6	0.24	500	0.0020	8.0856×10^{-5}	9.1899×10^{-4}
8	0.25	800	0.0013	1.3685×10^{-5}	1.7460×10^{-5}
10	0.25	1250	0.0008	3.5815×10^{-6}	2.8314×10^{-7}
12	0.24	1750	0.0006	1.3052×10^{-6}	3.6590×10^{-9}

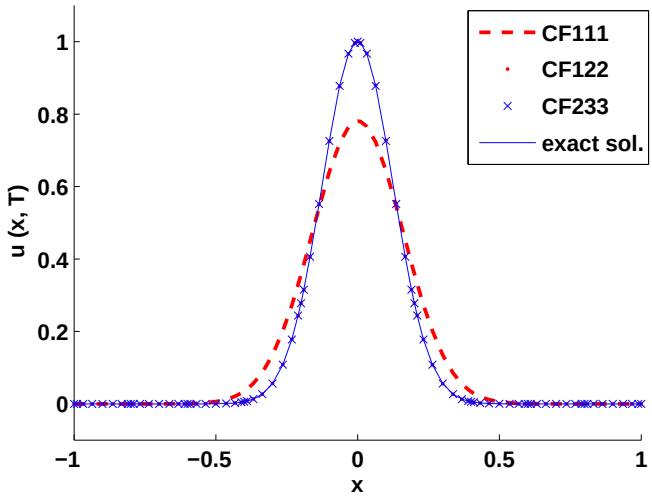
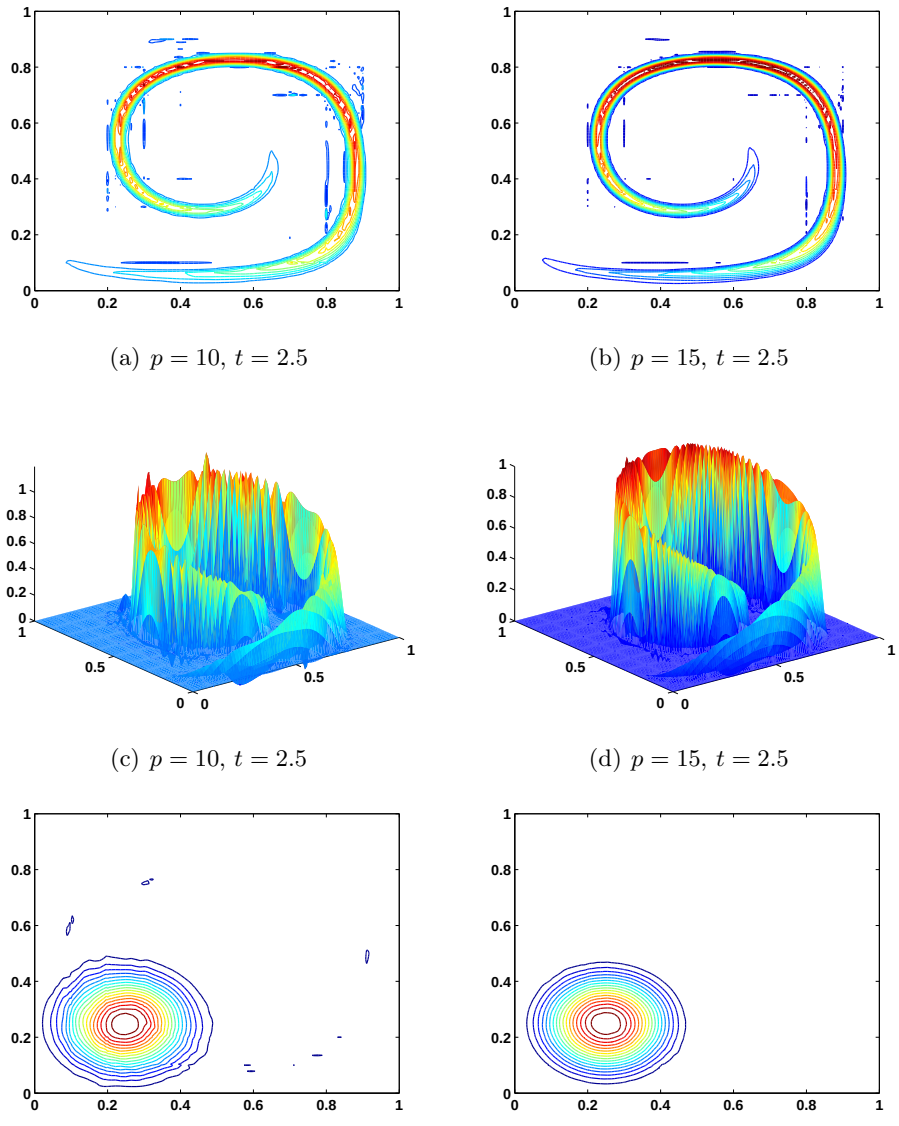


Figure 1: SLDG methods with $p = 8$, $N_e = 10$ for Gauss cone advection in 1D, using first, second and third order time integrators CF111, CF122 and CF233. The exact solution is represent by the solid curve. The respective errors in the relative L_2 norm are 1.50×10^{-1}, 2.00×10^{-3}, 8.38×10^{-6}. Meanwhile $T = 1, n_{steps} = 400$ and $C_r \approx 0.5$

We start by using $N_e = 10 \times 10$ elements with polynomial degree $p = 10$ in each element (see first colum in Figure 2). The time step chosen here is $\Delta t = 1/120$ which corresponds to a Courant nmber of $C_r = 2.5253$. The results show some overshoots and undershoots in the solution. Also the exact solution is not fully recovered. The L_2-error obtained in the case is 0.0063. Next we use a refined mesh, by choosing $p = 15$ and $N_e = 10 \times 10$. The time step used here is $\Delta t = 1/240$ which corresponds to a Courant nmber of $C_r = 2.7370$. The solution is greatly improved (see second column in Figure 2). We obtain an error of 0.0019 in this case. We also observed that at these choice of Courant numbers the RKDG schemes were unstable.



(a) $p = 10$, $t = 2.5$

(b) $p = 15$, $t = 2.5$

(c) $p = 10$, $t = 2.5$

(d) $p = 15$, $t = 2.5$

(e) $p = 10$, $t = 5$

(f) $p = 15$, $t = 5$

Figure 2: Tracer advection in 2D using CF233. In the first column, $p = 10$. In the second column, $p = 15$. The number of elements used in both case is $N_e = 10 \times 10$. First row: solution contours at $t = 2.5$; Second row: solutions at $t = 2.5$; third row: solution contours at $t = 5$. Contour levels: -36 to 36 by 18.

5 Conclusion

We have established a relation between the SLDG methods and the commutator-free (CF) methods for solving linear advection problems with divergence-free velocities. We have also shown (via numerical experiments) that the methods have better stability at high Courant numbers than the RKDG methods, and maintain the good spatial accuracy of DG methods. The model problems considered contain advection velocities that are constant or dependent on both time and space. The main advantage of the CF methods is that they provide routines for accurate approximations of linear pure advection.We believe that these preliminary results are promising and that it is worth pursuing the study of the SLDG combined with CF methods in the case where the advection term is nonlinear. This idea, however, has not been investigated here. Applications to nonlinear convection-diffusion models would also be of interest. A recent study on this is the work of Qiu and Shu in [17]

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