An Introduction to Baddeley's Delta metric

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Abstract

Metrics between images gives a measure of the discrepancy between two image is of basic interest among most fields within image processing and computer vision. Although its often easy visually to state that "Image A is closer to C than image B", it has shown to be difficult to bring this kind of knowledge into mathematics. In [1, 2] A. Baddeley proposed the *delta metric* for images with a few unordered colors or labels. The delta metric has reasonable visual performance, good theoretical properties in addition to being simple in structure. In this note, we give an introduction to this metric and a summary of its main properties.

Image metrics or error measures between two images are important within a large area of image processing and computer vision. Of basic interest are image metrics for binary images and images with a few unordered colors or labels, typically output from image segmentation, image classification or boundary detection algorithms. An image metric may *i*) provide a theoretical framework for deriving optimal algorithms (with respect to that metric), *ii*) quantify numerically the performance of different algorithms, and *iii*) give an objective measure of achieved quality. The spatial context in imaging makes it difficult to design good measures of the discrepancy between two images, which are firstly in reasonable agreement with our visual impression and which also are metrics with good theoretical properties. The *delta metric* proposed by Baddeley [1, 2] have reasonable visual performance, good theoretical properties in addition to being simple in structure.

To fix notation, let Λ be a finite lattice equipped with a metric ρ. For example, $\rho(i,j)$ might be the Euclidean distance between site i and j, or the shortest path length between site i and

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j. Binary images can take a binary value (zero or one) at each site $i \in \Lambda$, which we interpret as background and foreground respectively. Let $\mathbf{x}$ and $\mathbf{y}$ be two such binary images and let $\mathbf{x}'$ be the complement image of $\mathbf{x}$ found by switching the foreground and background. A binary image $\mathbf{x}$ can be uniquely identified by the set of foreground sites $b(\mathbf{x}) = \{i \in \Lambda : x_i = 1\}$. The distance from a site i to a set $A \subseteq \Lambda$ is defined as

$$d(i, A) = \min_{j \in A} \rho(i, j).$$

Denote by $|A|$ the number of sites in A. Let w be a strictly increasing and concave function defined on $[0, \infty]$ that satisfies $w(0) = 0$. For two binary images $\mathbf{x}$ and $\mathbf{y}$, Baddeley [1, 2] defines their delta distance as

$$\Delta_w^p(\mathbf{x}, \mathbf{y}) = \left(\frac{1}{|\Lambda|} \sum_{i \in \Lambda} \Big| w(d(i, b(\mathbf{x}))) - w(d(i, b(\mathbf{y}))) \Big|^p \right)^{1/p}, \quad 1 \leq p < \infty. \tag{1}$$

and names Δ_w^p the *delta metric*. The delta metric is a L_p average of the distances $w(d(i, b(\cdot)))$ from each site to the nearest foreground site in each image. It is clear that (1) is a metric with respect to the distances $w(d(i, b(\cdot)))$ and also a metric with respect to the binary images $\mathbf{x}$ and $\mathbf{y}$. To simplify the following discussion we take $p = 2$. We also assume that w is eventually constant, i.e. there exists a value t_c such that $w(t) = c$ for any $t \geq t_c$. The contribution to the delta distance from site i will then only depend on those sites in $\mathbf{x}$ and $\mathbf{y}$ within a ball of radius c centered at i. We follow [1, 2] and use in the following $w(t) = \min\{t, c\}$, where $c > 0$ is the *cut–off* value. We use the notation Δ_c^2 in this case.

Basic Properties

Let us from a practical point of view try to illuminate the basic properties of the delta metric.

The delta metric has two other important metrics as limits. First, if c is less than the smallest distance between two distinct sites, then $(\Delta_c^2/c)^2$ is the average sitewise error rate

$$\left(\Delta_c^2(\mathbf{x}, \mathbf{y})/c \right)^2 = \frac{1}{|\Lambda|} \sum_{i \in \Lambda} 1_{[x_i \neq y_i]}.$$

Secondly, if we let $p \to \infty$ and $w(t) = t$, we obtain the Hausdorff metric well known from the theory of mathematical morphology and stochastic geometry [5, 6].

The delta metric is unsymmetric with respect to foreground and background in the image, so the distance between $\mathbf{x}$ and $\mathbf{y}$ is different in general from the distance between the complement images $\mathbf{x}'$ and $\mathbf{y}'$. This follows from the definition itself. The delta metric emphasizes the set of foreground sites to determine the distance between two images, which is natural in many situations. The foreground may represent the important components in the image, for example objects, roads or edges. The background in the image is then of less interest. If the foreground and background are equally important, we can use the symmetrized version of the delta metric

$$\widetilde{\Delta}_c^2(\mathbf{x}, \mathbf{y}) = \left((\Delta_c^2(\mathbf{x}, \mathbf{y}))^2 + (\Delta_c^2(\mathbf{x}', \mathbf{y}'))^2 \right)^{1/2}. \tag{2}$$

As evident from the definition itself, (2) is not only a function of $\{1_{[x_i \neq y_i]}, i \in \Lambda\}$. A metric between two images with more than one foreground color, can be defined similarly to (2).

The cut–off value c can be interpretated as a parameter that determines the importance of misplacing a foreground object in the image. Let $b(\mathbf{x})$ be a single compact object centered at some origin in the image, and let $\mathbf{x}_\delta$ be the image where the object is misplaced and centered at δ. If δ and c is less than the size of the object, then

$$\Delta_c^2(\mathbf{x}, \mathbf{x}_\delta) \approx \Delta_\delta^2(\mathbf{x}, \mathbf{x}_c),$$

where identity holds for a one–dimensional continuous lattice. The cost of misplacing an object with offset δ and using a cut–off value c, is roughly the same as misplacing an object with offset c and using a cut–off value δ.

Images (*a*–*c*) demonstrate the sensitivity of the delta metric to add–on components to objects, and the reduced sensitivity to errors in the interior of objects. This is a reasonable behavior studying images of compact objects. We use white as the background color and grey as the foreground color. In each of the three images, we display those Euclidean distances which are less than the cut–off value 3, and the arrows go from each site to the nearest foreground site with the Euclidean distance as their length. The images (*b*) and (*c*) have delta distance 0.26 and 0.20 to the reference image (*a*). Image (*c*) is closer to image (*a*) than image (*b*) although the number of sitewise errors are 9 for (*c*) and 2 for (*b*). The extra add–on object in image (*b*) influences the distances at a number of sites and so is heavily penalized. A hole in the object as in image (*c*), is considered as less harmful as its influence on the distances is less.

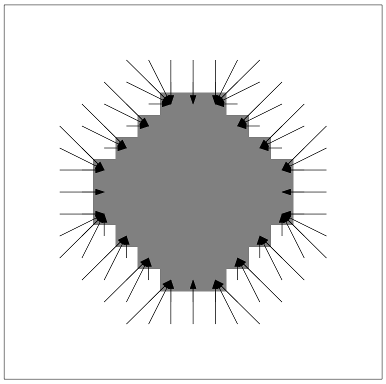

Image (*a*)

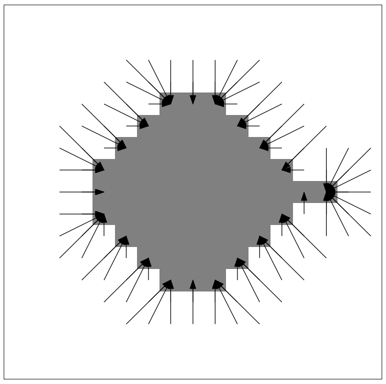

Image (*b*)

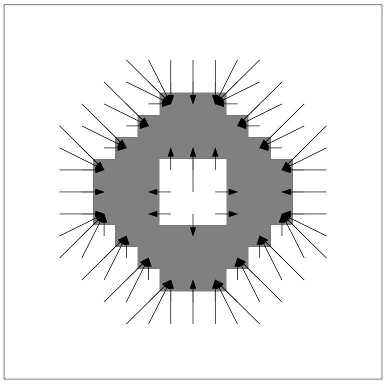

Image (c)

Further Results and Applications

The delta metric can be defined on a finite subset of the plane to be a metric between images of compact sets [2]. Under appropriate conditions Δ_w^p generates the myopic, or the hit–or–miss, topology which is an important property within image analysis as described by Serra [5]. The delta metric has been used to derive optimal Bayes estimators with application to image classification [3] and object recognition [4].

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